Genericity and randomness with ITTM's

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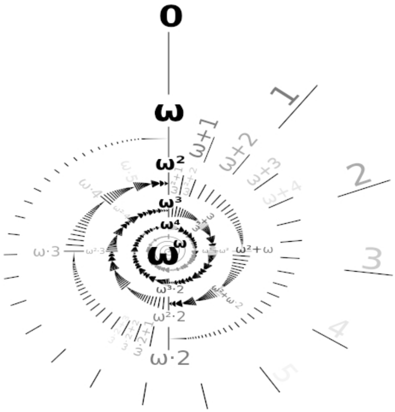
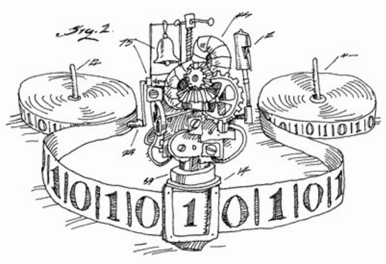
Infinite Time Turing Machine





Infinite Time Turing Machine

Definition (Hamkins, Lewis, 2000)

An Infinite Time Turing Machine is a Turing Machine with a special state called "limit state" and three tapes:

- The input tape,
- the working tape, and
- the output tape.

We now need to define a computation by an ITTM. Computations are indexed by ordinals.

- At successor step, the behaviour is the same as regular Turing Machines.
- We need to specify the behaviour at limit steps.

Limit steps

At limit steps:

- The state becomes the special "limit state".

```
function limit() {
    ...
}
```

- The value of each cells is the lim inf of its values at previous stage of computation:

Cell C_i: $\boxed{0} \rightarrow \boxed{1} \rightarrow \boxed{0} \rightarrow \boxed{1} \rightarrow \boxed{0} \rightarrow \boxed{1} \quad \cdots \xrightarrow{\liminf} \quad \boxed{0}$

Cell C_j: $\boxed{1} \rightarrow \boxed{1} \rightarrow \boxed{0} \rightarrow \boxed{0} \rightarrow \boxed{0} \rightarrow \boxed{0} \quad \cdots \xrightarrow{\liminf} \quad \boxed{0}$

Cell C_k: $\boxed{0} \rightarrow \boxed{0} \rightarrow \boxed{1} \rightarrow \boxed{1} \rightarrow \boxed{1} \rightarrow \boxed{1} \quad \cdots \xrightarrow{\liminf} \quad \boxed{1}$

Computing with an ITTM

We have a notion of computability for reals;

Definition (Writability)

A real x is writable if there is an ITTM M starting with blank input tape, which reach a halting state with x written on its output tape.

But also for classes of reals:

Definition (Decidability)

A class of reals $\mathcal{A}$ is ITTM-decidable if there exists an ITTM M such that $M(X)\downarrow = 1$ if $X \in \mathcal{A}$ and $M(X)\downarrow = 0$ otherwise.

The power of ITTM-decidability

Are ITTMs really strong?

Theorem

The class WO of codes for well-orders is ITTM-decidable.

Algorithm: Decide if $<$ is a well-order

```
while < is not empty do
    Look for the smallest element of <
    if There is no such element then
        return 0
    else
        Remove it from <
return 1;
```

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Algorithm: Look for a smallest element

```
min := any element of < ;
flag := 1;
for all elements e of Field(<) do
    if e < candidate then
        flag := 0
        flag := 1
        min := e
if flag = 0 then
    return "no minimum"
return min
```

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Corollary

All Π^1_1 sets (resp. class) are writable (resp. decidable).

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Kleene's $\mathcal{O}$, and $\mathcal{O}^{\mathcal{O}}$ and $\mathcal{O}^{(\mathcal{O}^{\mathcal{O}})}$ $\cdots$ are writable.

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Where does it stop?

Theorem

If an ITTM stops, it stops before ω_1.

Definition

We define $\gamma = \sup\{\alpha : \alpha \text{ is a halting time}\}$.

By cofinality, $\gamma < \omega_1$.

Definition (λ)

We call λ the supremum of the ordinals with writable codes.

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A real X is eventually writable if there is an ITTM that write X at some point X and never changes it.

Definition (ζ)

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Definition (ζ)

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A real X is accidentally writable if there is an ITTM that write X at some point X of its computation.

Definition (Σ)

We call Σ the supremum of the ordinals with accidentally writable codes.

Constructibility

Definition

Gödel's constructible are defined by induction over the ordinals:

$$
\begin{aligned}
L_0 &= \emptyset \\
L_{\alpha+1} &= \{\{x \in L_\alpha : L_\alpha \models \Phi(x)\} : \Phi \text{ a formula}\} \\
L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha
\end{aligned}
$$

Constructibility

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$$\begin{aligned} L_0[X] &= \{X\} \\ L_{\alpha+1}[X] &= \{\{x \in L_\alpha[X] : L_\alpha[X] \models \Phi(x)\} : \Phi \text{ a formula}\} \\ L_\lambda[X] &= \bigcup_{\alpha<\lambda} L_\alpha[X] \end{aligned}$$

Fundamental theorem for ITTMs

These ordinals λ, ζ and Σ are characterized in the following theorem:

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Theorem (Welch)

$(\lambda\ ,\zeta\ ,\Sigma\ )$ is the smallest triplet such that

$$L_\lambda \quad \prec_1 L_\zeta \quad \prec_2 L_\Sigma$$

Moreover $\gamma\ = \lambda\ .$

Definition (Stability)

$A \prec_n B$ if for every Σ_n formula ϕ with parameter in A, $A \models \Phi$ if and only if $B \models \Phi$.

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These ordinals λ, ζ and Σ are characterized in the following theorem:

Theorem (Welch)

Let x be any real.

$(\lambda^x, \zeta^x, \Sigma^x)$ is the smallest triplet such that

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Theorem (Welch)

$(\lambda\ ,\zeta\ ,\Sigma\ )$ are such that

L_λ	is the set of sets with writable code
L_ζ	is the set of sets with eventually writable code
L_Σ	is the set of sets with accidentally writable code

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$(\lambda^x, \zeta^x, \Sigma^x)$ are such that

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$L_{\zeta^x}[x]$ is the set of sets with eventually writable code

$L_{\Sigma^x}[x]$ is the set of sets with accidentally writable code


Celebrating Randomness

Randomness

We will use the following paradigm to define randomness:

Paradigm

A set Z is random if it avoids all the sufficiently simple null sets.

- Having countably many simple sets ensures that the randoms are co-null
- The more null sets are avoided, the more random the set is.

Let α be an ordinal.

Definition (randomness over L_α, Carl and Schlicht)

A set X is random over L_α if X is in no null Borel set with code in L_α.

Example

Randomness over $L_{\omega_1^{\mathrm{CK}}}$ corresponds to Δ^1_1-randomness

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Example

Randomness over $L_{\omega_1^{\mathrm{CK}}}$ corresponds to Δ^1_1-randomness

Definition (ITTM-decidable-randomness, Carl and Schlicht)

A set X is ITTM-decidable random if X is in no null ITTM-decidable set.

Theorem

Randomness over L_λ corresponds to ITTM-decidable-randomness

Definitions

Definition (α-ce open sets)

An open set U is α-ce if

$$U = \bigcup_{\substack{L_\alpha \models \Phi(\sigma) \\ \sigma \in 2^{<\omega}}} [\sigma]$$

for some Σ_1 formula Φ with parameters in L_α.

Definition (α-ML-randomness, Carl and Schlicht)

A set X is α-ML random if X is in no uniform intersection $\bigcap_n \mathcal{U}_n$ of uniformly α-ce open sets such that $\lambda(\mathcal{U}_n) \leq 2^{-n}$.

Example

Π^1_1-ML-randomness is also ω_1^{CK}-ML-randomness.

Comparison with higher randomness

In higher randomness, we have the following:

Theorem

Π^1_1-ML randomness is strictly stronger than Δ^1_1-randomness.

Could we generalize the results to other ordinals?

Question

For which ordinals α do we have:
"α-ML randomness is strictly stronger than randomness over L_α"?

- For $\alpha = \omega_1^{\mathrm{CK}}$, it is the case.
- What about $\alpha = \lambda$, or ζ, or Σ?

Projectibility

To answer this question, we need the concept of projectibility.

Definition (Projectible ordinals)

We say that an ordinal α is projectible into an ordinal β if there is an injective function from α to β that is Σ_1-definable in L_α.
We say that α is projectible if α is projectible into some $\beta < \alpha$.
The least such β is called the projectum of α.

Theorem (Anglès d'Auriac, Monin)

Let α be limit and such that $L_\alpha \models$"everything is countable". Then, the following are equivalent:

- α is projectible into ω,
- There is a universal α-ML random test,
- α-ML-randomness is strictly stronger than randomness over L_α.

Application with λ, ζ, Σ

Theorem (Friedman)

If $L_\alpha \models$"$\exists x : x$ is uncountable", then there exists $\beta, \gamma < \alpha$ such that $L_\beta \prec L_\gamma$.

Therefore, L_λ, L_ζ and L_Σ all satify "everything is countable".

Application with λ, ζ, Σ

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Theorem

The ordinal λ is projectible into ω.

Assign any $\alpha < \lambda$ to the code of the ITTM writing α.

Corollary

λ-ML-randomness is strictly stronger than ITTM-decidable randomness.

Application with λ, ζ, Σ

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Theorem

The ordinal ζ is not projectible into ω.

Suppose that an eventually writable parameter α can be used to have a projectum $f : \zeta \to \omega$. Then every eventually writable ordinals become writable using α. Then ζ becomes eventually writable using α. But then ζ is eventually writable.

Corollary

ζ-ML-randomness coincide with randomness over L_ζ, and there is no universal ζ-ML-test.

Application with λ, ζ, Σ

Theorem (Friedman)

If $L_\alpha \models$ "$\exists x : x$ is uncountable", then there exists $\beta, \gamma < \alpha$ such that $L_\beta \prec L_\gamma$.

Therefore, L_λ, L_ζ and L_Σ all satify "everything is countable".

Theorem

The ordinal Σ is projectible into ω, using ζ as a parameter.

Recall that Σ is not admissible!

Corollary

Σ-ML-randomness is strictly stronger than randomness over L_Σ.

ITTM randomness

What about equivalent of Π^1_1 randomness?

Definition (ITTM randomness)

A real X is said ITTM-random if it is in no ITTM-semi-decidable null set.

Theorem (Carl, Schlicht)

$$
\begin{aligned}
X \text{ is ITTM-random} \quad &\iff \quad X \text{ is random over } L_\Sigma \text{ and } \Sigma^X = \Sigma \\
&\iff \quad X \text{ is random over } L_\zeta \text{ and } \zeta^X = \zeta \\
&\iff \quad X \text{ is random over } L_\lambda \text{ and } \lambda^X = \lambda
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Compared with higher randomness:

Theorem

Let X be a real. Then

$$X \text{ is } \Pi^1_1\text{-random} \iff X \text{ is } \Delta^1_1\text{-random and } \omega_1^X = \omega_1^{\mathrm{CK}}$$

Diverging from higher randomness

In the higher randomness case, we have:

Theorem

$$\Delta^1_1\text{-}randomness \subsetneq \Pi^1_1\text{-}ML\text{-}randomness \subsetneq \Pi^1_1\text{-}randomness$$

However, in the ITTM case we have :

Theorem

$$\begin{array}{rcl} Randomness\ over\ L_\lambda & \subsetneq & \lambda\text{-}ML\text{-}randomness \subsetneq ITTM\text{-}randomness \\ Randomness\ over\ L_\zeta & = & \zeta\text{-}ML\text{-}randomness \subsetneq ITTM\text{-}randomness \\ Randomness\ over\ L_\Sigma & \subseteq & ITTM\text{-}randomness \subsetneq \Sigma\text{-}ML\text{-}randomness \end{array}$$

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Which leaves us with the question:

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Do we have?

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1. It is equivalent to the question: Does Σ-randomness for X implies $L_\zeta[X] \prec_2 L_\Sigma[X]$?
2. The problem comes from the fact that Σ is not admissible (ie. L_Σ is not a model of Σ_1-replacement)
3. What about genericity?

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3. What about genericity?

Generic objects corresponds to the typical objects with regard to Baire categoricity.

Definition (Meager sets)

A co-meager set is a countable intersection of dense open sets. The complement of a co-meager set is a meager set.

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A co-meager set is a countable intersection of dense open sets. The complement of a co-meager set is a meager set.

Definition (Genericity over L_α)

We say that X is generic over L_α if X is in every dense open set with code in L_α.

Definition (ITTM-genericity)

We say that X is ITTM-generic if X is in no ITTM-semi-decidable meager set.

The theorem relating ITTM-genericity and genericity over L_Σ still holds:

Theorem

Let X be a real. Then

$$X \text{ is ITTM-generic} \iff X \text{ is generic over } L_\Sigma \text{ and } \Sigma^X = \Sigma$$

But in fact...

The theorem relating ITTM-genericity and genericity over L_Σ still holds:

Theorem

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But in fact...

Theorem

If Z is generic over L_Σ, then $L_\zeta[Z] \prec_2 L_\Sigma[Z]$. In particular, $\Sigma^Z = \Sigma$

Corollary

ITTM-genericity and genericity over L_Σ are two equivalent notions.

there is no difference between the two notions!

To conclude:

Question

Do we have?

$$\text{randomness over } L_\Sigma \neq \text{ITTM-randomness}$$

is still unsolved...