Algorithmic randomness

Cuny logic worshop

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5 May 2017

Algorithmic randomness

Section 1

Algorithmic randomness

Defining randomness

Algorithmic randomness:

What does it mean for a binary sequence to be random ?

Intuitively : Is it reasonable to think that c_1, c_2 or c_3 could have been obtained by a fair coin tossing ?

c_1:000011000000000100010000001000010001010000001000100 . . .

c_2:101011000101100110100110001101011100100111001010 . . .

c_3:001001000011111101101010100010001000010110100011 . . .

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Law of large number : no

c_2:1010110001011001101001100011010111001001110010 10...

c_3:0010010000111111011010101000100010000101101000 11...

Defining randomness

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pattern repetition : no

c_3:0010010000111111011010101000100010000101101000011 . . .

Defining randomness

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pattern repetition : no

c_3:00100100001111110110101010001000100001011010001 1 . . .

$c_3 = \pi$: no

A general paragdigm

Intuition

A sequence of 2^ω should be random if it belongs to no set of measure 0 which is "simple to describe".

Fact

As long as at most countably many sets are "simple to describe", the set of randoms is of measure 1 (by countable additivty of measures).

The effective Borel hierarchy provides a range of natural candidates.

The Cantor space

What do we work with ?

Our playground	The Cantor space
Denoted by	2^ω
Topology	The one generated by the cylinders $[\sigma]$, the set of sequences extending σ, for every string σ
An open set $\mathcal{U}$ is	A union of cylinders
The measure λ	is the unique measure on 2^ω such that $\lambda([\sigma]) = 2^{-\lvert\sigma\rvert}$

Arithmetical complexity of sets

Following a work started by **Baire** in 1899 (Sur les fonctions de variables réelles), pursued by **Lebesgue** in his PhD thesis (1905), and many others (in particular **Lusin** and his student **Suslin**), we define the **Borel sets** on the Cantor space:

$\boldsymbol{\Sigma}^0_1$ sets are	Open sets
$\boldsymbol{\Pi}^0_1$ sets are	Closed sets
$\boldsymbol{\Sigma}^0_{n+1}$ sets are	Countable unions of $\boldsymbol{\Pi}^0_n$ sets
$\boldsymbol{\Pi}^0_{n+1}$ sets are	Complements of $\boldsymbol{\Sigma}^0_{n+1}$ sets

Effectivize the arithmetical complexity of sets

This has latter been effectivized, following a work of **Kleene** and **Mostowsky**:

Definition (Effectivization of open sets)

A set $\mathcal{U}$ is Σ^0_1, or **effectively open**, if there is a code e for a program enumerating strings such that so that $\mathcal{U}$ is the union of the cylinders corresponding to the enumerated strings.

Definition (Effectivization of closed sets)

A set $\mathcal{U}$ is Π^0_1, or **effectively closed**, if it is the complement of a Σ^0_1 set.

Effectivize the arithmetical complexity of sets

We can then continue inductively: $\left(\text{Notation} : [W_e] = \bigcup_{\sigma \in W_e} [\sigma]\right)$

Σ^0_1 sets are	of the form $[W_e]$
Π^0_1 sets are	of the form $[W_e]^c$
Σ^0_2 sets are	of the form $\bigcup_{n \in W_e} [W_n]^c$
Π^0_2 sets are	of the form $\bigcap_{n \in W_e} [W_n]$
...	...

Martin-Löf's definition

Definition (Martin-Löf randomness)

A sequence is **Martin-Löf random** if it belongs to no Π^0_2 set 'effectively of measure 0'. A Π^0_2 set 'effectively of measure 0' is called a **Martin-Löf test**.

Definition (Effectively of measure 0)

An intersection $\bigcap \mathcal{A}_n$ of sets is effectively of measure 0 if $\lambda(\mathcal{A}_n) \leqslant 2^{-n}$.

Fact

One can equivalently require that the function $f : n \to \lambda(\mathcal{A}_n)$ is bounded by a computable function going to 0.

Why Martin-Löf's definition ?

Question

Why don't we just take Π^0_2 sets of measure 0 ? How important is the 'effectively of measure 0' condition ?

Answer(1)

The 'effectively of measure 0' condition implies that there is a universal **Martin-Löf test**, that is a Martin-Löf test containing all the others.

Answer(2)

It is not true anymore if we drop the 'effectively of measure 0' condition. Instead we get a notion known as **weak-2-randomness**.

Algorithmic randomness

We can build a hierachy of randomness notions:

1-random	Every Π^0_2 sets 'effectively of measure 0'
weakly-2-random	Every Π^0_2 sets of measure 0
2-random	Every Π^0_3 sets 'effectively of measure 0'
weakly-3-random	Every Π^0_3 sets of measure 0
...	...

We have:

1-random ← w2-random ← 2-random ← w3-random ← ...

All implications are strict

Example

The set of sequences whose lim sup of the ratio of 0's and 1's is above $1/2+\varepsilon$, is the set $\bigcap_n \mathcal{U}_n$ where:

$$\mathcal{U}_n = \bigcup_{m \geqslant n} \mathcal{C}_m$$

and

$$\mathcal{C}_m = \left\{ \sigma \in 2^m \ : \ \frac{\#\{i \leqslant m \ : \ \sigma(i) = 0\}}{m} - \frac{1}{2} > \varepsilon \right\}$$

Using Hoeffding's inequality we have:

$$\lambda(\mathcal{C}_m) \leqslant e^{-2\varepsilon^2 m}$$

And thus:

$$\lambda(\mathcal{U}_n) \leqslant e^{-2\varepsilon^2 n}/(1 - e^{-2\varepsilon^2})$$

Kolmogorov complexity

Definition (Levin, Gács, Chaitin)

A prefix-free machine is a computable function $U : 2^{<\omega} \to 2^{<\omega}$ whose domain of definition is prefix-free : If $U(\sigma)\downarrow$, then $U(\tau)\uparrow$ for any $\tau \neq \sigma$.

Theorem (Levin, Gács, Chaitin)

There is a universal prefix-free machine $U : 2^{<\omega} \to 2^{<\omega}$:
The machine U is such that for any prefix-free machine M, we have a constant c_M for which $U(\sigma) \leqslant M(\sigma) + c_M$ for every σ.

Definition (Chaitin)

We define the prefix-free Kolmogorov complexity of σ by $K(\sigma) = \{\min |\tau| \ : \ U(\tau) = \sigma\}$.

Randomness and Kolmogorov complexity

Theorem (Levin, Schnorr)

A sequence X is Martin-Löf random iff there exists c such that $K(X\restriction_n) \geqslant n + c$ for every n.

Theorem (Chaitin)

The binary representation of the probability that a computer program halts, is Martin-Löf random :

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$$

Lowness for randomness

Definition

A sequence Z is **Martin-Löf random relative to** A if Z is in no $\Pi^0_2(A)$ set effectively of measure 0.

Definition

A sequence A is **low for Martin-Löf randomness** if the Martin-Löf randoms are all Martin-Löf random relative to A.

If A is computable, then A is low for Martin-Löf randomness.

How about the converse ?

Lowness for randomness

Theorem (Chaitin)

A sequence X is K-trivial if $K(X \upharpoonright_n) \leqslant^+ K(n)$ for every n.

Theorem (Solovay)

There are non-computable K-trivial sets.

Theorem (Chaitin)

Every K-trivial is computable from the halting problem (in particular there are at most countably many K-trivials).

Theorem (Hirschfield, Nies)

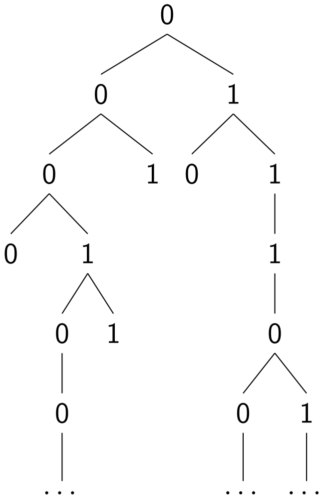
A sequence A is low for Martin-Löf randomness iff A is K-trivial.

Beyond arithmetic

∞ Section 2

Beyond arithmetic

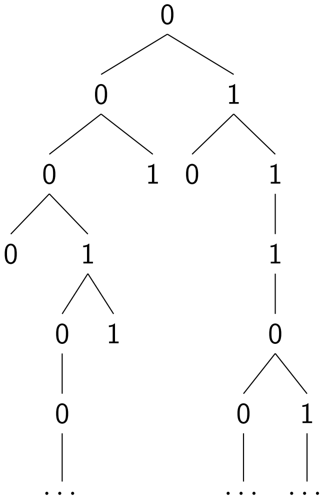
Motivation



Suppose T is a computable tree of 2^ω with **exactly** one infinite path.

Can we compute the path of the tree ?

Motivation

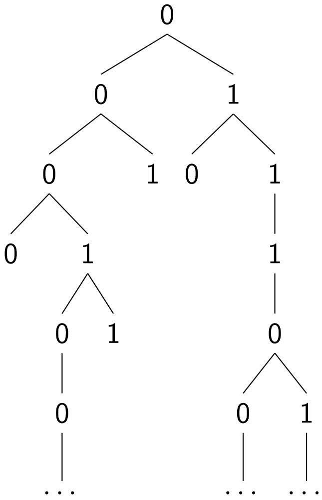


Suppose T is a computable tree of 2^ω with **exactly** one infinite path.

Can we compute the path of the tree ?

Yes. The path is computable

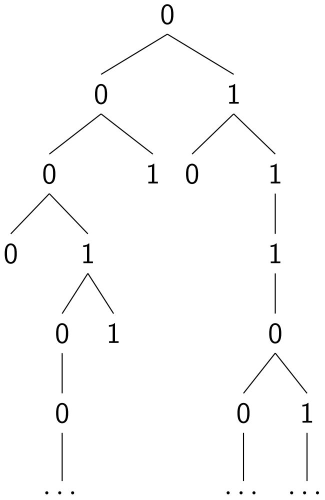
Motivation



Suppose T is a computable tree of 2^ω with perfectly many infinite paths.

Can we compute a path of the tree ?

Motivation



Suppose T is a computable tree of 2^ω with perfectly many infinite paths.

Can we compute a path of the tree ?

Not necessarily

Motivation

proposition

Every Π^0_1 class of 2^ω is the set of infinite paths of a computable tree $T \subseteq 2^{<\omega}$.

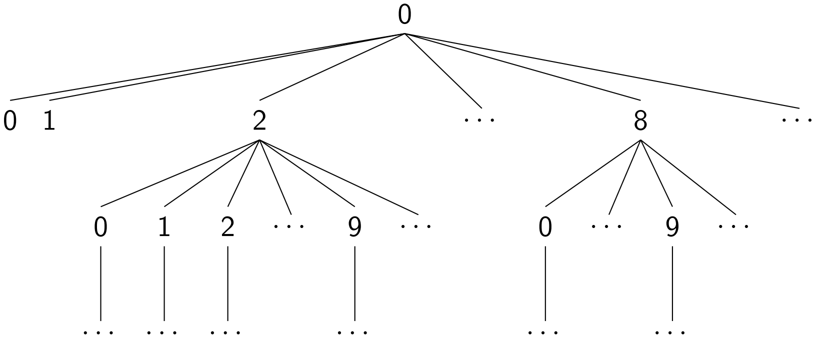
proposition

There is a Π^0_1 class which contains only Martin-Löf randoms.

proposition

The oracles which can compute a path in any Π^0_1 class are exactly the oracles which can compute a complete extention of Peano Arithmetic.

Motivation



Suppose T is a computable tree of ω^ω with **exactly** one infinite path. What computational power is needed to compute such a path ?

Computable ordinals

We define an order $<$ on a subset of ω as the smallest subset of $\omega \times \omega$ such that:

	$1 < 2$		
if	a is in the support of $<$	then	$a < 2^b$
if	$a_1 < a_2 \dots$ and if $\varphi_e(n) = a_n$	then	$a_n < 3.5^e \ \forall n$
if	$a < b$ and $b < c$	then	$a < c$

Let $\mathcal{O}$ be the support of $<$. For every $e \in \mathcal{O}$ we define $|e|$ to be the order type of $<$ restricted to elements smaller than e.

Definition (Kleene)

An ordinal α is computable if $\alpha = |e|$ for $e \in \mathcal{O}$.

Proposition

The computable ordinals form a strict initial segment of the countable ordinals.

Computable ordinals

We define a jump hierarchy by induction over $<$:

$$\begin{array}{rcl} H_0 & = & \varnothing \\ H_{2^a} & = & (H_a)' \\ H_{3.5^e} & = & \bigoplus_n H_{\varphi_e(n)} \end{array}$$

proposition

For $e_1 < e_2$ we have $H_{e_1} <_T H_{e_2}$.

proposition

For every $e \in \mathcal{O}$, the Turing degree of H_e is the Turing degree of the unique infinite path of a computable tree.

Hyperarithmetical complexity of sets

We can extend the definition of Borel sets by induction over the ordinals:

$\mathbf{\Sigma}^0_1$ sets are	Open sets
$\mathbf{\Pi}^0_1$ sets are	Closed sets
$\mathbf{\Sigma}^0_{\alpha+1}$ sets are	Countable unions of $\mathbf{\Pi}^0_\alpha$ sets
$\mathbf{\Sigma}^0_{\sup_n \alpha_n}$ sets are	Countable unions of $\mathbf{\Pi}^0_\beta$ sets for $\beta < \sup_n \alpha_n$
$\mathbf{\Pi}^0_\alpha$ sets are	Complements of $\mathbf{\Sigma}^0_\alpha$ sets

It is clear that no new sets are define at step ω_1, by uncountablity of ω_1. Before that one can prove that the hierarchy is strict.

Effective Hyperarithmetical complexity of sets

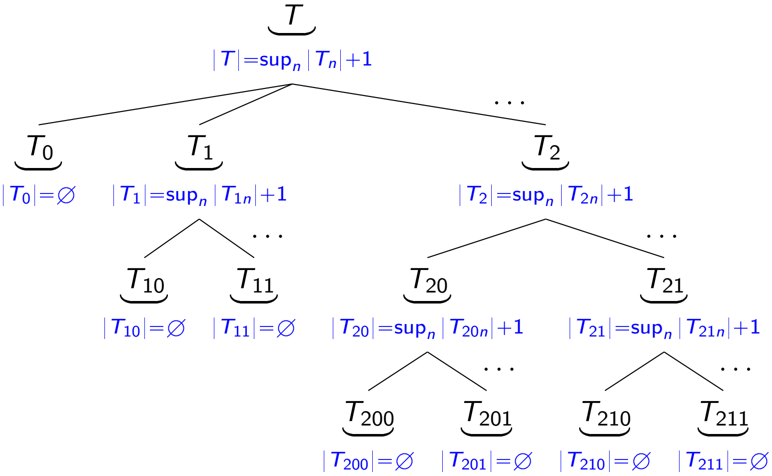
How about the effective version ? The challange is to be able to effectively 'unfold' all the components of a Σ^0_α set.

(Notation : The set of index n is denoted by $\{n\}$)

Σ^0_1 sets are	of the form $[W_e]$	with index	$\langle 0, e\rangle$
Π^0_α sets are	of the form $\{e\}^c$	with index	$\langle 1, e\rangle$
Σ^0_α sets are	of the form $\bigcup_{n\in W_e}\{n\}$ where n is an index for a Π^0_β set with $\beta < \alpha$	with index	$\langle 2, e\rangle$

Question : At what ordinal α no new set is added in the hierarchy ?

Order-type of well-founded trees



Computable ordinals and effective Borel sets

Definition (smallest non-computable ordinal)

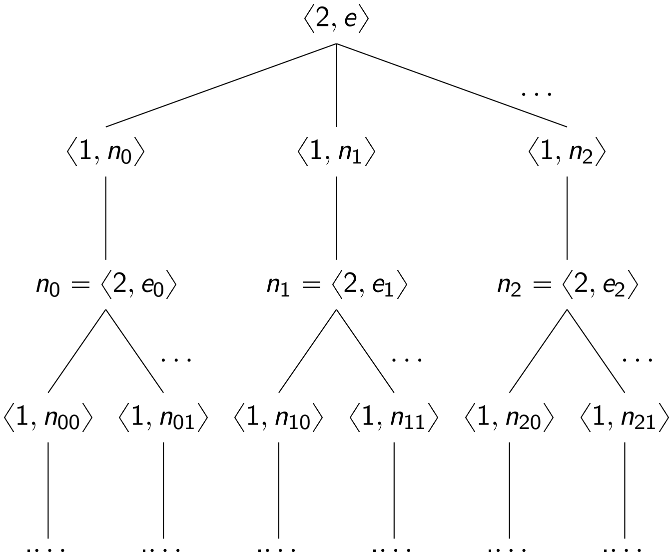
The smallest non-computable ordinal is denoted by ω_1^{ck}, where the *ck* stands for 'Church-Kleene'.

It is of historical interest to notice that the **Kleene's recursion theorem** has been 'cooked up' to work with codes of computable ordinals. Indeed, the theorem appear for the first time in 1938, in the paper called 'On notation for ordinal numbers'.

Claim

Indices of effective Borel sets are 'essentially' codes for computably enumerable well-founded trees.

Computable ordinals and effective Borel sets



Hyperarimthetical sets

It follows that every effective Borel set is Σ^0_α for $\alpha < \omega_1^{\mathsf{ck}}$. Again one can prove that the hierarchy is strict before ω_1^{ck}.

Definition (Hyperarithmetical sets)

The effective Borel sets are called **hyperarithmetical sets**.

Every Σ^0_n set for n finite is definable by a first-order formula of arithmetic. It is not the case anymore with Σ^0_ω and beyond. **We can however define them with second order formulas of arithmetic**.

Analytic and co-analytical sets

Definition (Σ^1_1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ is Σ^1_1 if it can be defined by a formula of arithmetic whose second order quantifiers are only existential (with no negation in front of them).

Definition (Π^1_1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ is Π^1_1 if it can be defined by a formula of arithmetic whose second order quantifiers are only univeral (with no negation in front of them).

Definition (Δ^1_1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ is Δ^1_1 if it is both Σ^1_1 and Π^1_1.

Analytic and co-analytical sets

Proposition

An effective Borel set is both Σ^1_1 and Π^1_1.

While this proposition is essentially a tedious but straightforward induction over the computable ordinals, the converse is less tedious but much clever. A non-effective version was first prove by Suslin in 1917 ("Sur une definition des ensembles mesurables B sans nombres transfinis"). Then the effective version was proved much latter (after the effective concepts were introduced) by Kleene in 1955 ('Hierarchies of number theoretic predicates').

Theorem (Suslin, Kleene)

An set is effective Borel iff it is both Σ^1_1 and Π^1_1.

Analytic and co-analytical sets

Notation

We denote by $\mathcal{O}$ the set of codes for computable ordinals, and $\mathcal{O}^X$ the set of X-codes for X-computable ordinals.
We denote by $\mathcal{O}_\alpha$ the set of codes for computable ordinals, coding for ordinal strictly smaller than α.

Example : we have $\mathcal{O} = \mathcal{O}_{\omega_1^{ck}}$ and $\mathcal{O}^X = \mathcal{O}^X_{\omega_1^X}$

We now have that the set $\mathcal{O}$, play the same role as $\emptyset'$, but for Π^1_1 predicates of ω.

Theorem (Complete Π^1_1 set)

A set of integers A is Π^1_1 iff there is a computable function $f : \omega \mapsto \omega$ so that $n \in A$ iff $f(n) \in \boldsymbol{W}$.

Analytic and co-analytical sets

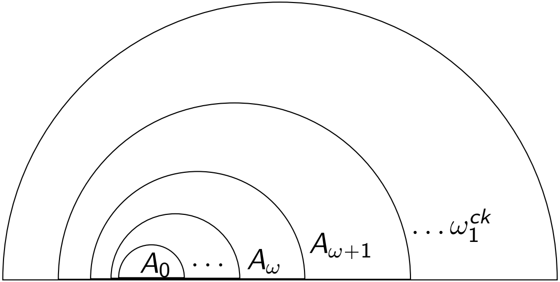
$\mathcal{A}$ is	a set of integer	a set of sequences
Π^1_1	$n \in \mathcal{A} \leftrightarrow f(n) \in \mathcal{O}$ for some computable function f	$X \in \mathcal{A} \leftrightarrow e \in \mathcal{O}^X$ for some e
Δ^1_1	$n \in \mathcal{A} \leftrightarrow f(n) \in \mathcal{O}_\alpha$ for some computable function f and some computable ordinal α	$X \in \mathcal{A} \leftrightarrow e \in \mathcal{O}^X_\alpha$ for some e and some ordinal α

Π^1_1 sets : Increasing union over the ordinals

Suppose $A \subseteq \omega$ is Π^1_1 with index e and let us denote

$$A_\alpha = \{n \ : \ \varphi_e(n) \in \mathcal{O}_\alpha\}$$

Then A is an increasing union of Δ^1_1 sets:

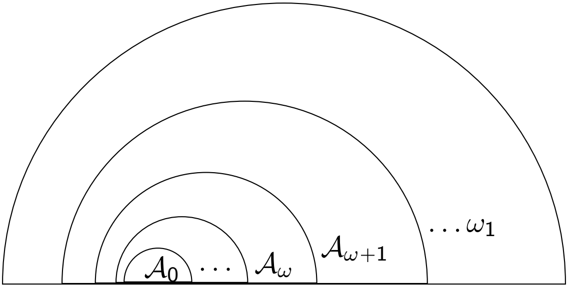


Π^1_1 sets : Increasing union over the ordinals

Suppose $\mathcal{A} \subseteq 2^\omega$ is Π^1_1 with index e and let us denote

$$\mathcal{A}_\alpha = \{X \ : \ e \in \mathcal{O}^X_\alpha\}$$

Then $\mathcal{A}$ is an increasing union of $\mathbf{\Delta}^1_1$ sets:



An important example of Π^1_1 set of sequences

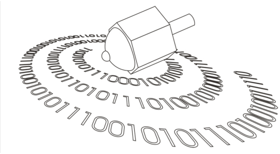
Definition

For a sequence $X \in 2^\omega$, the smallest non-X-computable ordinal is denoted by ω_1^X.

The set $\mathcal{C} = \{X \ : \ \omega_1^X > \omega_1^{ck}\}$ is a Π^1_1 set with the following properties:

- $\mathcal{C}$ is of measure 0 (Sacks).
- $\mathcal{C}$ is a meager set (Feferman).
- $\mathcal{C}$ contains no Σ^1_1 subset (Gandy).
- $\mathcal{C}$ is a $\boldsymbol{\Sigma}^0_{\omega_1^{ck}+2}$ set which is not $\boldsymbol{\Pi}^0_{\omega_1^{ck}+2}$ (Steel).

Higher randomness



Section 3

Higher randomness

Higher randomness

We can now define higher randomness notions

Definition Δ^1_1-random (Martin-Löf)

A sequence is Δ^1_1-random if it belongs to no Δ^1_1 set of measure 0.

Definition Π^1_1-random (Sacks)

A sequence is Π^1_1-random if it belongs to no Π^1_1 set of measure 0.

What about Σ^1_1-randomness ?

Theorem (Sacks)

A sequence is Σ^1_1-random iff it is Δ^1_1-random.

Π^1_1 randomness

The following theorems make Π^1_1-randomness an interesting notion of randomness:

Theorem (Keckris, Nies, Hjorth)

There is a universal Π^1_1 set of measure 0, that is one containing all the others.

As the set of $\{X : \omega_1^X > \omega_1^{ck}\}$ is a Π^1_1 set of measure 0. Therefore if something is Π^1_1-random, then $\omega_1^X = \omega_1^{ck}$. We have a very nice theorem about the converse:

Theorem (Chong, Yu, Nies)

A sequence X is Π^1_1-random iff it is Δ^1_1-random and $\omega_1^X = \omega_1^{ck}$.

Borel complexity of Π^1_1 randoms

Due to its universal nature, the set of Π^1_1 randoms is expected to have a higher Borel rank. But surprisingly we have:

Theorem (M.)

The set of Π^1_1 randoms is a $\mathbf{\Pi^0_3}$ set of the form:

$$\bigcap_n \bigcup_m \mathcal{F}_{n,m}$$

For each $\mathcal{F}_{n,m}$ a Σ^1_1 closed set.

where

Definition

A Π^1_1-**open set** is an open set $\mathcal{U}$ so that for a Π^1_1 set of strings A we have $\mathcal{U} = \bigcup\{[\sigma] \ : \ \sigma \in A\}$. A Σ^1_1-**closed set** is the complement of a Π^1_1-open set.

Lowness for Π^1_1-randomness

Definition

We say that A is low for Π^1_1-randomness if every $\Pi^1_1(A)$-random is also Π^1_1-random.

It is clear that any Δ^1_1 binary sequence is low for Π^1_1-randomness. Are there other sequences which are low for Π^1_1-randomness?

Theorem (Greenberg, M.)

The Δ^1_1 sequences are the only sequences that are low for Π^1_1-randomness.

Another hierarchy

Definition (Π^1_1 open set)

A Π^1_1 **open set** is an open set $\mathcal{U}$ so that for a Π^1_1 set of strings A we have $\mathcal{U} = \bigcup\{[\sigma] \ : \ \sigma \in A\}$.

Definition (Index for Π^1_1 open set)

For a Π^1_1 open set $\mathcal{U} = \bigcup\{[\sigma] \ : \ f(\sigma) \in \boldsymbol{W}\}$ with f a computable function, we say that a code e for f is an **index** for $\mathcal{U}$, and we write $\mathcal{U} = [W_e^{\omega_1^{ck}}]$.

Definition (Σ^1_1 closed set)

A Σ^1_1 **closed set** is the complement of a Π^1_1 open set.

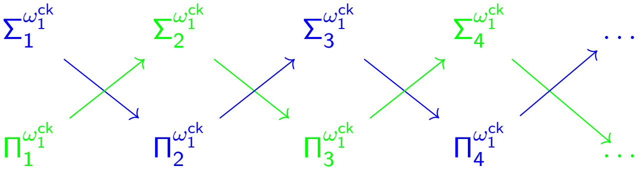
Another hierarchy

We can establish a new hierarchy by taking successive effective union and effective intersection of Π^1_1 open sets and Σ^1_1 closed sets.

$\boldsymbol{\Sigma}^{\omega_1^{ck}}_{\mathbf{1}}$ sets are	Π^1_1 open sets $[W_e^{\omega_1^{ck}}]$	with index	e
$\boldsymbol{\Pi}^{\omega_1^{ck}}_{\mathbf{1}}$ sets are	Σ^1_1 closed sets $[W_e^{\omega_1^{ck}}]^c$	with index	e
$\boldsymbol{\Sigma}^{\omega_1^{ck}}_{\boldsymbol{n+1}}$ sets are	$\bigcup_{m\in W_e}\{m\}$ where each m is an index for a $\Pi^{\omega_1^{ck}}_n$	with index	e
$\boldsymbol{\Pi}^{\omega_1^{ck}}_{\boldsymbol{n+1}}$ sets are	$\bigcap_{m\in W_e}\{m\}$ where each m is an index for a $\Sigma^{\omega_1^{ck}}_n$	with index	e

Another hierarchy

We can now define another hierarchy, starting with Π^1_1-open sets and Σ^1_1-closed sets.



The blue sets are Π^1_1 sets

The green sets are Σ^1_1 sets

Other higher randomness notions

Definition Π^1_1-MLR (Hjorth, Nies)

A sequence is Π^1_1-MLR if it belongs to no $\Pi^{\omega_1^{ck}}_2$ sets effectively of measure 0.

Definition strongly-Π^1_1-random (Nies)

A sequence is strongly-Π^1_1-MLR (or weakly-Π^1_1-random) if it belongs to no $\Pi^{\omega_1^{ck}}_2$ set of measure 0.

Definition strongly-Π^1_1-random (M.)

A sequence is strongly-$\Pi^{\omega_1^{ck}}_n$-MLR if it is in no $\Pi^{\omega_1^{ck}}_n$ set of measure 0.

Separation of randomness notions

Theorem (Hjorth, Nies)

Π^1_1-Martin-Löf randomness is strictly stronger than Δ^1_1-randomness.

Theorem (Chong, Yu 2012)

weakly-Π^1_1-randomness is strictly stronger than Π^1_1-Martin-Löf randomness.

Theorem (Greenberg, Bienvenu, M.)

Π^1_1-randomness is strictly stronger than weakly-Π^1_1-randomness.

Randomness notions along the hierarchy

Proposition

For a sequence X, the following are equivalent:

- X is in no $\Pi_3^{\omega_1^{ck}}$ set of measure 0.
- X is Δ_1^1-random.

Theorem (Greenberg, M.)

For a sequence X, the following are equivalent:

- X is in no $\Pi_4^{\omega_1^{ck}}$ set of measure 0.
- X is in no $\Pi_n^{\omega_1^{ck}}$ set of measure 0 for any n.
- X is in no Π_1^1 set of measure 0.

Other higher randomness notions

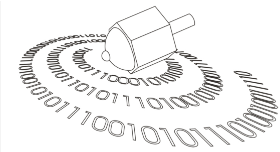
What is known:

$$\Delta^1_1 \mathit{random} \leftarrow \Pi^1_1 \mathit{MLR} \leftarrow \mathit{strongly} \Pi^1_1 \mathit{MLR} \leftarrow$$

$$\Pi^1_1 \mathit{random} = \text{`}\mathit{strongly} \Pi^{\omega_1^{\mathsf{ck}}}_4 \mathit{MLR}\text{'}$$

All the implications are strict. The proof of separation between $\Pi^{\omega_1^{\mathsf{ck}}}_2$ *random* and Π^1_1*random* was a difficult question, open for a while.

ITTM and randomness



Section 4

ITTM and randomness

ITTM

An infinite-time Turing machine is a Turing machine with three tapes whose cells are indexed by natural numbers:

- The input tape
- The output tape
- The working tape

It behaves like a standard Turing machine at successor step of computation.

At limit step of computation:

- The head goes back to the first cell
- The machine goes into a “limit” state.
- The value of each cell equals the $\liminf$ of the values at previous stages of computation.

Writable reals and decidable classes

What is the equivalent of computable for an ITTM:

definition

A real X is **writable** if there in an ITTM M such that $M(0)\downarrow[\alpha]=X$ for some ordinal α.

We also define the following notions:

definition

A class of real $\mathcal{A}$ is **decidable** if there is an ITTM M such that $M(X)\downarrow=1$ if $X\in\mathcal{A}$ and $M(X)\downarrow=0$ if $X\notin\mathcal{A}$.

definition

A class of real $\mathcal{A}$ is **semi-decidable** if there is an ITTM M such that $M(X)\downarrow$ if $X\in\mathcal{A}$.

Computational power of ITTM

Proposition (Hamkins, Lewis)

The Δ^1_1 reals are exactly the reals writable at step smaller than ω_1^{ck}.

Proposition (Hamkins, Lewis)

The class of reals coding for a well-order (with the code $X(\langle n, m\rangle) = 1$ iff $n < m$) is decidable.

Corollary (Hamkins, Lewis)

Every Π^1_1 set of real is decidable.

Corollary (Hamkins, Lewis)

The set of codes for computable ordinals $\mathcal{O}$ is writable.

Beyond writable ordinals

Proposition (Hamkins, Lewis)

Whatever an ITTM does, it does it before stage ω_1.

Definition (Hamkins, Lewis)

Let λ be the supremum of the writable ordinals.

Proposition (Hamkins, Lewis)

There is an ITTM which writes λ on its output tape, then leave the output tape unchanged without ever halting.

Definition (Hamkins, Lewis)

A real is eventually writable if there in an ITTM and a step α such that for every $\beta \geqslant \alpha$, the real is on the output tape at step β.
Let ζ be the supremum of the eventually writable ordinals.

Beyond eventually writable ordinals

Proposition (Hamkins, Lewis)

There is an ITTM which at some point writes ζ on its output tape.

Definition (Hamkins, Lewis)

A real is accidentally writable if there in an ITTM and a step α such that the real is on the output tape at step α.
Let Σ be the supremum of the accidentally writables.

Another notion will help us to understand better λ, ζ and Σ

Definition (Hamkins, Lewis)

An ordinal is clockable if there is an ITTM which halts at stage α.

What is the supremum of the clockable ordinals ?

Understanding ITTM

Theorem (Welch)

The whole state of an ITTM at step ζ is the same than its state at step Σ. In particular, it enters an infinite loop at stage ζ.

Corollary (Welch)

λ is the supremum of the clockable ordinals.

Corollary (Welch)

The writable reals are exactly the reals of L_λ.

$$\begin{aligned} L_\emptyset &= \emptyset \\ L_{\alpha+1} &= \{A \subseteq L_\alpha \ : \ \text{first order definable in } L_\alpha\} \\ L_{\sup_n \alpha_n} &= \bigcup_n L_{\alpha_n} \end{aligned}$$

Understanding ITTM

$$\begin{aligned} L_{\emptyset} &= \emptyset \\ L_{\alpha+1} &= \{A \subseteq L_\alpha \ : \ \text{first order definable in } L_\alpha\} \\ L_{\sup_n \alpha_n} &= \bigcup_n L_{\alpha_n} \end{aligned}$$

Corollary (Welch)

- The writable reals are exactly the reals of L_λ.
- The eventually writable reals are exactly the reals of L_ζ.
- The accidentally writable reals are exatly the reals of L_Σ.

Corollary (Welch)

(λ, ζ, Σ) is the lexicographically smallest triplet such that:

$$L_\lambda \prec_1 L_\zeta \prec_2 L_\Sigma$$

ITTM and randomness

Definition (Carl, Schlicht)

A sequence X is ITTM-random if X is in no semi-decidable set of measure 0.

Definition (Carl, Schlicht)

A sequence X is ITTM-decidable random iff X is in no decidable set of measure 0.

Definition (Carl, Schlicht)

A sequence X is α-random if X is in no set whose Borel code is in L_α.

ITTM and randomness

Theorem (Carl, Schlicht)

The following are equivalent for a sequence X:

1. X is ITTM-random
2. X is Σ-random and $\Sigma^X = \Sigma$
3. X is λ-random and $\Sigma^X = \Sigma$

Theorem (Carl, Schlicht)

The following are equivalent for a sequence X:

1. X is ITTM-decidable random
2. X is λ-random

Thank you