Randomness and ITTM

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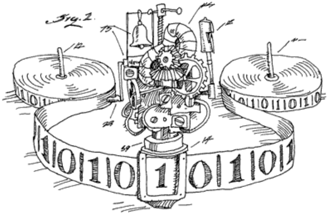
Infosys



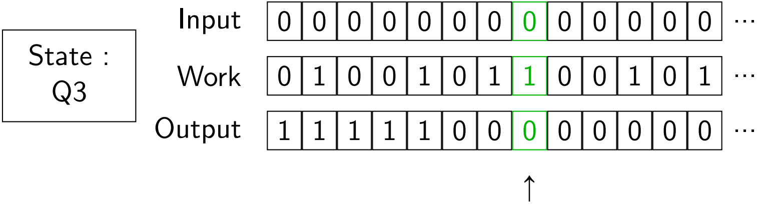
Section 1

The ITTM model

Infinite time Turing machines

An infinite-time Turing machine is a Turing machine with three tapes whose cells are indexed by natural numbers :

- The input tape
- The output tape
- The working tape

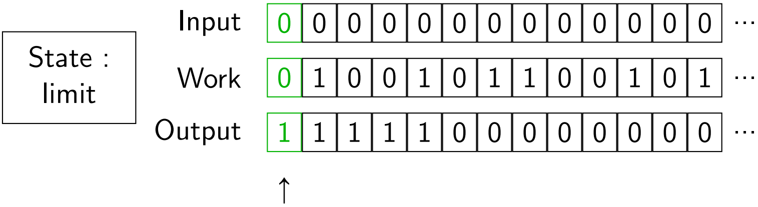


It behaves like a standard Turing machine at successor steps of computation.

Infinite time Turing machines

At limit steps of computation :

- The head goes back to the first cell.
- The machine goes into a “limit” state.
- The value of each cell equals the lim inf of the values at previous stages of computation.

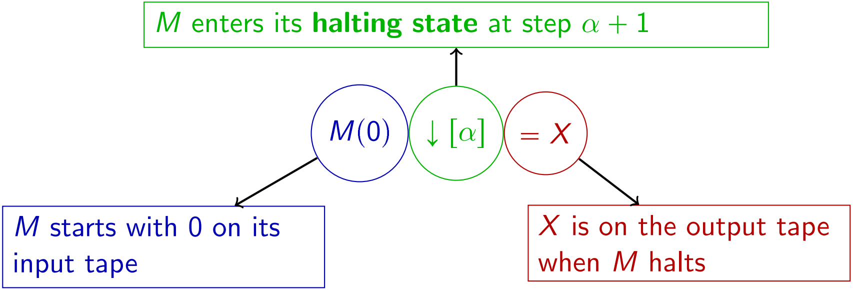


Writable reals

What is the equivalent of computable for an ITTM ?

Definition

A real X is **writable** if there in an ITTM M such that :
$M(0) \downarrow [\alpha] = X$ for some ordinal α.



Decidable classes

Which reals are writable ?

Definition

A class of real $\mathcal{A}$ is **decidable** if there is an ITTM M such that $M(X)\downarrow= 1$ if $X \in \mathcal{A}$ and $M(X)\downarrow= 0$ if $X \notin \mathcal{A}$.

Proposition (Hamkins, Lewis)

The class of reals coding for a well-order (with the code $X(\langle n, m\rangle) = 1$ iff $n < m$) is decidable.

Decide well-orders

Proposition (Hamkins, Lewis)

The class of reals coding for a well-order (with the code $X(\langle n, m\rangle) = 1$ iff $n < m$) is decidable.

The algorithm is as follow, where $<$ is the order coded by X :

Algorithm to decide well-orders

```
while < is not empty do
    Look for the smallest element a of < (coded by X)
    if there is no smaller element then
        write 0 and halts
    else
        remove a from <
    end
end
When < is empty, write 1 and halts.
```

Decide well-orders

How to find the smallest element ?

Algorithm to find the smallest element

Write 1 on the first cell. Set the current element $c = +\infty$
if state is successor **then**
  if there exists $a < c$ **then**
    Update $c = a$
    Flip the first cell to 0 and then back to 1
  end
else
  if If the first cell is 0 **then**
    c is the smallest element
  else
    There is no smallest element
  end
end

Decidable and writable sets

Proposition (Hamkins, Lewis)

The class of reals coding for a well-order (with the code $X(\langle n, m\rangle) = 1$ iff $n < m$) is decidable.

Corollary (Hamkins, Lewis)

Every Π^1_1 set is decidable.

Corollary (Hamkins, Lewis)

Every Π^1_1 set of integers is writable.

Computational power of ITTM

ω_1^{ck} step of computations are enough to write any Π_1^1 set of integers. But there is no bound in the ordinal step of computation an ITTM can use.

Using a program that writes Kleene's O, we can design a program which writes the double hyperjump O^O and then $O^{(O^O)}$ and so on.

Where does it stop ?

Proposition (Hamkins, Lewis)

Whatever an ITTM does, it does it before stage ω_1.

Computational power of ITTM

Proposition (Hamkins, Lewis)

Whatever an ITTM does, it does it before stage ω_1.

The configuration of an ITTM is given by :

1. Its tapes
2. Its state
3. The position of the head.

Let $C(\alpha) \in 2^\omega$ be a canonical encoding of the tapes of an ITTM at stage α.

There must be some *limit ordinal* $\alpha < \omega_1$ such that $C(\alpha) = C(\omega_1)$. The full configuration of the machine at step ω_1 is then the same than the one step α.

Computational power of ITTM

ω_1	0	1	0	0	0	0	0	0	0	1	0	0	1	···
							···							
$\sup_n \alpha_n^+$	0	1	0	0	0	0	0	0	0	1	0	0	1	···
							···							
$\alpha_2^+ > \alpha_1^+$	0	1	0	1	0	1	0	0	1	1	1	1	1	···
$\alpha_1^+ > \alpha_0$	1	1	1	1	0	1	0	0	1	1	1	1	1	···
α_0	0	1	1	1	0	1	0	0	1	1	1	1	1	···

α_0 : The smallest ordinal such that every cell converging at step ω_1 (in green) will never change pass that point.

α_{n+1}^+ : The smallest ordinal $> \alpha_n^+$ such that the $n+1$ non-converging cells (in red) change value at least once in the interval $[\alpha_n^+, \alpha_{n+1}^+]$

Beyond the writable ordinals

Definition (Hamkins, Lewis)

An ordinal α is **writable** if there is an ITTM which writes an encoding of a well-order of ω with order-type α.

Proposition (Hamkins, Lewis)

The writables are all initial segments of the ordinals.

Definition (Hamkins, Lewis)

Let λ be the supremum of the writable ordinals.

Proposition (Hamkins, Lewis)

There is an ITTM which writes λ on its output tape, then leave the output tape unchanged without ever halting.

Beyond the writable ordinals

Proposition (Hamkins, Lewis)

There is a univeral ITTM U which runs simultaneously all the ITTM computations $P_e(0)$ for every $e \in \omega$.

Algorithm to eventually write λ

for every stage s **do**
    Run the universal machine U for one step.
    Compute the sum α_s of all ordinals which are on the output tapes of programs simulated by $U[s]$ and which have terminated.
    Write α_s on the output tape.
end

Let s be the smallest stage such that every halting ITTM have halted by stage s in the simulation U.

1. We clearly have $\alpha_s \geqslant \lambda$.
2. We clearly have that $\alpha_t = \alpha_s$ for every $s \geqslant t$.

Beyond the eventually writable ordinals

Definition (Hamkins, Lewis)

A real is **eventually writable** if there in an ITTM and a step α such that for every $\beta \geqslant \alpha$, the real is on the output tape at step β.

Proposition (Hamkins, Lewis)

The eventually writable ordinals are an initial segments of the ordinals.

Definition (Hamkins, Lewis)

Let ζ be the supremum of the eventually writable ordinals.

Proposition (Hamkins, Lewis)

There is an ITTM which at some point writes ζ on its output tape.

Beyond the eventually writable ordinals

Algorithm to accidentally write ζ

---

for every stage s **do**
  Run the universal machine U for one step.
  Compute the sum α_s of all ordinals which are on the output tapes of programs simulated by $U[s]$.
  Write α_s on the output tape.
end

Let s be the smallest stage such that every ITTM writting an eventually writable ordinal, have done so by stage s in the simulation U. We clearly have $\alpha_s \geqslant \zeta$.

Beyond the eventually writable ordinals

Definition (Hamkins, Lewis)

A real is **accidentally writable** if there in an ITTM and a step α such that the real is on the output tape at step α.

Proposition (Hamkins, Lewis)

The accidentally writables are all initial segments of the ordinals.

Definition (Hamkins, Lewis)

Let Σ be the supremum of the accidentally writables.

Proposition (Hamkins, Lewis)

We have $\lambda < \zeta < \Sigma$.

Section 2

ITTM
and constructibility

The constructibles

Definition (Godel)

The **constructible universe** is defined by induction over the ordinals as follow :

$$\begin{array}{rcl} L_{\varnothing} & = & \varnothing \\ L_{\alpha^+} & = & \{X \subseteq L_\alpha \ : \ X \text{ is f.o. definable with param. in } L_\alpha\} \\ L_{\sup_n \alpha_n} & = & \bigcup_n L_{\alpha_n} \end{array}$$

Theorem (Hamkins, Lewis)

- If α is writable and $X \in 2^\omega \cap L_\alpha$ then X is writable.
- If α is eventually writable and $X \in 2^\omega \cap L_\alpha$ then X is eventually writable.
- If α is accidentally writable and $X \in 2^\omega \cap L_\alpha$ then X is accidentally writable.

The admissibles

Definition (Admissibility)

An ordinal α is **admissible** if L_α is a model of Σ_1-replacement. Formally for any Σ_1 formula Φ with parameters and any $N \in L_\alpha$ we must have :

$$\begin{array}{rcl} L_\alpha & \models & \forall n \in N\ \exists z\ \Phi(n,z) \\ \rightarrow\ L_\alpha & \models & \exists Z\ \forall n \in N\ \exists z \in Z\ \Phi(n,z) \end{array}$$

$\omega, \omega_1^{ck}, \omega_2^{ck}, \omega_3^{ck}, etc\ldots$ are the first admissible ordinals. But $\sup_n \omega_n^{ck}$ is not admissible.

We define :

$$\begin{array}{rcl} \tau_0 & = & \omega \\ \tau_\beta & = & \text{The smallest admissible strictly greater than } \tau_\gamma \text{ for } \gamma < \beta \end{array}$$

The admissibles

Proposition (Hamkins, Lewis)

The ordinals λ and ζ are admissible.

Suppose that for some $N \in L_\lambda$ and a Σ_1 formula Φ we have :

$$L_\lambda \models \forall n \in N \; \exists z \; \Phi(n, z)$$

We define the following ITTM :

Algorithm to write λ

Write a code for N
for every $n \in N$ **do**
  Look for the first writable α_n such that $L_{\alpha_n} \models \exists z \; \Phi(n, z)$
  Write α_n somewhere.
end
Write $\sup_{n \in N} \alpha_n$

The admissibles

Proposition (Hamkins, Lewis)

The ordinals λ is the λ-th admissible.
The ordinals ζ is the ζ-th admissible.

Suppose λ is the α-th admissible for $\alpha < \lambda$.

Algorithm to write λ

Write α
while $\alpha > 0$ **do**
    Look for the smallest element *e* of α and remove it from α
    Look for the next admissible writable ordinal and write it to the *e*-th tape
end
Write the smallest admissible greater than all the one written previously.

The recursively inaccessible

Definition

An ordinal is **recursively inaccessible** if it is admissible and limit of admissible.

Proposition

An ordinal α is recursively inaccessible iff $\alpha = \tau_\alpha$.

Corolarry

The ordinals λ and ζ are recursively inaccessible.

The recursively inaccessible

1. $J_1 = \emptyset$, $J_2 = O$.
   $\rightarrow$ Define $1 < 2$
2. If a is in the field of $<$, let $J_{2^a} = O^{J_a}$.
   $\rightarrow$ Define $a < 2^a$ and $b < 2^a$ for any $b < a$.
3. If a is in the field of $<$ and if e is the code of a computable functional such that

$$\varphi_e(0, J_a) = a \text{ and } \varphi_e(n, J_a) < \varphi_e(n+1, J_a)$$

   then $J_{3^e 5^a} = \oplus_{m_i} J_{m_i}$ for $m_i = \varphi_e(i, J_a)$.
   $\rightarrow$ Define $b < 3^e 5^a$ for any $b < m_i$ for some i.

Fact

The smallest ordinal which does not have a code in the field of $<$ is the first recursively inaccessible.

How big is λ

Proposition (Hamkins, Lewis)

The ordinals λ is the λ-th recursively admissible.
The ordinals ζ is the ζ-th recursively admissible.

Definition

An ordinal is **meta-recursively inaccessible** if it is admissible and a limit of recursively inaccessible.

Proposition (Hamkins, Lewis)

The ordinals λ is the λ-th meta recursively admissible.
The ordinals ζ is the ζ-th meta recursively admissible.

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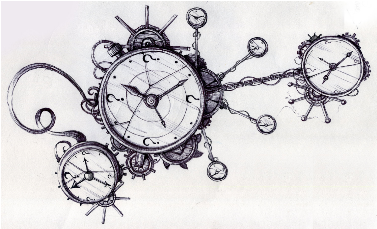


Section 3

The clockable ordinals

The clockable ordinals

Another notion will help us to understand better λ, ζ and Σ

Definition (Hamkins, Lewis)

An ordinal α is **clockable** if there is an ITTM which halts at stage α (at stage α it decides to go into the halting state).

What is the supremum of the clockable ordinals ?

Definition (Hamkins, Lewis)

Let γ be the supremum of the clockable ordinals.

Proposition (Hamkins, Lewis)

We have $\lambda \leqslant \gamma$.

The clockable ordinals

Proposition (Hamkins, Lewis)

We have $\lambda \leqslant \gamma$.

Suppose the ITTM M writes α. Then one can easily create an ITTM which does the following :

Algorithm to countdown α

Use M to write α

while $\alpha > 0$ **do**

  Find the smallest element of α and remove it from α.

end

Enter the halting state.

It is easy to see that the above algorithm takes at least α step before it ends.

Understanding the clockables

Theorem (Hamkins, Lewis)

The clockable ordinals are not an initial segment of the ordinals :
If α is admissible then no ITTM halts in α steps.

For α limit to be clockable we need for some $i \in \{0, 1\}$ to have both :

1. A transition rule of the form : (limit, i) $\rightarrow$ halt
2. The first cell to contain i at step α

If $\{C_i(\gamma)\}_{\gamma<\alpha}$ converges we have a limit $\beta < \alpha$ s.t. $C_i(\beta) = C_i(\alpha)$
$\rightarrow$ We have (1) and (2) for $\beta < \alpha$

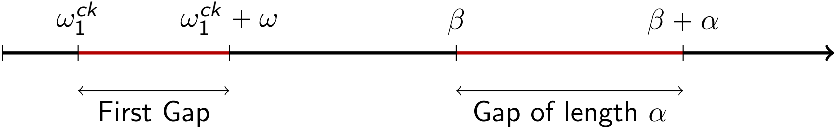
If $\{C_i(\gamma)\}_{\gamma<\alpha}$ diverges, let :
$f(n+1) =$ the smallest $\alpha > f(n)$ s.t. $C_0(\beta)$ changes for $\beta \in [f(n), \alpha]$
$\rightarrow$ By admissibility $\sup_n f(n) < \alpha$ and we have (1) and (2) for $\sup_n f(n)$

In both cases the machine stopped before stage α.

Understanding the clockables

Definition (Hamkins, Lewis)

A **gap of size** α in the clockable ordinals is an interval of ordinals $[\alpha_0, \alpha_0 + \alpha]$ such that no ITTM halts in this interval, but some halt after that.



Theorem (Hamkins, Lewis)

For any writable α, there is a gap of size at least α in the clockable ordinals.

Understanding the clockables

Algorithm to witness large gaps

Run the universal ITTM
while true **do**
  if a new ordinal α_0 is written on a tape **then**
    if no ITTM halts in the interval $[\alpha_0, \alpha_0 + \alpha]$ **then**
      Write $\alpha_0 + \alpha$ and halt.
    end
  end
end

Note that if α is writabe then $\lambda + \alpha < \zeta < \Sigma$. Suppose there is no gap of size α.

→ Then the algorithm will at some point :

1. Eventually write λ and will see that no ITTM halts in $[\lambda, \lambda + \alpha]$
2. Write $\lambda + \alpha$ and halts

This is a contradiction.

Understanding λ, ζ, Σ

Lemma (Welch)

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha<\lambda}$ converges, then for every $\alpha \in [\lambda, \Sigma]$ we have $C_i(\alpha) = C_i(\lambda)$.

Suppose w.l.o.g. that $\{C_i(\alpha)\}_{\alpha<\lambda}$ converges to 0.
Let β be the smallest such that for all $\alpha \in [\beta, \lambda]$ we have $C_i(\alpha) = 0$.

Algorithm

for every $\alpha > \beta$ written by U **do**
  Simulate another run of U for α steps
  if $C_i(\gamma) = 1$ for $\gamma \in [\beta, \alpha]$ **then**
    Write α and halt.
  end
end

Suppose there is an accidentally writable ordinal $\alpha > \beta$ s.t. $C_i(\alpha) = 1$. Then U will write such an ordinal at some point, and the above program will then write $\alpha > \lambda$ and halt. This is a contradiction.

Understanding λ, ζ, Σ

Theorem (Welch)

The whole state of an ITTM at step ζ is the same than its state at step Σ. In particular, it enters an infinite loop at stage ζ.

The theorem follows from the two following lemmas :

Lemma (Welch)

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha<\zeta}$ converges, then for every $\alpha \in [\zeta, \Sigma]$ we have $C_i(\alpha) = C_i(\zeta)$.

Lemma (Welch)

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha<\zeta}$ diverges, then the sequence $\{C_i(\alpha)\}_{\alpha<\Sigma}$ diverges.

Understanding λ, ζ, Σ

Suppose w.l.o.g. that $\{C_i(\alpha)\}_{\alpha<\zeta}$ converges to 0.
Let β be the smallest such that for all $\alpha \in [\beta, \zeta]$ we have $C_i(\alpha) = 0$.
The ordinal β is eventually writable through different versions $\{\beta_s\}_{s \in ORD}$

Algorithm

for every s and every $\alpha > \beta_s$ written by U **do**
  Simulate another run of U for α steps
  if $C_i(\gamma) = 1$ for $\gamma \in [\beta_s, \alpha]$ and β_s has changed **then**
    Write α on the output tape.
  end
end

Suppose there is an accidentally writable ordinal $\alpha > \beta$ s.t. $C_i(\alpha) = 1$. Then some ordinal $\alpha' \geqslant \alpha$ will be written at some stage at which β_s has stabilized. Thus the above program will then eventually write some $\alpha' > \zeta$. This is a contradiction.

Understanding λ, ζ, Σ

Suppose $\{C_i(\alpha)\}_{\alpha<\Sigma}$ converges.

Algorithm

Set $\beta = 0$
for every $\alpha > \beta$ written by U **do**
  Simulate another run of U for α steps
  if $C_i(\gamma)$ changes for $\gamma \in [\beta, \alpha]$ **then**
    Let $\beta = \alpha$
    Write α
  end
end

The algorithm will eventually write some ordinal α s.t. $\{C_i(\gamma)\}$ does not change for $\gamma \in [\alpha, \Sigma]$. But then α is eventually writable and $\{C_i(\alpha)\}_{\alpha<\zeta}$ converges.

Understanding λ, ζ, Σ

Theorem (Welch)

The whole state of an ITTM at step ζ is the same than its state at step Σ. In particular, it enters an infinite loop at stage ζ.

Corollary (Welch)

λ is the supremum of the clockable ordinals.

Indeed, suppose that we have $M(0) \downarrow [\alpha]$ for some M and α accidentally writable. Then we can run $M(0)[\beta]$ for every β accidentally writable until we find one for which M halts, and then write β. Thus α must be writable.

Suppose now that $M(0) \uparrow [\Sigma]$. Then M will never halt. Thus if M halts, it halts at a writable step.

Understanding λ, ζ, Σ

Theorem (Welch)

The whole state of an ITTM at step ζ is the same than its state at step Σ. In particular, it enters an infinite loop at stage ζ.

Corollary (Welch)

- The writable reals are exactly the reals of L_λ.
- The eventually writable reals are exactly the reals of L_ζ.
- The accidentally writable reals are exatly the reals of L_Σ.

We can construct every successive configurations of a running ITTM. Also to compute a writable reals, there are less than λ steps of computation and then less than λ steps of construction. Thus every writable real is in L_λ.

The argumet is similar for ζ and Σ.

Understanding λ, ζ, Σ

Definition

Let $\alpha \leqslant \beta$. We say that L_α is ***n*-stable** in L_β and write $L_\alpha \prec_n L_\beta$ if

$$L_\alpha \models \Phi \leftrightarrow L_\beta \models \Phi$$

For every Σ_n formula Φ with parameters in L_α.

Theorem (Welch)

(λ, ζ, Σ) is the lexicographically smallest triplet such that :

$$L_\lambda \prec_1 L_\zeta \prec_2 L_\Sigma$$

Understanding λ, ζ, Σ

Theorem (Welch)

The ordinal Σ is not admissible.

To see this, we define the following function $f : \omega \to \Sigma$:

$$
\begin{aligned}
f(0) &= \zeta \\
f(n) &= \text{the smallest } \alpha > f(n+1) \text{ s.t. } C(\alpha)\restriction_n = C(\zeta)\restriction_n
\end{aligned}
$$

It is not very hard to show that we must have $\sup_n f(n) = \Sigma$

Theorem (Welch)

The ordinal Σ is a limit of admissible.

Otherwise, if α is the greatest admissible smaller than Σ, one could compute $\Sigma \leqslant \omega_1^{\alpha}$.

Section 4

ITTM and randomness

ITTM and randomness

Definition (Carl, Schlicht)

X is α-**random** if X is in no set whose Borel code is in L_α.

Definition

An open set U is α-**c.e.** if $U = \bigcap_{\sigma \in A}[\sigma]$ for a set $A \subseteq 2^{<\omega}$ such that :

$$\sigma \in A \leftrightarrow L_\alpha \models \Phi(\sigma)$$

for some Σ_1 formula Φ with parameters in L_α.

Definition (Carl, Schlicht)

X is α-**ML-random** if X is in no set uniform intersection $\bigcap_n U_n$ of α-open set, with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$.

Toolbox for the constructibles

Theorem (satisfaction is uniform)

There is a Σ_1 formula $\Phi(e, x)$ such that for any $e \in \omega$, any α limit and any $x \in L_\alpha$ we have :

$$L_\alpha \models \Phi(e, x) \leftrightarrow L_\alpha \models \Phi_e(x)$$

Theorem (L is absolute)

There is a Σ_1 formula $\Phi(\beta, X)$ such that for any α limit and any $\beta < \alpha$ we have :

$$L_\alpha \models \Phi(\beta, X) \leftrightarrow L_\beta = X$$

Theorem (axiom of choice)

There is a Σ_1 formula $\Phi(x, y)$ such that for any α limit and any $x, y \in L_\alpha$ we have :

$$x < y \leftrightarrow L_\alpha \models \Phi(x, y)$$

is a well-order of L_α

Toolbox for the constructibles

Theorem (measure is uniform)

There is a Σ_1 formula $\Phi(e, q)$ such that for any $e \in \omega$, any α limit, any parameter $x \in L_\alpha$, and any rational q we have :

$$L_\alpha \models \Phi(x, e, q) \leftrightarrow \lambda\left(\bigcup\{[\sigma] \ : \ L_\alpha \models \Phi_e(x, \sigma)\}\right) > q$$

Theorem (measure is uniform)

There is a Σ_1 formula $\Phi(e, x)$ such that for any α limit, any Borel code $x \in L_\alpha$, and any rational q we have :

$$L_\alpha \models \Phi(x, q) \leftrightarrow \lambda([x]) > q$$

where $[x]$ is the Borel set coded by x.

Projectibles and ML-randomness

Definition

We say that α is **projectible** into $\beta < \alpha$ if there is an injective function $f : \alpha \to \beta$ that is Σ_1-definable in L_α.
The least β such that α is projectible into β is called the **projectum** of α and denoted by α^*.

Theorem (Angles d'Auriac, Monin)

The following are equivalent for α limit such that $L_\alpha \models$ everything is countable :

- α is projectible into ω.
- There is a universal α-ML-test.
- α-ML-randomness is strictly stronger than α-randomness.

Projectibles and Universal ML-Test

Suppose α is projectible into ω. It is then possible to effectively assign an integer all the parameters in L_α and then to each α-c.e. open sets.
Suppose we have an intersection of open sets $\bigcap_k U_k$, each U_k being the union of clopens $[\sigma]$ given by the Σ_1 formula

$$\exists x \ \Phi_m(x, k, p, \sigma)$$

with p a parameter. Let f be the projectum. One can uniformly define the Σ_1 formula $\Psi(n, m, k, \sigma)$:

$$\exists p \ f(n) = p \wedge \exists x \ \Phi_{g(m)}(x, k, p, \sigma)$$

where :

$$\begin{aligned} \exists x \ \Phi_{g(m)}(x, k, p, \sigma) \quad \equiv \quad & \exists x \ \Phi_m(x, k, p, \sigma) \wedge \\ & \lambda(\bigcup\{[\sigma] \ : \ \forall z \leqslant x \ \Phi_m(\beta, z, \sigma)\}) \leqslant 2^{-k} \end{aligned}$$

Using this one easily define a universal test like in the lower case.

Projectibles and ML-randomness

Suppose that α is projectible into ω. Then one can perform a separation between α-ML-randomness and α-randomness in a similar way one separates Δ^1_1-randomness from Π^1_1-ML-randomness : we create a α-Δ_2 element X such that :

- $X \upharpoonright_n$ changes at most n times
- X is α-random

To do so we use the fact that :

- The set of Borel codes of L_α for Borel sets of measure 1 is projectible into ω
- For each Borel set B of L_α on can find a Borel code for a union $\bigcup_n F_n \subseteq B$ of closed set of measure 1

Projectibles and ML-randomness

Indices	…					
0	…					
1	$\bigcup$	F^1_0	F^1_1	F^1_2	…	$\lambda(\bigcup_m F^1_m) = 1$
2	…					
3	…					
4	…					
5	…					
6	…					
7	…					
8	…					

…

Projectibles and ML-randomness

Indices	...					
0	...					
1	$\bigcup$	F^1_0	F^1_1	F^1_2	...	$\lambda(\bigcup_m F^1_m) = 1$
2	...					
3	...					
4	...					
5	...					
6	$\bigcup$	F^6_0	F^6_1	F^6_2	...	$\lambda(\bigcup_m F^6_m) = 1$
7	...					
8	...					

...

Projectibles and ML-randomness

Indices	...					
0	...					
1	$\bigcup$	F^1_0	F^1_1	F^1_2	...	$\lambda(\bigcup_m F^1_m) = 1$
2	...					
3	$\bigcup$	F^3_0	F^3_1	F^3_2	...	$\lambda(\bigcup_m F^3_m) = 1$
4	...					
5	...					
6	$\bigcup$	F^6_0	F^6_1	F^6_2	...	$\lambda(\bigcup_m F^6_m) = 1$
7	...					
8	...					

...

Randomness with the non-projectibles

Theorem

If α is not projectible into $\beta < \alpha$, then α is recursively inaccessible. Thus if α is not recursively inaccessible and if $L_\alpha \models$ everything is countable, then α is projectible into ω.

Theorem

Suppose $L_\alpha \models$ everything is countable. If α is not projectible into ω then $L_\alpha \models \Sigma_1$-comprehension for subsets of ω.

Suppose α not project. into ω. Let $A \subseteq \omega$ s.t. $n \in A \leftrightarrow L_\alpha \models \exists x\ \Phi(x, n)$
$f(n) = \min \beta > f(n-1)\ s.t. L_\alpha \models \exists x \in L_\beta\ \Phi(x, n)$
Suppose for contradiction that $f(\omega)$ unbounded in α. Define :
$g(\beta) = \min \gamma \in f(\omega)\ s.t. \gamma \notin \{g(\gamma')\ :\ \gamma' < \gamma\}$
As α admissible, g is a bijection between $f(\omega)$ and α. Thus
$h(\beta) = f^{-1}(g(\beta))$
is a projectum into ω. Contradiction.

Randomness with the non-projectibles

If α such that $L_\alpha \models$ everything is countable is not projectible into ω, then the randomness notions collapses.

Suppose α is not projectible into ω. Then every open α-c.e. set is a set with a Borel code in L_α. It follows that α-randomness coincides with α-ML-randomness.

It follows that there is no universal α-ML-test, because there is never a universal α-test for α limit.

Theorem (Friedman)

For every limit accidentally writable α we have that :
$L_\alpha \models$ *everything is countable.*

If β is not countable in α, one could build in L_α a countable ordinal γ (smaller than β) such that $L_\gamma \prec_2 L_\beta$ which contradicts minimality of the pair (ζ, Σ).

λ-ML-randomness

Theorem

The ordinal λ is projectible into ω without using any parameters.

Each writable ordinal can be effectively assigned to the code of the ITTM writting it.

Corollary

Most work in Δ^1_1 and Π^1_1-ML-randomness still work with λ-ML-randomness and λ-randomness. In particular λ-ML-randomness is strictly weaker than λ-randomness.

ζ-ML-randomness

Theorem

The ordinal ζ is not projectible into ω.

Suppose that an eventually writable parameter α can be used to have a projuctum $f : \zeta \to \omega$. Then every eventually writable ordinals become writable using α. Then ζ becomes eventually writable using α. But then ζ is eventually writable.

Corollary

ζ-randomness coincides with ζ-ML-randomness. An analogue of Ω for ζ-randomness does not exists.

ζ-ML-randomness

Theorem

The ordinal ζ is not projectible into ω.

Corollary

For many writable ordinals α we have that α-randomness coincides with α-ML-randomness.

$$L_\Sigma \models \exists \alpha \text{ not projectible into } \omega$$

By the fact that $L_\lambda \prec_1 L_\Sigma$ we must have :

$$L_\lambda \models \exists \alpha \text{ not projectible into } \omega$$

Σ-ML-randomness

Theorem

The ordinal Σ is projectible into ω, using ζ as a parameter.

We can use the fact that (ζ, Σ) is the least pair such that : $C(\zeta) = C(\Sigma)$, with the function :

$$\begin{aligned} f(0) &= \zeta \\ f(n) &= \text{the smallest } \alpha > f(n+1) \text{ s.t. } C(\alpha)\restriction_n = C(\zeta)\restriction_n \end{aligned}$$

Every ordinal $f(n)$ is then Σ_1-definable with ζ as a parameter.
As $L_\Sigma \models$ "everything is countable", it follows that every ordinal smaller than $f(n)$ for some n is Σ_1-definable with ζ as a parameter.
As $\sup_n f(n) = \Sigma$, it follows that every accidentally writable is Σ_1-definable with ζ as a parameter.
The projectum is then a code for the formula defining each ordinal.

ITTM-random and ITTM-decidable random

Definition (Hamkins, Lewis)

A class of real $\mathcal{A}$ is **semi-decidable** if there is an ITTM M such that $M(X)\downarrow$ if $X \in \mathcal{A}$.

Definition (Carl, Schlicht)

A sequence X is **ITTM-random** if X is in no semi-decidable set of measure 0.

Definition (Carl, Schlicht)

A sequence X is **ITTM-decidable random** iff X is in no decidable set of measure 0.

Lowness for λ, ζ, Σ

Definition

We say that X is low for λ if $\lambda^X = \lambda$.
We say that X is low for ζ if $\zeta^X = \zeta$.
We say that X is low for Σ if $\zeta^X = \Sigma$.

Theorem

For any ordinal α with $\lambda \leqslant \alpha < \zeta$ we have $\lambda^\alpha > \lambda$ but :

1. $\zeta^\alpha = \zeta$.
2. $\Sigma^\alpha = \Sigma$.

(1) Indeed, suppose ζ is eventually writable using α and the machine M. As α is also eventually writable, we can run M on every version of α and eventually write ζ which is a contradiction. (2) Same argument.

Lowness for λ, ζ, Σ

Theorem

The following are equivalent :

1. $\zeta^X > \zeta$.
2. $\Sigma^X > \Sigma$.
3. $\lambda^X > \Sigma$.

(1) $\rightarrow$ (2) : We can again use the function :

$$\begin{aligned} f(0) &= \zeta \\ f(n) &= \text{the smallest } \alpha > f(n+1) \text{ s.t. } C(\alpha)\upharpoonright_n = C(\zeta)\upharpoonright_n \end{aligned}$$

To show that every ordinal $f(n)$ becomes eventually writable uniformly in n. Thus $\Sigma = \sup_n f(n)$ is also eventually writable.

(2) $\rightarrow$ (3) : Define the machine that looks for the first pair of ordinals $\alpha < \beta$ such that $L_\alpha \prec_2 L_\beta$. Then write β. These ordinals must be ζ and Σ.

Lowness for λ, ζ, Σ and randomness

Theorem

For any X the triplet $(\lambda^X, \zeta^X, \Sigma^X)$ is the lexicographically least pair such that $L_{\lambda^X}[X] \prec_1 L_{\zeta^X}[X] \prec_2 L_{\Sigma^X}[X]$.

Theorem (Carl, Schlicht)

If X is $(\Sigma + 1)$-random, then $L_\lambda[X] \prec_1 L_\zeta[X] \prec_2 L_\Sigma[X]$. In particular $\Sigma^X = \Sigma$, $\zeta^X = \zeta$ and $\lambda^X = \lambda$.

Corollary (Carl, Schlicht)

The set $\{X : \Sigma^X > \Sigma\}$ and $\{X : \lambda^X > \lambda\}$ are included in Borel sets of measure 0.

ITTM-decidable randomness

Theorem (Carl, Schlicht)

The following are equivalent for a sequence X :

1. X is ITTM-decidable random
2. X is λ-random

Suppose some machine M decides a set of measure 0 that X belongs to. In particular it decides a set of measure 1 X does not belong to. We have :

$$\lambda(\{X \ : \ M(X)\downarrow = 0\}) = 1$$

We then have

$$\lambda(\{X \ : \ M(X)\downarrow [\lambda] = 0\}) = 1$$

as the set of X s.t. $\lambda^X = \lambda$ has measure 1. But then by admissibility :

$$\lambda(\{X \ : \ M(X)\downarrow [\alpha] = 0\}) = 1$$

already for some writable α. The complement of this set is a Borel set of measure 0, with a writable code, and containing X.

ITTM-randomness

Theorem (Carl, Schlicht)

The following are equivalent for a sequence X :

1. X is ITTM-random
2. X is Σ-random and $\Sigma^X = \Sigma$
3. X is ζ-random and $\Sigma^X = \Sigma$

Lemma (Carl, Schlicht)

If $\Sigma^X > \Sigma$, then X is not ITTM-random.

The set $\{X \ : \ \Sigma^X > \Sigma\}$ is an ITTM-semi-decidable set of measure 0. We saw that it is of measure 0. To see that it is ITTM-decidable, one can designe the machine which halts whenever it founds two X-accidentally writable ordinals $\alpha < \beta$ such that $L_\alpha \prec_1 L_\beta$.

ITTM-randomness

Lemma (Carl, Schlicht)

If X is not Σ-random, then X is not $ITTM$-random.

If X is not Σ-random, then with X as an oracle, we can look for the first accidentally writable code for a Borel set of measure 0 containing X.

Lemma (Carl, Schlicht)

If X is ζ-random, but not ITTM-random, then $\Sigma^X > \Sigma$.

Suppose there is a ITTM M which semi-decide a set of measure 0 containing X. Suppose $M(X) \downarrow [\alpha]$. Then we must have $\alpha \geqslant \zeta$ as otherwise the set $\{X \ : \ M(X) \downarrow [\alpha]\}$ would ba set set of measure 0 with a Borel code in L_ζ. Thus we must have $\lambda^X > \zeta$ and then $\Sigma^X > \Sigma$.

ITTM-randomness

Question

Does there exists X such that X is Σ-random but not ITTM random ?

Question

If X is Σ-random, do we have $L_\zeta[X] \prec_2 L_\Sigma[X]$?