Higher randomness

Algorithmic Randomness Interacts with Analysis and Ergodic Theory

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Introduction

Section 1

Introduction

Randomness

Paradigm

An element $X \in 2^\omega$ is random if it belongs to no set of measure 0 among a given countable class of sets/.

Which countable class of sets should we pick ?

Class of sets	Randomness notion
Π_1^0 sets of measure 0	weak-randomness
Π_2^0 sets effectively of measure 0	Martin-Löf randomness
Π_2^0 sets of measure 0	weak-2-randomness
...	...

Effective Hyperarithmetical complexity of sets

We define the effective Borel set by induction over the ordinals:

(Notation : The set of index n is denoted by $\{n\}$)

Name	Definition	Indices
Σ^0_1 sets are	of the form $[W_e]$	with index $\langle 0, e\rangle$
Π^0_α sets are	of the form $\{e\}^c$ where e is an index for a Σ^0_α set	with index $\langle 1, e\rangle$
Σ^0_α sets are	of the form $\bigcup_{n\in W_e}\{n\}$ where n is an index for a Π^0_β set with $\beta < \alpha$	with index $\langle 2, e\rangle$

Question : What is the level α at which no new set is added in the hierarchy?

Computable ordinals

Definition

An ordinal α is **computable** if there is a c.e. well-order $R \subseteq \omega \times \omega$ so that $|R|$, the order-type of R, is equal to α.

Definition (Church, Kleene)

The smallest non-computable ordinal is denoted by ω_1^{ck}, where the *ck* stands for 'Church-Kleene'.

Proposition

Every effective Borel set is Σ^0_α for $\alpha < \omega_1^{\mathrm{ck}}$. The hierarchy is strict before ω_1^{ck}.

Computable ordinals

Definition (Hyperarithmetical sets)

The effective Borel sets are called **hyperarithmetical sets**.

Every Σ^0_n set for n finite is definable by a first-order formula of arithmetic. It is not the case anymore with Σ^0_ω and beyond. **We can however define them with second order formulas of arithmetic**.

Analytic and co-analytical sets

Definition (Σ^1_1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ (or of ω) is Σ^1_1 if it can be defined by a formula of arithmetic whose second order quantifiers are only existential.

Definition (Π^1_1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ (or of ω) is Π^1_1 if it can be defined by a formula of arithmetic whose second order quantifiers are only universal.

Definition (Δ^1_1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ (or of ω) is Δ^1_1 if it is both Σ^1_1 and Π^1_1.

Theorem (Suslin 1917, Kleene 1955)

A set is hyperarithmetic iff it is Δ^1_1.

An important example of Π^1_1 set of sequences

Definition

For a sequence $X \in 2^\omega$, the smallest non-X-computable ordinal is denoted by ω_1^X.

The set $\mathcal{C} = \{X \ : \ \omega_1^X > \omega_1^{ck}\}$ is a Π^1_1 set with the following properties:

- $\mathcal{C}$ is of measure 0 (Sacks).
- $\mathcal{C}$ is a meager set (Feferman).
- $\mathcal{C}$ contains no Σ^1_1 subset (Gandy).
- $\mathcal{C}$ is a $\mathbf{\Sigma}^0_{\omega_1^{ck}+2}$ set which is not $\mathbf{\Pi}^0_{\omega_1^{ck}+2}$ (Steel).

A universal Π_1^1 set of integers

Notation

We denote by $\mathcal{O}$ the set of codes for computable ordinals, and $\mathcal{O}^X$ the set of X-codes for X-computable ordinals.
We denote by $\mathcal{O}_\alpha$ the set of codes for computable ordinals, coding for ordinals strictly smaller than α.

Example : we have $\mathcal{O} = \mathcal{O}_{\omega_1^{ck}}$ and $\mathcal{O}^X = \mathcal{O}^X_{\omega_1^X}$

The set $\mathcal{O}$, plays the same role as $\emptyset'$, but for Π_1^1 predicates:

Theorem (Complete Π_1^1 set)

A set of integers A is Π_1^1 iff there is a computable function $f : \omega \mapsto \omega$ so that $n \in A$ iff $f(n) \in \mathcal{O}$.

Π^1_1 sets with Kleene's $\mathcal{O}$

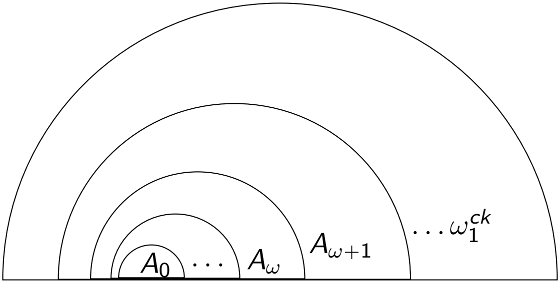
$\mathcal{A}$ is	a set of integers	a set of sequences
Π^1_1	$n \in \mathcal{A} \leftrightarrow f(n) \in \mathcal{O}$ for some computable function f	$X \in \mathcal{A} \leftrightarrow e \in \mathcal{O}^X$ for some e
Δ^1_1	$n \in \mathcal{A} \leftrightarrow f(n) \in \mathcal{O}_\alpha$ for some computable function f and some computable ordinal α	$X \in \mathcal{A} \leftrightarrow e \in \mathcal{O}^X_\alpha$ for some e and some ordinal α

Π^1_1 sets of integers

Suppose $A \subseteq \omega$ is Π^1_1 of index e and let us denote

$$A_\alpha = \{n \;:\; \varphi_e(n) \in \mathcal{O}_\alpha\}$$

Then A is an increasing union of Δ^1_1 sets:



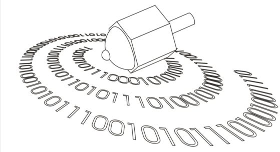
Π^1_1 is the higher analogue of c.e.

The higher analog of c.e. sets of integers is Π^1_1 sets of integers.

This has been sketched in previous slides : one can think of a Π^1_1 set of integers as being given by an enumeration with stages $\{s \mid s < \omega_1^{\mathrm{ck}}\}$.

Bottom setting	Higher analogue
c.e.	Π^1_1
finite c.e.	Δ^1_1
computable	Δ^1_1
$\varnothing'$	$\mathcal{O}$

Π^1_1-randomness



Section 2

Π^1_1-randomness

Higher randomness

We can now define higher randomness notions

Definition (Martin-Löf 1970)

A sequence is Δ^1_1-**random** if it belongs to no Δ^1_1 set of measure 0.

Definition (Sacks)

A sequence is Π^1_1-**random** if it belongs to no Π^1_1 set of measure 0.

What about Σ^1_1-randomness?

Theorem (Sacks)

A sequence is Σ^1_1-random iff it is Δ^1_1-random.

Π^1_1 randomness

The following theorems make Π^1_1-randomness an interesting notion of randomness:

Theorem (Kechris 1975, Hjorth, Nies 2007)

There is a universal Π^1_1 set of measure 0, that is, one containing all the others.

The set of $\{X : \omega_1^X > \omega_1^{ck}\}$ is a Π^1_1 set of measure 0. Therefore if X is Π^1_1-random, then $\omega_1^X = \omega_1^{ck}$. We also have some converse:

Theorem (Chong, Nies, Yu 2008)

A sequence X is Π^1_1-random iff it is Δ^1_1-random and $\omega_1^X = \omega_1^{ck}$.

Borel complexity of Π^1_1 randoms

Due to its universal nature, the set of Π^1_1 randoms is expected to have a higher Borel rank. But surprisingly we have:

Theorem (M.)

The set of Π^1_1 randoms is a $\mathbf{\Pi^0_3}$ set of the form:

$$\bigcap_n \bigcup_m \mathcal{F}_{n,m}$$

For each $\mathcal{F}_{n,m}$ a Σ^1_1 closed set.

where

Definition

A Π^1_1-**open set** is an open set $\mathcal{U}$ so that for a Π^1_1 set of strings A we have $\mathcal{U} = \bigcup\{[\sigma] \ : \ \sigma \in A\}$. A Σ^1_1-**closed set** is the complement of a Π^1_1-open set.

Sketch of the proof : Π^1_1 randoms is $\mathbf{\Pi^0_3}$

We define:

Definition (M.)

A sequence X is Σ^1_1-Solovay-generic if for every uniform union of Σ^1_1 closed sets $\bigcup_n \mathcal{F}_n$, either X is in $\bigcup_n \mathcal{F}_n$ or X belongs to a Σ^1_1 closed set of positive measure, included in the complement of $\bigcup_n \mathcal{F}_n$.

And we have:

Theorem (M.)

A sequence is Σ^1_1-Solovay-generic iff it is Π^1_1-random.

Lowness for Π^1_1-randomness

Definition

We say that A is low for Π^1_1-randomness if every $\Pi^1_1(A)$-random is also Π^1_1-random.

It is clear that any Δ^1_1 binary sequence is low for Π^1_1-randomness. Are there other sequences which are low for Π^1_1-randomness?

Theorem (Greenberg, M.)

The Δ^1_1 sequences are the only sequences that are low for Π^1_1-randomness.

The proof uses the equivalence between Π^1_1-randomness and Σ^1_1-Solovay-genericity.

Other characterization of Π^1_1-randomness

Theorem (Downey, Nies, Weber and Yu 2006)

A Martin-Löf random binary sequence is weakly-2-random iff it computes no non-computable c.e. binary sequence.

There is an analogue characterization of Π^1_1-randomness, using the notion of higher Turing reduction.

Theorem (Greenberg, M.)

For a Π^1_1-Martin-Löf random sequence X, the following are equivalent:

- X is Π^1_1-random.
- X higher Turing computes no (non Δ^1_1) Π^1_1 sequence.

Other characterization of Π^1_1-randomness

Theorem (Yu 2012, Franklin, Ng 2010)

For a sequence X, the following are equivalent:

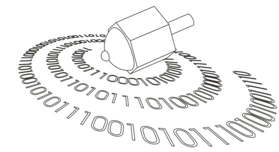
- X is Π^1_1-Martin-Löf random and does not higher Turing compute Kleene's $\mathcal{O}$.
- X is in no set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{F} \cap \mathcal{U}_n) \leqslant 2^{-n}$ where $\mathcal{F}$ a Σ^1_1-closed set and each $\mathcal{U}_n$ a Π^1_1 open set uniformly in n.

Theorem (Greenberg, M.)

For a sequence X, the following are equivalent:

- X is Π^1_1-random.
- X is in no set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{F} \cap \bigcap_n \mathcal{U}_n) = 0$ where $\mathcal{F}$ a Σ^1_1-closed set and each $\mathcal{U}_n$ a Π^1_1 open set uniformly in n.

Σ^1_1-genericity



Section 3

Σ^1_1-genericity

Σ^1_1-genericity

Definition (Greenberg, M.)

A sequence is Δ^1_1-**generic** if it is in every dense Δ^1_1-open set.

Definition (Greenberg, M.)

A sequence is **weakly-**Π^1_1**-generic** if it is in every dense Π^1_1-open set.
A sequence is Π^1_1-**generic** if for every Π^1_1-open set $\mathcal{U}$, either X is in $\mathcal{U}$ or X is in the interior of the complement of $\mathcal{U}$.

Definition (Greenberg, M.)

A sequence is **weakly-**Σ^1_1**-generic** if it is in every dense Σ^1_1-open set.
A sequence is Σ^1_1-**generic** if for every Σ^1_1-open set $\mathcal{U}$, either X is in $\mathcal{U}$ or X is in the interior of the complement of $\mathcal{U}$.

Σ^1_1-genericity

Proposition (Greenberg, M.)

For a sequence X the following are equivalent:

- X is Δ^1_1-generic.
- X is weakly-Π^1_1-generic.

There is a Π^1_1-generic which is not weakly-Π^1_1-generic.

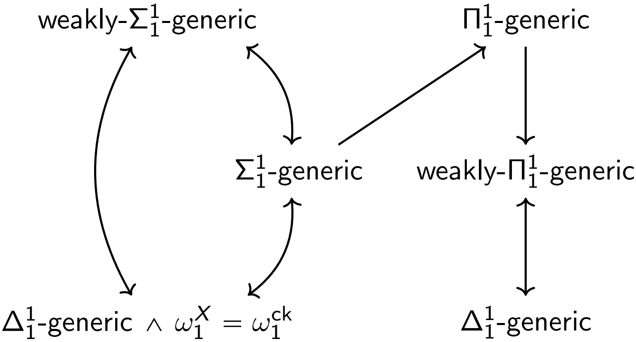
Theorem (Greenberg, M.)

For a sequence X the following are equivalent:

- X is Σ^1_1-generic.
- X is weakly-Σ^1_1-generic.
- X is Δ^1_1-generic and $\omega_1^X = \omega_1^{\mathrm{ck}}$.

Σ^1_1-genericity (3)

Summary:



Open questions

Question

What is lowness for Σ_1^1-genericty ?

Definition

An approximation $\{f_s\}_{s<\omega_1^{ck}}$ of a function f is finite-change if each $f_s(n)$ changes only finitely often over time.

Question

Does there exists a sequence X such that if X computes a finite-change approximation of a funtion f, then there is a finite-change approximation of a function g such that $f < g$.

Other higher randomness notions

Section 4

Other higher randomness notions

Π^1_1-Martin-Löf randomness

Definition

A Π^1_1-**open set** is an open set $\mathcal{U}$ so that for a Π^1_1 set of strings A we have $\mathcal{U} = \bigcup\{[\sigma] : \sigma \in A\}$.

Definition

A Σ^1_1-**closed set** is the complement of a Π^1_1-open set.

Definition (Hjorth, Nies 2007)

A sequence is Π^1_1-Martin-Löf random if it belongs to no set of the form $\bigcap_n \mathcal{U}_n$ where each $\mathcal{U}_n$ is a Π^1_1-open set, uniformly in n, with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$.

Higher weak-2-randomness

We now transfer to the higher setting the difference between weak-2-randomness and Martin-Löf randomness.

Definition (Nies 2009)

A set is weakly-Π^1_1-random if it is in no set $\bigcap_n \mathcal{U}_n$ with $\lambda(\bigcap_n \mathcal{U}_n) = 0$ where each $\mathcal{U}_n$ is a Π^1_1-open set uniformly in n.

The following justifies the terminology weak-Π^1_1-randomness:

Definition (M.)

A set is weakly-Σ^1_1-Solovay-generic if it is in every uniform union of Σ^1_1-closed sets which intersects with positive measure every Σ^1_1-closed sets of positive measure.

Theorem (M.)

A sequence is weakly-Σ^1_1-Solovay-generic iff it is weakly Π^1_1-random.

Separation of weak-Π^1_1-randomness from Π^1_1-randomness

Theorem (Chong, Yu 2012)

There are sequences which are Π^1_1-Martin-Löf random but not weakly-Π^1_1-random.

To prove this theorem, Chong and Yu proved that no sequence with a left-c.e. approximation is weakly-Π^1_1-random. Also it is well known that some sequence with a higher left-c.e. approximation is Π^1_1-Martin-Löf random.

Theorem (Greenberg, Bienvenu, M.)

There are sequences which are weakly-Π^1_1-random but not Π^1_1-random.

To prove this theorem, we define other restrictions of higher Δ^0_2 approximations.

Higher approximations

Definition (Greenberg, Bienvenu, M.)

An approximation $\{X_s\}_{s<\omega_1^{ck}}$ of X is **closed** if the set $\{X_s\}_{s<\omega_1^{ck}} \cup \{X\}$ is a closed set.

Theorem (Greenberg, Bienvenu, M.)

No sequence with a closed approximation is weakly-Π^1_1-random.

Definition (Greenberg, Bienvenu, M.)

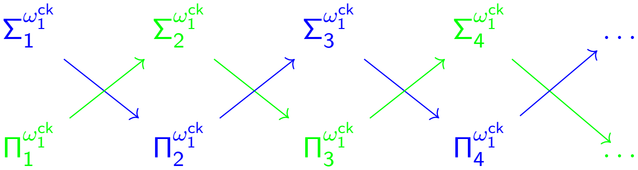
An approximation $\{X_s\}_{s<\omega_1^{ck}}$ of X is **collapsing** if for every $s < \omega_1^{ck}$, the sequence X is not in the closure of $\{X_t\}_{t<s}$.

Theorem (Greenberg, Bienvenu, M.)

No sequence with a collapsing approximation is Π^1_1-random. But such sequences can be weakly-Π^1_1-random.

Another hierarchy

We can now define another hierarchy, starting with Π^1_1-open sets and Σ^1_1-closed sets.



The blue sets are Π^1_1 sets

The green sets are Σ^1_1 sets

Randomness notions along the hierarchy

Fact

A sequence is Π^1_1-MLR if it belongs to no $\Pi^{\omega_1^{ck}}_2$ set effectively of measure 0.

Fact

A sequence is weakly-Π^1_1-random if it belongs to no $\Pi^{\omega_1^{ck}}_2$ set of measure 0.

Proposition

For a sequence X, the following are equivalent:

- X is in no $\Pi^{\omega_1^{ck}}_3$ set of measure 0.
- X is Δ^1_1-random.

Randomness notions along the hierarchy

Theorem (Greenberg, M.)

For a sequence X, the following are equivalent:

- X is in no $\Pi^{\omega_1^{ck}}_4$ set of measure 0.
- X is in no $\Pi^{\omega_1^{ck}}_n$ set of measure 0 for any n.
- X is in no Π^1_1 set of measure 0.

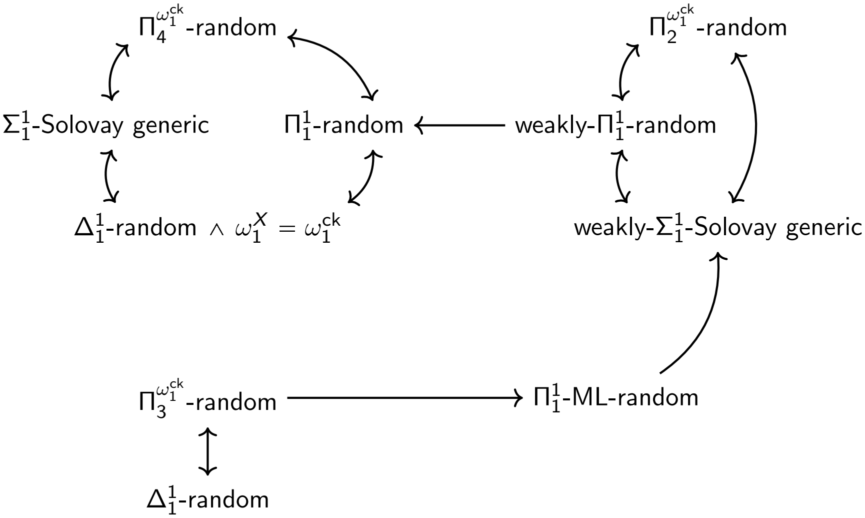
Theorem (Greenberg, M.)

The set of Π^1_1-randoms is $\Pi^{\omega_1^{ck}}_5$.

Question

Is there some X which is in no $\Sigma^{\omega_1^{ck}}_3$ set of measure 0 and not Π^1_1-random?

Summary


$\Pi_4^{\omega_1^{ck}}$-random
$\Pi_2^{\omega_1^{ck}}$-random
Σ^1_1-Solovay generic
Π^1_1-random
weakly-Π^1_1-random
Δ^1_1-random $\wedge$ $\omega_1^X = \omega_1^{ck}$
weakly-Σ^1_1-Solovay generic
$\Pi_3^{\omega_1^{ck}}$-random
Π^1_1-ML-random
Δ^1_1-random

Open questions

Question

Can the set of Π^1_1-random be $\Sigma_4^{\omega_1^{\mathrm{ck}}}$ set ?

Question

Is there a $\Pi_3^{\omega_1^{\mathrm{ck}}}$ set which contains only Π^1_1-randoms?

Equivalent characterization for Π^1_1-randomness

The following are equivalent:

- X is Π^1_1-random.
- X is in no $\Pi^{\omega_1^{ck}}_4$-null set.
- X is not in the largest $\Sigma^{\omega_1^{ck}}_5$ nullset.
- X is Σ^1_1-Solovay generic.
- X is Δ^1_1-random and $\omega_1^X = \omega_1^{ck}$.
- X is Π^1_1-Martin-Löf random and higher Turing computes no non trivial Π^1_1 sequence.
- X is in no set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{F} \cap \bigcap_n \mathcal{U}_n) = 0$ where $\mathcal{F}$ a Σ^1_1-closed set and $\bigcap_n \mathcal{U}_n$ a $\Pi^{\omega_1^{ck}}_2$ set.

Thank you