A unifying approach to the Gamma question

Benoit Monin
joint work with André Nies

Université Paris-Est Créteil

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Coarse computability

Given a set $A \subseteq \mathbb{N}$. How close is A to being computable ?

A recent paradigm : A is coarsely computable. This means there is a computable set R such that the asymptotic density of

$$\{n\colon\ A(n) = R(n)\}$$

equals 1.

Reference : *Downey, Jockusch, and Schupp, Asymptotic density and computably enumerable sets, Journal of Mathematical Logic, 13, No. 2 (2013)*

The γ-value of a set $A \subseteq \mathbb{N}$

A computable set R tries to approximate a complicated set A :

A : 100100100100 000101001001 010101111010 101010100111
R : 000010110111 010101000101 010001011010 101010100111

≈1/2 correct

≈2/3 correct

≈3/4 correct

≈4/5 correct

Take sup of the asymptotic correctness over all computable R's :

$$\begin{aligned} \gamma(A) &= \sup_{R \text{ computable}} \underline{\rho}\{n\colon\ A(n) = R(n)\} \\ \text{where } \underline{\rho}(Z) &= \liminf_{n} \frac{|Z \cap [0,n)|}{n}. \end{aligned}$$

Some examples of values $\gamma(A)$

Recall

$$\begin{aligned} \gamma(A) &= \sup_{R \text{ computable}} \underline{\rho}\{n \colon A(n) = R(n)\} \\ \text{where } \underline{\rho}(Z) &= \liminf_n \frac{|Z \cap [0,n)|}{n}. \end{aligned}$$

Some possible values

$$\begin{aligned} A \text{ computable} &\Rightarrow \gamma(A) = 1 \\ A \text{ random} &\Rightarrow \gamma(A) = 1/2. \end{aligned}$$

Γ-value of a Turing degree

Andrews, Cai, Diamondstone, Jockusch and Lempp (2013) looked at Turing degrees, rather than sets. They defined

$$\Gamma(A) = \inf\{\gamma(B) : \ B \text{ has the same Turing degree as } A\}$$

A smaller Γ value means that A is further away from computable.

Example

An oracle A is called computably dominated if every function that A computes is below a computable function. *They show :*

- If A is random and computably dominated, then $\Gamma(A) = 1/2$.
- If A is not computably dominated then $\Gamma(A) = 0$.

$\Gamma(A) > 1/2$ implies $\Gamma(A) = 1$

Fact (Hirschfeldt, Jockusch, McNicholl and Schupp)

If $\Gamma(A) > 1/2$ then A is computable (so that $\Gamma(A) = 1$).

The idea is to obtain B of the same Turing degree as A by "padding" :

- "Stretch" the value $A(n)$ over the whole interval $I_n = [(n-1)!, n!)$.
- Since $\gamma(B) > 1/2$ there is a computable R agreeing with B on more than half of the bits in almost every interval I_n.
- So for almost all n, the bit $A(n)$ equals the majority of values $R(k)$ where $k \in I_n$.

The Γ-question

Question (Γ-question, Andrews et al., 2013)

Is there a set $A \subseteq \mathbb{N}$ *such that* $0 < \Gamma(A) < 1/2$ *?*

$\bullet$	? ? ? ? ? ? ? ? ? ? ?	$\bullet$	$\times$ $\times$ $\times$ $\times$ $\times$ $\times$ $\times$	$\bullet$
$\Gamma = 0$		$\Gamma = 1/2$		$\Gamma = 1$

New examples towards answering the question

Recall : Γ-question, Andrews et al., 2013

Is there a set $A \subseteq \mathbb{N}$ such that $0 < \Gamma(A) < 1/2$?

Summary of previously known examples :

$\Gamma(A) = 0$	A non computably dominated or A PA
$\Gamma(A) = 1/2$	A low for Schnorr ; A random & comp. dominated
$\Gamma(A) = 1$	A computable

- Towards answering the question, we obtain natural classes of oracles with Γ value 1/2, and with Γ value 0.
- This yields new examples for both cases.

Weakly Schnorr engulfing

- We view oracles as infinite bit sequences, that is, elements of Cantor space $2^{\mathbb{N}}$.
- A Σ^0_1 set has the form $\bigcup_i [\sigma_i]$ for an effective sequence $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ of strings. $[\sigma]$ denotes the sequences extending σ.
- A Schnorr test is an effective sequence $(S_m)_{m \in \mathbb{N}}$ of Σ^0_1 sets in $2^{\mathbb{N}}$ such that
  - each λS_m is a computable real uniformly in m
  - $\lambda S_m \leqslant 2^{-m}$ (λ is the usual uniform measure on 2^ω).
- Fact : $\bigcap_m S_m$ fails to contain all computable sets.

We can relativize these notions to an oracle A.

We say that A is weakly Schnorr engulfing if A computes a Schnorr test containing all the computable sets.

This highness property of oracles was introduced by Rupprecht (2010), in analogy with 1980s work in set theory (cardinal characteristics).

Examples of A such that $\Gamma(A) \geqslant 1/2$

- The two known properties of A implying $\Gamma(A) \geqslant 1/2$ were :
  1. Computably dominated random, and
  2. low for Schnorr : every A-Schnorr test is covered by a plain Schnorr test.
- Both properties imply non-weakly Schnorr engulfing.
- There is a non-weakly Schnorr engulfing set without any of these properties. (Kjos-Hanssen, Stephan and Terwijn, 2015).

So the following result yields new examples of degrees with a Gamma of $1/2$.

Theorem

Let A be not weakly Schnorr engulfing. Then $\Gamma(A) \geqslant 1/2$.

Proof sketch : Given $B \leqslant_{\mathrm{T}} A$ and rational $\varepsilon > 0$, build an A-Schnorr test so that any set R passing it approximates B with asymptotic correctness $\geqslant 1/2 - \varepsilon$ (this uses Chernoff bounds).

Characterization of w.S.e. via traces

An obvious question is whether conversely, $\Gamma(A) \geqslant 1/2$ implies that A is not weakly Schnorr engulfing. We characterised w.S.e. towards obtaining an answer.

Let $H : \mathbb{N} \mapsto \mathbb{N}$ be computable with $\sum 1/H(n)$ finite. $\{T_n\}_{n\in\omega}$ is a *small computable H-trace* if

- T_n is a uniformly computable finite set
- $\sum_n |T_n|/H(n)$ is finite and computable.

Theorem

A is weakly Schnorr engulfing iff for some computable function H, there is an A-computable small H-trace capturing every computable function bounded by H.

Examples of $\Gamma(A) = 0$: infinitely often equal

We know that $A \subseteq \mathbb{N}$ not computably dominated implies $\Gamma(A) = 0$.

- We say $g : \mathbb{N} \to \mathbb{N}$ is infinitely often equal (i.o.e.) if $\exists^\infty n\ f(n) = g(n)$ for each computable function $f : \mathbb{N} \to \mathbb{N}$.
- We say that $A \subseteq \mathbb{N}$ is i.o.e. if A computes function g that is i.o.e.

Surprising fact : A is i.o.e $\Leftrightarrow$ A not computably dominated.

$\Rightarrow$ Suppose A computes a function g that equals infinitely often to every computable function. Then no computable function bounds g.

$\Leftarrow$ *Idea.* Suppose A computes a function g that is dominated by no computable function. Then g is infinitely often above the halting time of any computable total function.

New Examples of $\Gamma(A) = 0$: weaken infinitely often equal

We know A not computably dominated implies $\Gamma(A) = 0$.

Recall

We say that A is infinitely often equal (i.o.e.) if A computes a function g such that $\exists^\infty n\ f(n) = g(n)$ for each computable function $f : \mathbb{N} \to \mathbb{N}$.

We can weaken this :

Let $H\colon \mathbb{N} \to \mathbb{N}$ be computable. We say that A is H-infinitely often equal if A computes a function g such that $\exists^\infty n\ f(n) = g(n)$ for each computable function f bounded by H.

This appears to get harder for A the faster H grows.

New example of $\Gamma(A) = 0$

Let $H\colon \mathbb{N} \to \mathbb{N}$ be computable. We say that $A \subseteq \mathbb{N}$ is H-infinitely often equal if A computes a function g such that $\exists^\infty n\ f(n) = g(n)$ for each computable function f bounded by H.

Theorem

Let A be $2^{(\alpha^n)}$-i.o.e. for some $\alpha > 1$. Then $\Gamma(A) = 0$.

Previously known examples of sets A with $\Gamma(A) = 0$:

- not computably dominated, and
- degree of a completion of Peano arithmetic (PA for short).

If A is in one of these classes, for any computable bound H, A can compute an H-i.o.e. function.

Given a computable $H \geqslant 2$, we can build an H-i.o.e. set A that is computably dominated, and not PA. So we have a new example $\Gamma(A) = 0$ (using Rupprecht (2010)).

New example of $\Gamma(A) = 0$

Recall : A is H-infinitely often equal if A computes a function g such that $\exists^\infty n\ f(n) = g(n)$ for each computable function f bounded by H.

Theorem

Let A be $2^{(\alpha^n)}$-i.o.e. for some computable $\alpha > 1$. Then $\Gamma(A) = 0$.

Proof sketch. First step : Let f be $2^{(\alpha^n)}$-i.o.e. Then for any $k \in \mathbb{N}$, f computes a function g that is $2^{(k^n)}$-i.o.e.

f(0) f(1) f(2) f(3) f(4) f(5) . . . i.o.e. every comp. funct. $\leqslant 2^{(\alpha^n)}$

$$\begin{array}{lll} \to & f(0)f(2)f(4)\ldots & \text{i.o.e. every comp. funct. } \leqslant n \mapsto 2^{(\alpha^{2n})} \\ \text{or} & f(1)f(3)f(5)\ldots & \text{i.o.e. every comp. funct. } \leqslant n \mapsto 2^{(\alpha^{2n+1})} \end{array}$$

Iterating this $\to f \geqslant_T g$ which i.o.e. every comp. funct. $\leqslant 2^{(k^n)}$

Proof sketch. Second step : g is $2^{(k^n)}$-i.o.e. implies $g \geqslant_T Z$ with $\Gamma(Z) \leqslant 1/k$.

$$
\begin{array}{rccccc}
 & g(0) & g(1) & \dots & g(n) & \dots \\
 & = & = & \dots & = & \dots \\
Z: & \underbrace{\sigma_0}_{|\sigma_0|=k^0} & \underbrace{\sigma_1}_{|\sigma_1|=k^1} & \dots & \underbrace{\sigma_n}_{|\sigma_n|=k^n} & \dots \\
 & & & & & \\
\text{Computable } R: & \tau_0 & \tau_1 & \dots & \tau_n & \dots \\
 & & & \downarrow \text{(bit flip)} & & \\
\overline{R}: & \overline{\tau_0} & \overline{\tau_1} & \dots & \overline{\tau_n} & \dots \\
 & = & = & = & = & \\
 & j(0) & j(1) & \dots & j(n) & \dots
\end{array}
$$

j equals g infinitely often. Then for infinitely many n, $\tau_n(i) \neq \sigma_n(i)$ everywhere. We have

$$|\tau_n| \geqslant (k-1) \sum_{i<n} |\tau_i|$$

Then the lim inf of fraction of places where R agrees with Z is bounded by $1/k$.

Infinitely often equal : hierarchy

It is interesting to study infinite often equality for its own sake.

Question

Let H be a computable bound. Can we always find $H' >> H$ such that some f is H-i.o.e. but f computes no function that is H'-i.o.e. ?

First step : What about H-i.o.e. for H constant ?

X computable $\rightarrow$ X not 2-i.o.e. $\rightarrow$ X not c-i.o.e. for $c \in \mathbb{N}$

X not 2-i.o.e. $\rightarrow$ X computable.

X not 3-i.o.e. $\rightarrow$?

$$\begin{array}{rl}
Z \in 2^{\mathbb{N}} : & 0010101000100100101 \\
R \text{ computable} : & 1101010111011011010 \\
\hline
Z \in 3^{\mathbb{N}} : & 0210122002100102122 \\
R \text{ computable} : & 1102010111011211210
\end{array}$$

Infinitely often equal : constant bound

For any $c \in \mathbb{N}$, we can show X not c-i.o.e. $\rightarrow$ X computable. Let $c = 3$.

For $Z \in 2^\omega$, let $\#_2^Z : \omega^2 \rightarrow \omega$ the function which on $a, b \in \mathbb{N}$ returns $|Z \cap \{a, b\}|$. Note that $\#_2^Z$ can take three different values : $0, 1$ and 2.

Theorem (Kummer)

Suppose Z is an oracle such that $\#_3^Z$ is traceable via some trace $\{T_n\}_{n \in \omega}$, where each T_n is c.e. uniformly in n and $|T_n| \leqslant 3$. Then Z is computable.

Example :

$$
\begin{array}{cc}
 & 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7 \cdots \\
Z = & 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1 \cdots
\end{array}
\qquad
\begin{array}{rcl}
\#_3^Z(2,3) & \in & \{0,2\} \\
\#_3^Z(1,4) & \in & \{1,2\} \\
\#_3^Z(3,7) & \in & \{0,1\} \\
 & \cdots &
\end{array}
$$

Infinitely often equal : implications

Known implications :

$$
\begin{array}{ccl}
c\text{-i.o.e. for } c \geqslant 2 & \leftarrow & H(n)\text{-i.o.e with } H \text{ computable} \\
 & & \text{order function s.t. } \sum_n \frac{1}{H(n)} = \infty \\
\updownarrow & & \qquad\qquad\qquad \uparrow \\
\text{not computable} & & H(n)\text{-i.o.e with } H \text{ computable} \\
 & & \text{order function s.t. } \sum_n \frac{1}{H(n)} < \infty
\end{array}
$$

We don't know that there is a proper hierarchy for functions H with $\infty > \sum_n 1/H(n)$.

Summary

X is $2^{(\alpha^n)}$-i.o.e. (for some $\alpha > 1$)	$\rightarrow$	$\Gamma(X) = 0$
$\Gamma(X) < 1/2$	$\rightarrow$	X computes a small H-trace capturing all computable functions bounded by H (for some computable bound H)
X computable	$\leftrightarrow$	$\Gamma(X) > 1/2$
	$\leftrightarrow$	$\Gamma(X) = 1$