A unifying approach to the Gamma question

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Lowness paradigms

Given a set $A \subseteq \mathbb{N}$. How close is A to being computable?

Several paradigms have been suggested and studied.

- A has little power as a Turing oracle.
- Many oracles compute A.

A recent paradigm: A is coarsely computable.
This means there is a computable set R such that the asymptotic density of $\{n\colon\ A(n) = R(n)\}$ equals 1.

Reference: *Downey, Jockusch, and Schupp, Asymptotic density and computably enumerable sets, Journal of Mathematical Logic, 13, No. 2 (2013)*

The γ-value of a set $A \subseteq \mathbb{N}$

A computable set R tries to approximate a complicated set A:

A : 100100100100 000101001001 010101111010 101010100111

R : 000010110111 010101000101 010001011010 101010100111

≈1/2 correct

≈2/3 correct

≈3/4 correct

≈4/5 correct

Take sup of the asymptotic correctness over all computable R's:

$$\gamma(A) = \sup_{R \text{ computable}} \underline{\rho}\{n\colon\ A(n) = R(n)\}$$

$$\text{where } \underline{\rho}(Z) = \liminf_n \frac{|Z \cap [0,n)|}{n}.$$

Some examples of values $\gamma(A)$

Recall

$$\begin{aligned} \gamma(A) &= \sup_{R \text{ computable}} \underline{\rho}\{n\colon\ A(n) = R(n)\} \\ \text{where } \underline{\rho}(Z) &= \liminf_n \frac{|Z \cap [0,n)|}{n}. \end{aligned}$$

Some possible values

$$\begin{aligned} A \,\text{computable} &\Rightarrow \gamma(A) = 1 \\ A \,\text{random} &\Rightarrow \gamma(A) = 1/2. \end{aligned}$$

Γ-value of a Turing degree

Andrews, Cai, Diamondstone, Jockusch and Lempp (2013) looked at Turing degrees, rather than sets. They defined

$$\Gamma(A) = \inf\{\gamma(B) \colon B \text{ has the same Turing degree as } A\}.$$

A smaller Γ value means that A is further away from computable.

Example

An oracle A is called computably dominated if every function that A computes is below a computable function. *They show:*

- If A is random and computably dominated, then $\Gamma(A) = 1/2$.
- If A is not computably dominated then $\Gamma(A) = 0$.

$\Gamma(A) > 1/2$ implies $\Gamma(A) = 1$

Fact (Hirschfeldt et al., 2013)

If $\Gamma(A) > 1/2$ then A is computable (so that $\Gamma(A) = 1$).

Idea:

- Obtain B of the same Turing degree as A by "padding":
- "Stretch" the value $A(n)$ over the whole interval $I_n = [(n-1)!, n!)$.
- Since $\gamma(B) > 1/2$ there is a computable R agreeing with B on more than half of the bits in almost every interval I_n.
- So for almost all n, the bit $A(n)$ equals the majority of values $R(k)$ where $k \in I_n$.

The Γ-question

Question (Γ-question, Andrews et al., 2013)

Is there a set $A \subseteq \mathbb{N}$ such that $0 < \Gamma(A) < 1/2$?

$\bullet$??????????? $\bullet$ $\times\ \times\ \times\ \times\ \times\ \times\ \times$ $\bullet$

$\Gamma = 0$ $\Gamma = 1/2$ $\Gamma = 1$

New examples towards answering the question

Recall: Γ-question, Andrews et al., 2013

Is there a set $A \subseteq \mathbb{N}$ such that $0 < \Gamma(A) < 1/2$?

Summary of previously known examples:

$\Gamma(A) = 0$	A non computably dominated or A PA
$\Gamma(A) = 1/2$	A low for Schnorr; A random & comp. dominated
$\Gamma(A) = 1$	A computable

- Towards answering the question, we obtain natural classes of oracles with Γ value $1/2$, and with Γ value 0.
- This yields new examples for both cases.

Weakly Schnorr engulfing

- We view oracles as infinite bit sequences, that is, elements of Cantor space $2^{\mathbb{N}}$.
- A Σ_1^0 set has the form $\bigcup_i [\sigma_i]$ for an effective sequence $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ of strings. $[\sigma]$ denotes the sequences extending σ.
- A Schnorr test is an effective sequence $(S_m)_{m \in \mathbb{N}}$ of Σ_1^0 sets in $2^{\mathbb{N}}$ such that
  - each λS_m is a computable real uniformly in m
  - $\lambda S_m \leqslant 2^{-m}$. ($\lambda$ is the usual uniform measure on $2^{\mathbb{N}}$.)
- Fact: $\bigcap_m S_m$ fails to contain all computable sets.

We can relativize these notions to an oracle A.

We say that A is weakly Schnorr engulfing if
A computes a Schnorr test containing all the computable sets.

This highness property of oracles was introduced by Rupprecht (2010), in analogy with 1980s work in set theory (cardinal characteristics).

Examples of A such that $\Gamma(A) \geqslant 1/2$

- The two known properties of A implying $\Gamma(A) \geqslant 1/2$ were:
  (1) Computably dominated random, and
  (2) low for Schnorr test:
  every A-Schnorr test is covered by a plain Schnorr test.
- Both properties imply non-weakly Schnorr engulfing.
- There is a non-weakly Schnorr engulfing set without any of these properties. (Kjos-Hanssen, Stephan and Terwijn, 2015).

So the following result yields new examples, answering Question 5.1 in Andrews et al.

Theorem

Let A be not weakly Schnorr engulfing. Then $\Gamma(A) \geqslant 1/2$.

Proof: Given $B \leqslant_{\mathrm{T}} A$ and rational $\epsilon > 0$, build an A-Schnorr test so that any set R passing it approximates B with asymptotic correctness $\geqslant 1/2 - \epsilon$. This uses Chernoff bounds.

Characterization of w.S.e. via traces

An obvious question is whether conversely, $\Gamma(A) \geqslant 1/2$ implies that A is not weakly Schnorr engulfing. We characterised w.S.e. towards obtaining an answer. Again this is analogous to earlier work in cardinal characteristics.

Let $H : \mathbb{N} \mapsto \mathbb{N}$ be computable with $\sum 1/H(n)$ finite.
$\{T_n\}_{n\in\omega}$ is a *small computable H-trace* if

- T_n is a uniformly computable finite set
- $\sum_n |T_n|/H(n)$ is finite and computable.

Theorem

A is weakly Schnorr engulfing iff for some computable function H, there is an A-computable small H-trace capturing every computable function bounded by H.

Version of Γ in computational complexity

Fix an alphabet Σ. For $Z, A \subseteq \Sigma^*$ let

$$\begin{aligned}
\underline{\rho}(Z) &= \liminf_n \frac{\left|Z \cap \Sigma^{\leqslant n}\right|}{\left|\Sigma^{\leqslant n}\right|} \\
\gamma_{\texttt{poly}}(A) &= \sup_{R \text{ poly time computable}} \underline{\rho}(\{w \colon A(w) = R(w)\}) \\
\Gamma_{\texttt{poly}}(A) &= \inf\{\gamma_{\texttt{poly}}(B) \colon B \equiv^p_T A\}.
\end{aligned}$$

- The basic facts from computability used above need to be re-examined in the context of complexity theory.
- We only know at present that the values $\Gamma_{\texttt{poly}}(A)$ can be each of $0, \frac{1}{|\Sigma|}, 1$.

Examples of $\Gamma(A) = 0$: infinitely often equal

We know that $A \subseteq \mathbb{N}$ not computably dominated implies $\Gamma(A) = 0$.

- We say $g : \mathbb{N} \to \mathbb{N}$ is infinitely often equal (i.o.e.) if $\exists^\infty n\ f(n) = g(n)$ for each computable function $f : \mathbb{N} \to \mathbb{N}$.
- We say that $A \subseteq \mathbb{N}$ is i.o.e. if A computes function g that is i.o.e.

Surprising fact: A is i.o.e $\Leftrightarrow$ A not computably dominated.

$\Rightarrow$ Suppose A computes a function g that equals infinitely often to every computable function. Then no computable function bounds g.

$\Leftarrow$ *Idea.* Suppose A computes a function g that is dominated by no computable function. Then g is infinitely often above the halting time of any computable total function.

New Examples of $\Gamma(A) = 0$: weaken infinitely often equal

We know A not computably dominated implies $\Gamma(A) = 0$.

Recall

We say that A is infinitely often equal (i.o.e.) if A computes a function g such that $\exists^\infty n\ f(n) = g(n)$ for each computable function $f : \mathbb{N} \to \mathbb{N}$.

We can weaken this:

Let $H : \mathbb{N} \to \mathbb{N}$ be computable. We say that A is H-infinitely often equal if A computes a function g such that $\exists^\infty n\ f(n) = g(n)$ for each computable function f bounded by H.

This appears to get harder for A the faster H grows.

New example of $\Gamma(A) = 0$

Let $H\colon \mathbb{N} \to \mathbb{N}$ be computable. We say that $A \subseteq \mathbb{N}$ is H-infinitely often equal if A computes a function g such that $\exists^\infty n f(n) = g(n)$ for each computable function f bounded by H.

Theorem

Let A be $2^{(\alpha^n)}$-i.o.e. for some $\alpha > 1$.

Then $\Gamma(A) = 0$.

Previously known examples of sets A with $\Gamma(A) = 0$:

- not computably dominated, and
- degree of a completion of Peano arithmetic (PA for short).

If A is in one of these classes, for any computable bound H, A can compute an H-i.o.e. function.

Given a computable $H \geqslant 2$, we can build an H-i.o.e. set A that is computably dominated, and not PA. So we have a new example of $\Gamma(A) = 0$ (using Rupprecht (2010)).

New example of $\Gamma(A) = 0$

(Recall: A is H-infinitely often equal if A computes a function g such that $\exists^\infty n f(n) = g(n)$ for each computable function f bounded by H.)

Theorem

Let A be $2^{(\alpha^n)}$-i.o.e. for some computable $\alpha > 1$.

Then $\Gamma(A) = 0$.

Proof sketch. First step: Let f be $2^{(\alpha^n)}$-i.o.e. Then for any $k \in \mathbb{N}$, f computes a function g that is $2^{(k^n)}$-i.o.e.

f(0) f(1) f(2) f(3) f(4) f(5) ... i.o.e. every comp. funct. $\leqslant 2^{(\alpha^n)}$

$$
\begin{array}{lll}
\rightarrow & f(0)f(2)f(4)\ldots & \text{i.o.e. every comp. funct.} \leqslant n \mapsto 2^{(\alpha^{2n})} \\
\text{or} & f(1)f(3)f(5)\ldots & \text{i.o.e. every comp. funct.} \leqslant n \mapsto 2^{(\alpha^{2n+1})}
\end{array}
$$

Iterating this $\rightarrow$ $f \geqslant_T g$ which i.o.e. every comp. funct. $\leqslant 2^{(k^n)}$

Proof sketch. Second step: g is $2^{(k^n)}$-i.o.e. implies $g \geqslant_T Z$ with $\Gamma(Z) \leqslant 1/k$.

$$
\begin{array}{rccccc}
 & g(0) & g(1) & \dots & g(n) & \dots \\
 & = & = & \dots & = & \dots \\
Z: & \underbrace{\sigma_0}_{|\sigma_0|=k^0} & \underbrace{\sigma_1}_{|\sigma_1|=k^1} & \dots & \underbrace{\sigma_n}_{|\sigma_n|=k^n} & \dots \\
\text{Computable } R: & \tau_0 & \tau_1 & \dots & \tau_n & \dots \\
 & & & \downarrow \text{(bit flip)} & & \\
\overline{R}: & \overline{\tau_0} & \overline{\tau_1} & \dots & \overline{\tau_n} & \dots \\
 & = & = & = & = & \\
 & j(0) & j(1) & \dots & j(n) & \dots
\end{array}
$$

j equals g infinitely often. Then for infinitely many n, $\tau_n(i) \neq \sigma_n(i)$ everywhere. We have

$$|\tau_n| \geqslant (k-1) \sum_{i<n} |\tau_i|$$

Then the $\liminf$ of fraction of places where R agrees with Z is bounded by $1/k$.

Infinitely often equal: hierarchy

It is interesting to study infinite often equality for its own sake.

Question

Let H be a computable bound. Can we always find $H' >> H$ such that some f is H-i.o.e. but f computes no function that is H'-i.o.e. ?

First step : What about H-i.o.e. for H constant?

X computable $\rightarrow$ X not 2-i.o.e. $\rightarrow$ X not c-i.o.e. for $c \in \mathbb{N}$

X not 2-i.o.e. $\rightarrow$ X computable.

X not 3-i.o.e. $\rightarrow$?

$$\begin{array}{rl} Z \in 2^{\mathbb{N}} : & 001010100010010 0101 \\ R \text{ computable} : & 110101011101101 1010 \\ \hline Z \in 3^{\mathbb{N}} : & 021012200210010 2122 \\ R \text{ computable} : & 110201011101121 1210 \end{array}$$

Infinitely often equal: constant bound

For any $c \in \mathbb{N}$, we can show X not c-i.o.e. $\to$ X computable.
Let $c = 3$.
For $Z \in 2^\omega$, let $\#_2^Z : \omega^2 \to \omega$ the function which on $a, b \in \mathbb{N}$ returns $|Z \cap \{a, b\}|$. Note that $\#_2^Z$ can take three different values : $0, 1$ and 2.

Theorem (Kummer)

Suppose Z is an oracle such that $\#_3^Z$ is traceable via some trace $\{T_n\}_{n\in\omega}$, where each T_n is c.e. uniformly in n and $|T_n| \leqslant 3$. Then Z is computable.

Example:

$$\begin{array}{rl} & 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\cdots \\ Z = & 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\cdots \end{array} \qquad \begin{array}{rcl} \#_3^Z(2,3) & \in & \{0,2\} \\ \#_3^Z(1,4) & \in & \{1,2\} \\ \#_3^Z(3,7) & \in & \{0,1\} \\ & \cdots & \end{array}$$

Infinitely often equal: implications

Known implications:

$$\begin{array}{ccc}
c\text{-i.o.e. for } c \geqslant 2 & \leftarrow & H(n)\text{-i.o.e with } H \text{ computable} \\
 & & \text{order function s.t. } \sum_n \frac{1}{H(n)} = \infty \\
\updownarrow & & \uparrow \\
\text{not computable} & & H(n)\text{-i.o.e with } H \text{ computable} \\
 & & \text{order function s.t. } \sum_n \frac{1}{H(n)} < \infty
\end{array}$$

We don't know that there is a proper hierarchy for functions H with $\infty > \sum_n 1/H(n)$.

References

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