Higher randomness

Benoit Monin - LIAFA - University of Paris VII

Victoria university - 16 April 2014

Introduction

Section 1

Introduction

The Cantor space

What do we work with ?

Our playground	The Cantor space
Denoted by	2^ω
Topology	The one generated by the cylinders $[\sigma]$, the set of sequences extending σ, for every string σ
An open set $\mathcal{U}$ is	A union of cylinders

Algorithmic randomness (1)

What does it mean for a binary sequence to be random ?

Intuitively : Is it reasonable to think that c_1, c_2 or c_3 could have been obtained by a fair coin tossing ?

c_1:0000110000000001000100000100001000101000001000100 . . .

c_2:1010110001011001101001100011010111001001110010 10 . . .

c_3:0010010000111111011010101000100010000101101000 11 . . .

Algorithmic randomness (1)

What does it mean for a binary sequence to be random ?

Intuitively : Is it reasonable to think that c_1, c_2 or c_3 could have been obtained by a fair coin tossing ?

c_1:0000110000000001000100000010000010001010000001000100 ...

Law of large number : no

c_2:1010110001011001101001100011010111001001110010 10 ...

c_3:0010010000111111011010101000100010000101101000 11 ...

Algorithmic randomness (1)

What does it mean for a binary sequence to be random ?

Intuitively : Is it reasonable to think that c_1, c_2 or c_3 could have been obtained by a fair coin tossing ?

c_1:0000110000000001000100000010000100010100000010001 00...

Law of large number : no

c_2:1010 1100 0101 1001 1010 0110 0011 0101 1100 1001 1100...

pattern repetition : no

c_3:0010010000111111011010101000100010000101101000 11...

Algorithmic randomness (1)

What does it mean for a binary sequence to be random ?

Intuitively : Is it reasonable to think that c_1, c_2 or c_3 could have been obtained by a fair coin tossing ?

c_1:0000110000000001000100000010000100010100000010001 00 . . .

Law of large number : no

c_2:1010 1100 0101 1001 1010 0110 0011 0101 1100 1001 1100 . . .

pattern repetition : no

c_3:001001000011111101101010100010001000010110100011 . . .

$c_3 = \pi$: no

Algorithmic randomness (2)

Kolmogorov had the idea that our intuition of randomness for finite strings, corresponds to the incompressibility of finite strings.

Definition (Kolmogorov)

For a given machine $M : 2^{<\omega} \to 2^{<\omega}$, the M-Kolmogorov complexity $C_M(\sigma)$ of a string σ if the size of the smallest program which outputs σ via M.

Proposition/Definition (Kolmorogov)

There is a machine $U : 2^{<\omega} \to 2^{<\omega}$, universal in the sense that for any machine M we have $C_U(\sigma) \leqslant C_M(\sigma) + c_M$ with c_M a constant depending on M. The value $C(\sigma) = C_U(\sigma)$ is the **Kolmogorov complexity** of the string σ, well defined up to a constant.

Algorithmic randomness (3)

Proposition/Definition (Kolmorogov)

A string σ is d-incompressible if $C(\sigma) > |\sigma| - d$. The smallest d is, the more random σ is.

How to extend this notion of randomness for strings, to infinite sequences ? A first idea : A sequence X should be random if there is some d so that each prefix of X is d-incompressible.

But that fails, as for any d we have:

$$\underbrace{0100\ldots1010}_{=\sigma \text{ with } |\sigma|=d}\underbrace{010000010101010\ldots010000101001010}_{=\tau \text{ with } |\tau|=\sigma \text{ seen as an integer}}$$

Then the machine $M(\tau) = |\tau|\hat{}\tau$ can d-compress some prefix of X, for any d.

Complexity of sets

Section 2

Complexity of sets

Arithmetical complexity of sets

Following a work started by **Baire** in 1899 (Sur les fonctions de variables réelles), pursued by **Lebesgue** in his PhD thesis (1905), and many others (in particular **Lusin** and his student **Suslin**), we define the **Borel sets** on the Cantor space:

$\mathbf{\Sigma}^0_1$ sets are	Open sets
$\mathbf{\Pi}^0_1$ sets are	Closed sets
$\mathbf{\Sigma}^0_{n+1}$ sets are	Countable unions of $\mathbf{\Pi}^0_n$ sets
$\mathbf{\Pi}^0_{n+1}$ sets are	Complements of $\mathbf{\Sigma}^0_{n+1}$ sets

Effectivize the arithmetical complexity of sets (1)

This has latter been effectivized, following a work of **Kleene** and **Mostowsky**:

Definition (Effectivization of open sets)

A set $\mathcal{U}$ is Σ_1^0, or **effectively open**, if there is a code e for a program enumerating string such that so that $\mathcal{U}$ is the union of the cylinders corresponding to the enumerated strings.

Definition (Effectivization of closed sets)

A set $\mathcal{U}$ is Π_1^0, or **effectively closed**, if is the complement of a Σ_1^0 set.

Effectivize the arithmetical complexity of sets (2)

We can then continue inductively: $\left(\text{Notation} : [W_e] = \bigcup_{\sigma \in W_e} [\sigma]\right)$

Σ^0_1 sets are	of the form $[W_e]$
Π^0_1 sets are	of the form $[W_e]^c$
Σ^0_2 sets are	of the form $\bigcup_{n \in W_e} [W_n]^c$
Π^0_2 sets are	of the form $\bigcap_{n \in W_e} [W_n]$
...	...

Algorithmic randomness

Section 3

Algorithmic randomness

Martin-Löf's intuition

The first satisfactory definition of randomness for infinite sequences has been made by Martin-Löf in 1966.

Intuition

A sequence of 2^ω should be random if it belongs to no set of measure 0 (using Lebesgue measure, the uniform measure).

Problem

Any sequence X belongs to the set $\{X\}$, which is of measure 0.

Solution

We can pick countably many sets of measure 0. **The effective hierarchy provides a range of natural candidates**.

Martin-Löf's definition

Definition (Martin-Löf randomness)

A sequence is **Martin-Löf random** if it belongs to no Π^0_2 set 'effectively of measure 0'. A Π^0_2 set 'effectively of measure 0' is called a **Martin-Löf test**.

Definition (Effectively of measure 0)

An intersection $\bigcap \mathcal{A}_n$ of sets is effectively of measure 0 if $\lambda(\mathcal{A}_n) \leqslant 2^{-n}$.

Fact

One can equivalently require that the function $f : n \to \lambda(\mathcal{A}_n)$ is bounded by a computable function going to 0.

Why Martin-Löf's definition ?

Question

Why don't we just take Π^0_2 sets of measure 0 ? How important is the 'effectively of measure 0' condition ?

Answer(1)

The 'effectively of measure 0' condition implies that there is a universal **Martin-Löf test**, that is a Martin-Löf test containing all the others.

Answer(2)

It is not true anymore if we drop the 'effectively of measure 0' condition. Instead we get a notion known as **weak-2-randomness**.

Algorithmic randomness

We can build a hierachy of randomness notions:

1-random	Every Π^0_2 sets 'effectively of measure 0'
weakly-2-random	Every Π^0_2 sets of measure 0
2-random	Every Π^0_3 sets 'effectively of measure 0'
weakly-3-random	Every Π^0_3 sets of measure 0
...	...

We have:

$$\text{1-random} \leftarrow \text{w2-random} \leftarrow \text{2-random} \leftarrow \text{w3-random} \leftarrow \ldots$$

All implications are strict

Beyond arithmetic

∞

Section 4

Beyond arithmetic

Hyperarithmetical complexity of sets

We can extend the definition of Borel sets by induction over the ordinals:

$\mathbf{\Sigma}^0_1$ sets are	Open sets
$\mathbf{\Pi}^0_1$ sets are	Closed sets
$\mathbf{\Sigma}^0_{\alpha+1}$ sets are	Countable unions of $\mathbf{\Pi}^0_\alpha$ sets
$\mathbf{\Sigma}^0_{\sup_n \alpha_n}$ sets are	Countable unions of $\mathbf{\Pi}^0_\beta$ sets for $\beta < \sup_n \alpha_n$
$\mathbf{\Pi}^0_\alpha$ sets are	Complements of $\mathbf{\Sigma}^0_\alpha$ sets

It is clear that no new sets are define at step ω_1, by uncountablity of ω_1. Before that one can prove that the hierarchy is strict.

Effective Hyperarithmetical complexity of sets

How about the effective version ? The challange is to be able to effectively ‘unfold’ all the components of a Σ^0_α set.

(Notation : The set of index n is denoted by $\{n\}$)

Σ^0_1 sets are	of the form $[W_e]$	with index	$\langle 0, e\rangle$
Π^0_α sets are	of the form $\{e\}^c$	with index	$\langle 1, e\rangle$
Σ^0_α sets are	of the form $\bigcup_{n\in W_e}\{n\}$ where n is an index for a Π^0_β set with $\beta < \alpha$	with index	$\langle 2, e\rangle$

Question : At what ordinal α no new set is added in the hierarchy ?

Computable ordinals

Definition (Computable ordinals)

An ordinal α is computable if there is a c.e. well-order $R \subseteq \omega \times \omega$ so that $|R| = \alpha$.

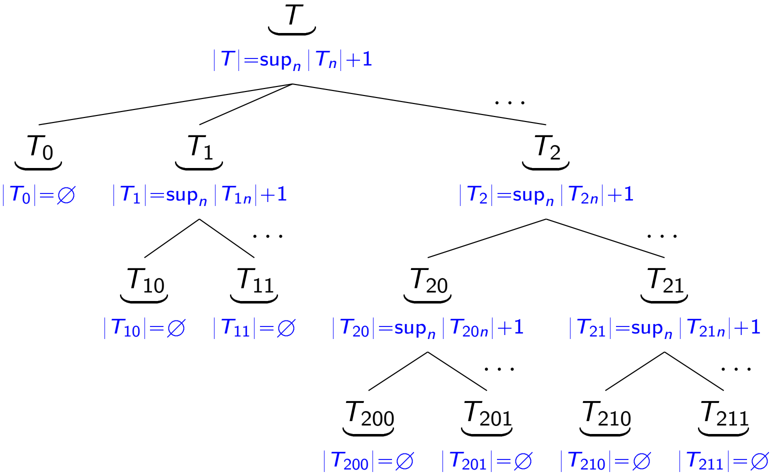
Proposition (Strict initial segment)

The computable ordinals form a strict initial segment of the countable ordinals.

Proposition (well-founded trees)

An ordinal is computable iff it is the order-type of a c.e. well-founded tree (tree with no infinite path).

Order-type of well-founded trees



Computable ordinals and effective Borel sets (1)

Definition (smallest non-computable ordinal)

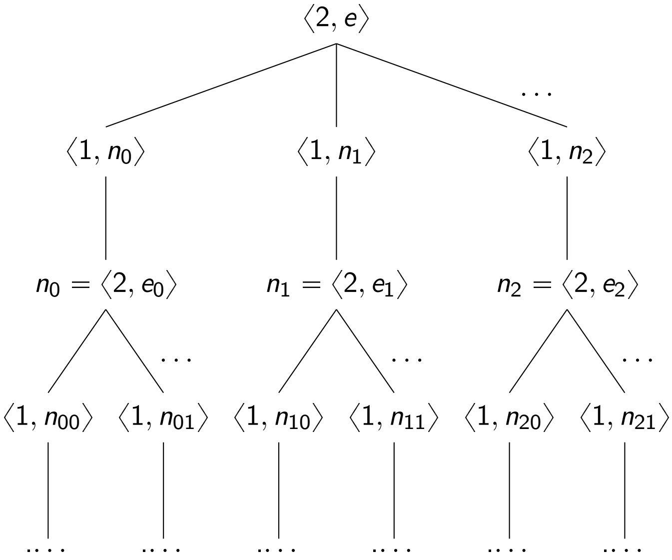
The smallest non-computable ordinal is denoted by ω_1^{ck}, where the *ck* stands for 'Church-Kleene'.

It is of historical interest to notice that the **Kleene's recursion theorem** has been 'cooked up' to work with codes of computable ordinals. Indeed, the theorem appear for the first time in 1938, in the paper called 'On notation for ordinal numbers'.

Claim

Indices of effective Borel sets are 'essentially' codes for computably enumerable well-founded trees.

Computable ordinals and effective Borel sets (1)



Computable ordinals

It follows that every effective Borel set is Σ^0_α for $\alpha < \omega_1^{\mathrm{ck}}$. Again one can prove that the hierarchy is strict before ω_1^{ck}.

Definition (Hyperarithmetical sets)

The effective Borel sets are called **hyperarithmetical sets**.

Every Σ^0_n set for n finite is definable by a first-order formula of arithmetic. It is not the case anymore with Σ^0_ω and beyond. **We can however define them with second order formulas of arithmetic**.

Analytic and co-analytical sets (1)

Definition (Σ_1^1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ is Σ_1^1 if it can be defined by a formula of arithmetic whose second order quantifiers are only existential (with no negation in front of them).

Definition (Π_1^1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ is Π_1^1 if it can be defined by a formula of arithmetic whose second order quantifiers are only univeral (with no negation in front of them).

Definition (Δ_1^1 sets)

A subset $\mathcal{A} \subseteq 2^\omega$ is Δ_1^1 if it is both Σ_1^1 and Π_1^1.

Analytic and co-analytical sets (2)

Proposition

An effective Borel set is both Σ^1_1 and Π^1_1.

While this proposition is essentially a tedious but straightforward induction over the computable ordinals, the converse is less tedious but much clever. A non-effective version was first prove by Suslin in 1917 ("Sur une definition des ensembles mesurables B sans nombres transfinis"). Then the effective version was proved much latter (after the effective concepts were introduced) by Kleene in 1955 ('Hierarchies of number theoretic predicates').

Theorem (Suslin, Kleene)

An set is effective Borel iff it is both Σ^1_1 and Π^1_1.

Analytic and co-analytical sets (3)

Notation

We denote by $\boldsymbol{W}$ the set of codes for computable ordinals, and $\boldsymbol{W}^X$ the set of X-codes for X-computable ordinals.
We denote by $\boldsymbol{W}_\alpha$ the set of codes for computable ordinals, coding for ordinal strictly smaller than α.

Example : we have $\boldsymbol{W} = \boldsymbol{W}_{\omega_1^{ck}}$ and $\boldsymbol{W}^X = \boldsymbol{W}_{\omega_1^X}$

We now have that the set $\boldsymbol{W}$, play the same role as $\emptyset'$, but for Π^1_1 predicates

Theorem (Complete Π^1_1 set)

A set of integers A is Π^1_1 iff there is a computable function $f : \omega \mapsto \omega$ so that $n \in A$ iff $f(n) \in \boldsymbol{W}$.

Analytic and co-analytical sets (4)

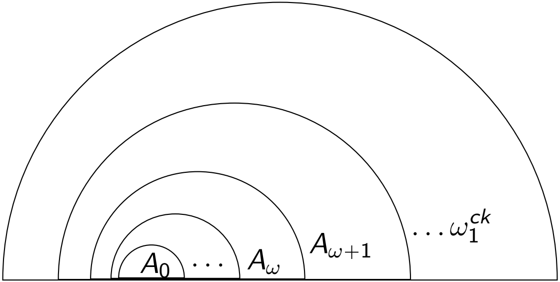
$\mathcal{A}$ is	a set of integer	a set of sequences
Π^1_1	$n \in \mathcal{A} \leftrightarrow f(n) \in \boldsymbol{W}$ for some computable function f	$X \in \mathcal{A} \leftrightarrow e \in \boldsymbol{W}^X$ for some e
Δ^1_1	$n \in \mathcal{A} \leftrightarrow f(n) \in \boldsymbol{W}_\alpha$ for some computable function f and some computable ordinal α	$X \in \mathcal{A} \leftrightarrow e \in \boldsymbol{W}^X_\alpha$ for some e and some ordinal α

Π^1_1 sets : Increasing union over the ordinals (1)

Suppose $A \subseteq \omega$ is Π^1_1 with index e and let us denote

$$A_\alpha = \{n \ : \ \varphi_e(n) \in \boldsymbol{W}_\alpha\}$$

Then A is an increasing union of Δ^1_1 sets:

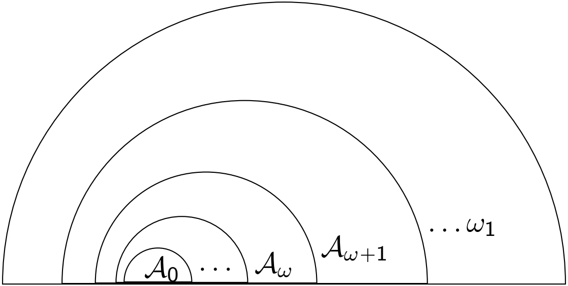


Π^1_1 sets : Increasing union over the ordinals (2)

Suppose $\mathcal{A} \subseteq 2^\omega$ is Π^1_1 with index e and let us denote

$$\mathcal{A}_\alpha = \{X \ : \ e \in \boldsymbol{W}^X_\alpha\}$$

Then $\mathcal{A}$ is an increasing union of $\boldsymbol{\Delta}^1_1$ sets:



An important example of Π^1_1 set of sequences

The set $\left\{X \ : \ \omega_1^X > \omega_1^{ck}\right\}$ is a Π^1_1 set :

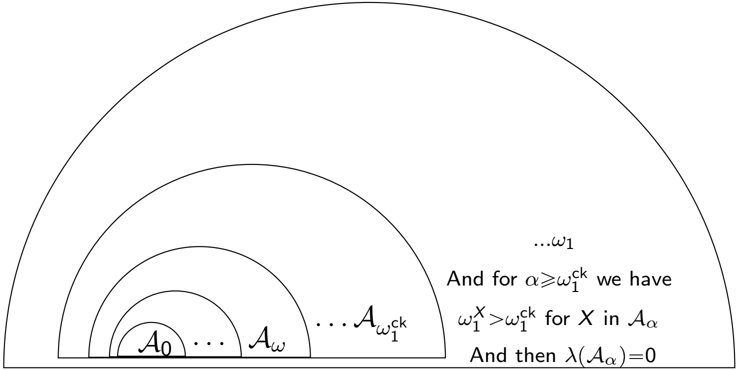
$$\left\{ X : \begin{array}{l} \exists e \in \boldsymbol{W}^X \quad \forall n \quad \forall f \\ f \text{ is not a bijection between the relation coded by } n \\ \text{and the one coded by } e \end{array} \right\}$$

Theorem (Sacks)

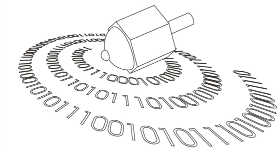
The set $\{X \ : \ \omega_1^X > \omega_1^{ck}\}$ is of measure 0.

Consequences of Sacks theorem

Suppose $\mathcal{A} \subseteq 2^\omega$ is Π^1_1 with index e, we then have :



Higher randomness



Section 5

Higher randomness

Higher randomness

We can now define higher randomness notions

Definition Δ^1_1-random (Martin-Löf)

A sequence is Δ^1_1-random if it belongs to no Δ^1_1 set of measure 0.

Definition Π^1_1-random (Sacks)

A sequence is Π^1_1-random if it belongs to no Π^1_1 set of measure 0.

What about Σ^1_1-randomness ?

Theorem (Sacks)

A sequence is Σ^1_1-random iff it is Δ^1_1-random

Π^1_1 randomness

Theorem (Keckris, Nies, Hjorth)

There is a universal Π^1_1 set of measure 0, that is one containing all the others.

As the set of $\{X : \omega_1^X > \omega_1^{\mathrm{ck}}\}$ is a Π^1_1 set of measure 0. Therefore if something is Π^1_1-random, then $\omega_1^X = \omega_1^{\mathrm{ck}}$. We have a very nice theorem about the converse:

Theorem (Chong, Yu, Nies)

A sequence X is Π^1_1-random iff it is Δ^1_1-random and $\omega_1^X = \omega_1^{\mathrm{ck}}$.

Another hierarchy (1)

Definition (Π^1_1 open set)

A Π^1_1 **open set** is an open set $\mathcal{U}$ so that for a Π^1_1 set of strings A we have $\mathcal{U} = \bigcup\{[\sigma] \ : \ \sigma \in A\}$.

Definition (Index for Π^1_1 open set)

For a Π^1_1 open set $\mathcal{U} = \bigcup\{[\sigma] \ : \ f(\sigma) \in \boldsymbol{W}\}$ with f a computable function, we say that a code e for f is an **index** for $\mathcal{U}$, and we write $\mathcal{U} = [W_e^{\omega_1^{\mathrm{ck}}}]$.

Definition (Σ^1_1 closed set)

A Σ^1_1 **closed set** is the complement of a Π^1_1 open set.

Another hierarchy (2)

We can establish a new hierarchy by taking successive effective union and effective intersection of Π^1_1 open sets and Σ^1_1 closed sets.

$\boldsymbol{\Sigma}_{\mathbf{1}}^{\boldsymbol{\omega}_1^{\mathrm{ck}}}$ sets are	Π^1_1 open sets $[W_e^{\omega_1^{\mathrm{ck}}}]$	with index	e
$\boldsymbol{\Pi}_{\mathbf{1}}^{\boldsymbol{\omega}_1^{\mathrm{ck}}}$ sets are	Σ^1_1 closed sets $[W_e^{\omega_1^{\mathrm{ck}}}]^c$	with index	e
$\boldsymbol{\Sigma}_{\boldsymbol{n}+\mathbf{1}}^{\boldsymbol{\omega}_1^{\mathrm{ck}}}$ sets are	$\bigcup_{m \in W_e} \{m\}$ where each m is an index for a $\Pi_n^{\omega_1^{\mathrm{ck}}}$	with index	e
$\boldsymbol{\Pi}_{\boldsymbol{n}+\mathbf{1}}^{\boldsymbol{\omega}_1^{\mathrm{ck}}}$ sets are	$\bigcap_{m \in W_e} \{m\}$ where each m is an index for a $\Sigma_n^{\omega_1^{\mathrm{ck}}}$	with index	e

Another hierarchy (3)

Unlike what we have with the other hierarchies, we now have that a $\Sigma_n^{\omega_1^{\mathrm{ck}}}$ is not necessarily $\Sigma_{n+1}^{\omega_1^{\mathrm{ck}}}$:

$$
\begin{tikzcd}
\Sigma_1^{\omega_1^{\mathrm{ck}}} \arrow[dr] & \Sigma_2^{\omega_1^{\mathrm{ck}}} \arrow[dr] & \Sigma_3^{\omega_1^{\mathrm{ck}}} \arrow[dr] & \Sigma_4^{\omega_1^{\mathrm{ck}}} \arrow[dr] & \cdots \\
\Pi_1^{\omega_1^{\mathrm{ck}}} \arrow[ur] & \Pi_2^{\omega_1^{\mathrm{ck}}} \arrow[ur] & \Pi_3^{\omega_1^{\mathrm{ck}}} \arrow[ur] & \Pi_4^{\omega_1^{\mathrm{ck}}} \arrow[ur] & \cdots
\end{tikzcd}
$$

The blue sets are Π_1^1 sets

The green sets are Σ_1^1 sets

Other higher randomness notions (1)

We can now define:

Definition Π^1_1-MLR (Hjorth, Nies)

A sequence is Π^1_1-MLR if it belongs to no $\Pi^{\omega_1^{\mathrm{ck}}}_2$ sets effectively of measure 0

Definition weakly-Π^1_1-random (Nies)

A sequence is strongly-Π^1_1-MLR if it belongs to no $\Pi^{\omega_1^{\mathrm{ck}}}_2$ set of measure 0.

Other higher randomness notions (2)

We cannot straightforwardly continue to define randomness notions along the hierarchy, because for example, Π^0_3 sets of measure 0 are Σ^1_1 sets ($\bigcap_n \bigcup_m F_{n,m}$ where each $F_{n,m}$ is Σ^1_1 closed), and then fail to capture even Π^1_1-MLR.

And we can however define the notion of being captured by $\Pi^{\omega_1^{\mathrm{ck}}}_n$ sets of measure 0 for n even. But it collapse for $n = 4$:

Theorem

A sequence is no $\Pi^{\omega_1^{\mathrm{ck}}}_4$ sets of measure 0 iff it is in no Π^1_1 sets of measure 0 iff it is in no $\Pi^{\omega_1^{\mathrm{ck}}}_n$ sets of measure 0 for any n.

The question has not been investigated for $\Pi^{\omega_1^{\mathrm{ck}}}_4$ sets effectively of measure 0.

Other higher randomness notions (3)

What is known:

$$\Delta^1_1 \textit{random} \leftarrow \Pi^1_1 \textit{MLR} \leftarrow \textit{strongly}\Pi^1_1 \textit{MLR} \leftarrow$$

$$\Pi^1_1 \textit{random} = \text{`}\textit{strongly}\Pi^{\omega_1^{ck}}_4 \textit{MLR}'$$

All the implications are strict. The proof of separation between $\Pi^{\omega_1^{ck}}_2$ *random* and Π^1_1*random* requires a refinement of the notion of being Δ^0_2 that would not make sense in the lower world.

Higher Δ^0_2 (1)

In the lower case: A Δ^0_2 sequence X:

$$X = \lim_{s \in \omega} X_s$$

we have that $\{X_s\}_{s \leqslant \omega}$ is a closed set, by definition of convergence

In the higher case: A $\Delta_2^{\omega_1^{\mathrm{ck}}}$ sequence X:

$$X = \lim_{s \in \omega_1^{\mathrm{ck}}} X_s$$

There is no reason for $\{X_s\}_{s \leqslant \omega_1^{\mathrm{ck}}}$ to be a closed set...

Higher Δ^0_2 (2)

Definition (closed approximation)

A sequence X has a **closed approximation** if $X = \lim_{s \in \omega_1^{\text{ck}}} X_s$ where each X_s is Δ^1_1 uniformly in s and where $\{X_s\}_{s \leq \omega_1^{\text{ck}}}$ is a closed set.

Definition (wicked approximation)

A sequence X has a **wicked approximation** if $X = \lim_{s \in \omega_1^{\text{ck}}} X_s$ where each X_s is Δ^1_1 uniformly in s and where for any $t < \omega_1^{\text{ck}}$ we have $X \notin \overline{\{X_s\}_{s \leq t}}$

Topological differences

Section 6

Topological differences

The first Δ^1_1 continuous reduction

In the bottom world, the following four definitions are equivalent:

1. $A \geqslant_T X$
2. There is a Σ^0_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent on prefixes of** A, such that $\forall n \;\; \exists \tau \prec X \;\; \exists \sigma \prec A \;\; |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R$
3. There is a Σ^0_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent everywhere**, such that $\forall n \;\; \exists \tau \prec X \;\; \exists \sigma \prec A \;\; |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R$
4. There is a Σ^0_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent everywhere and closed under prefixes**, such that $\forall n \;\; \exists \tau \prec X \;\; \exists \sigma \prec A \;\; |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R$

The first Δ^1_1 continuous reduction

A first attempt to use continuous version of hyperarithmetic reducibility was made by Hjorth and Nies in order to study higher analogue of Kucera-Gacs and Higher analogue of Base for randomness. They defined fin-*h* reductions, corresponding to the strongest notion among those defined in the previous slides:

Definition

A fin-h reduction is a Π^1_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent everywhere and closed under prefixes**, such that
$\forall n \ \ \exists \tau \prec X \ \ \exists \sigma \prec A \ \ |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R$

We say that $A \geqslant_{fin-h} X$ if for a *fin-h* reduction M we have
$\forall n \ \ \exists \tau \prec X \ \ \exists \sigma \prec A \ \ |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in M$.

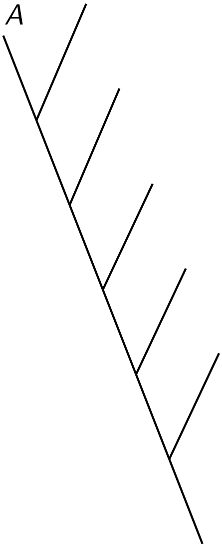
A topological difference

The bottom world	The higher world
At any time t of the enumeration, the set of strings mapped so far is a **clopen set**	At any time α of the enumeration, the set of strings mapped so far is an **open set**.

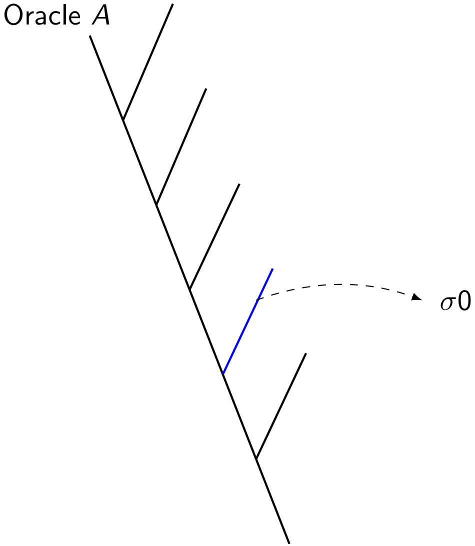
This make the three previous notions different in the higher world.

The Fishbone

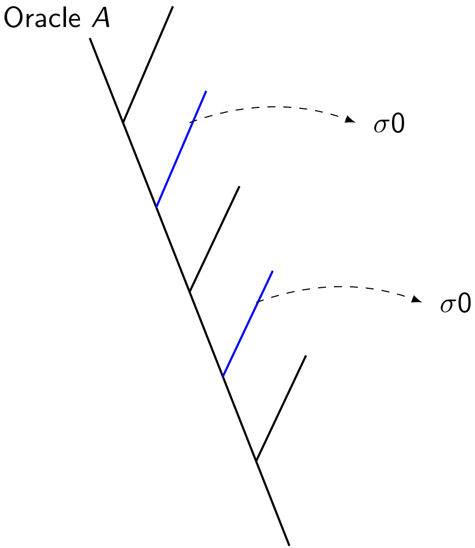
Oracle A



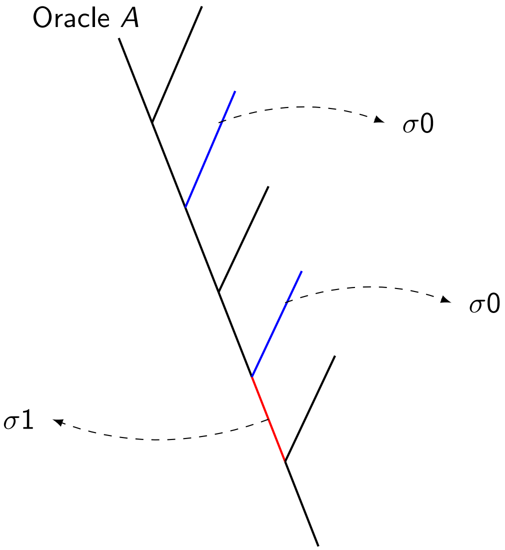
The Fishbone



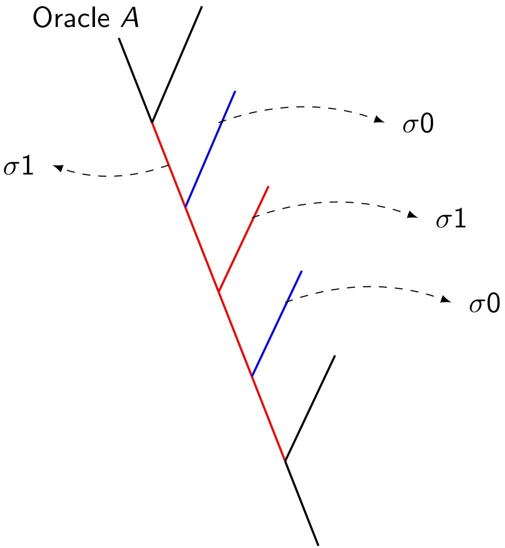
The Fishbone



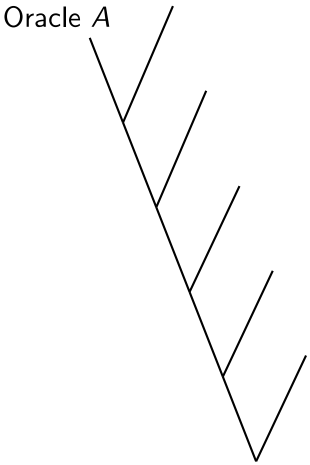
The Fishbone



The Fishbone

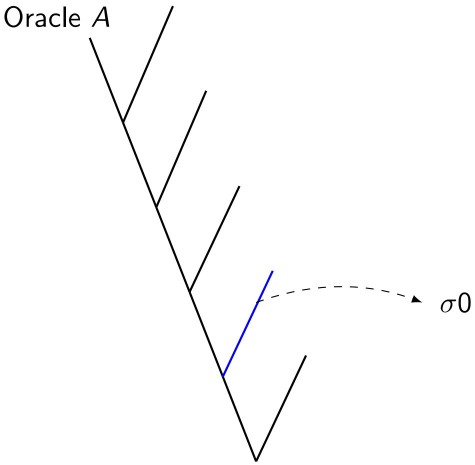


Defeating fin-h



Basic strategy:

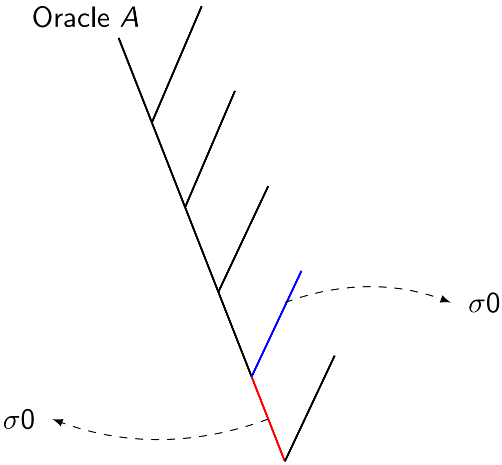
Defeating fin-h



Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Defeating fin-h

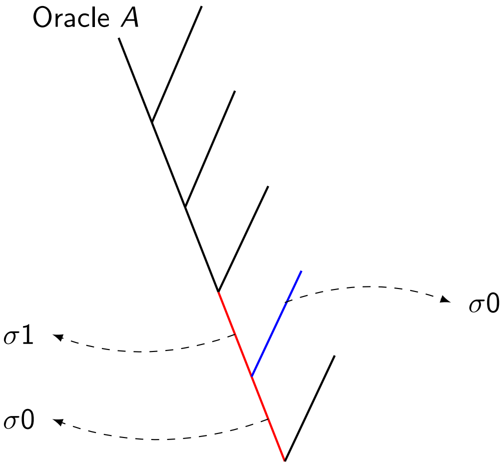


Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Suppose it matches one
prefix to $\sigma 0$ as well...

Defeating fin-h



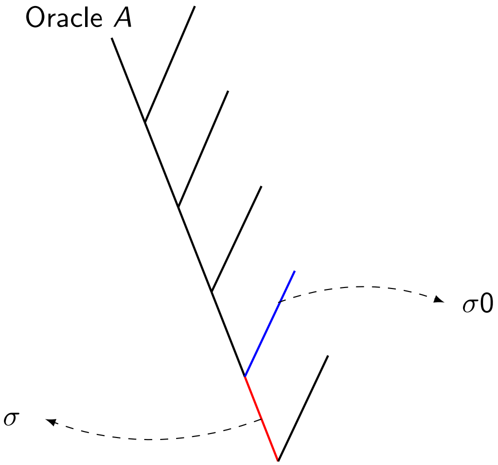
Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Suppose it matches one
prefix to $\sigma 0$ as well...

Then you win

Defeating fin-h

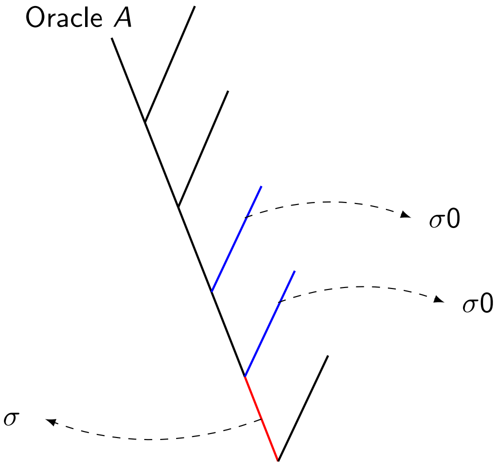


Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Otherwise...

Defeating fin-h

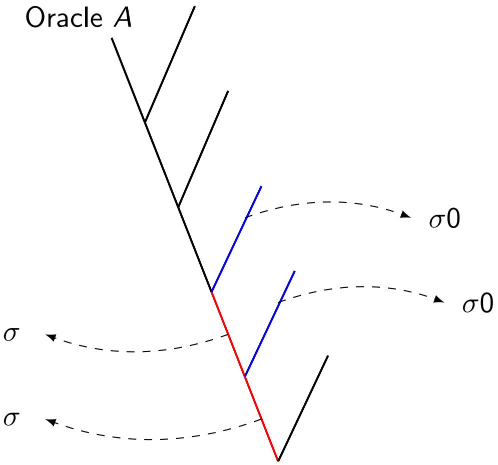


Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Otherwise...

Defeating fin-h

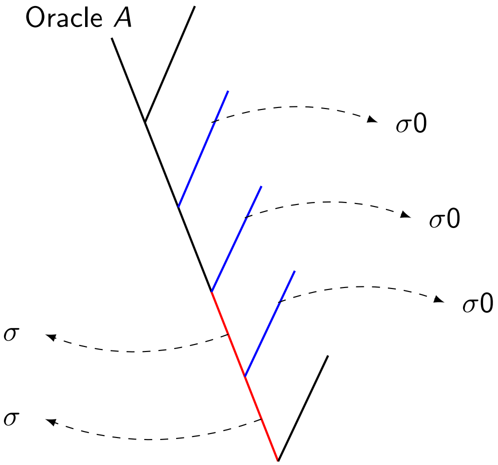


Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Otherwise...

Defeating fin-h

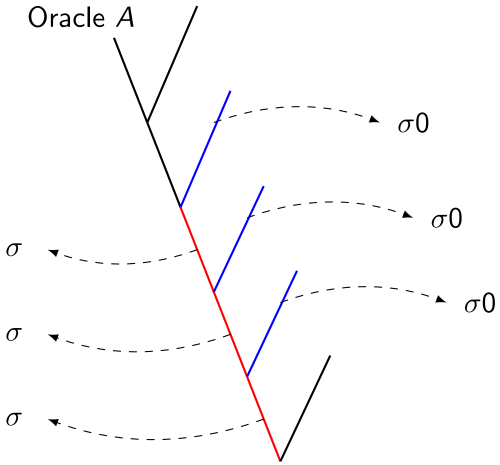


Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Otherwise...

Defeating fin-h

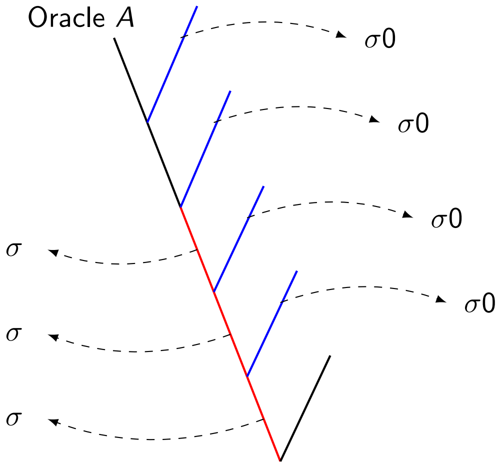


Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Otherwise...

Defeating fin-h

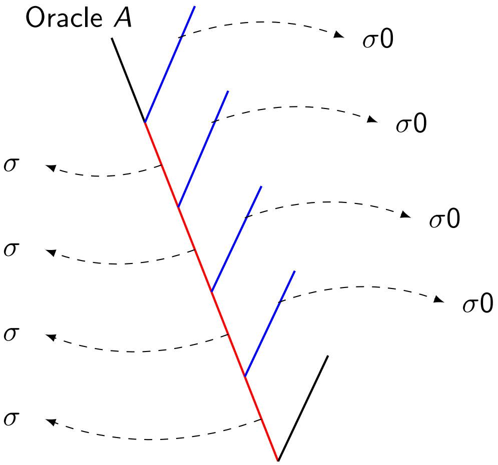


Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

Otherwise...

Defeating fin-h



Basic strategy:

Wait for
the opponent to decide
sth. on all the prefixes.

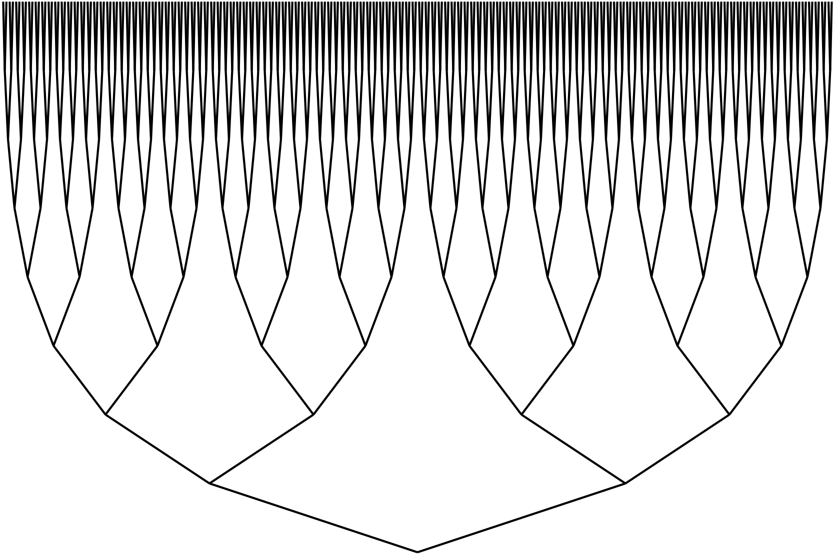
Otherwise...

Defeating fin-h

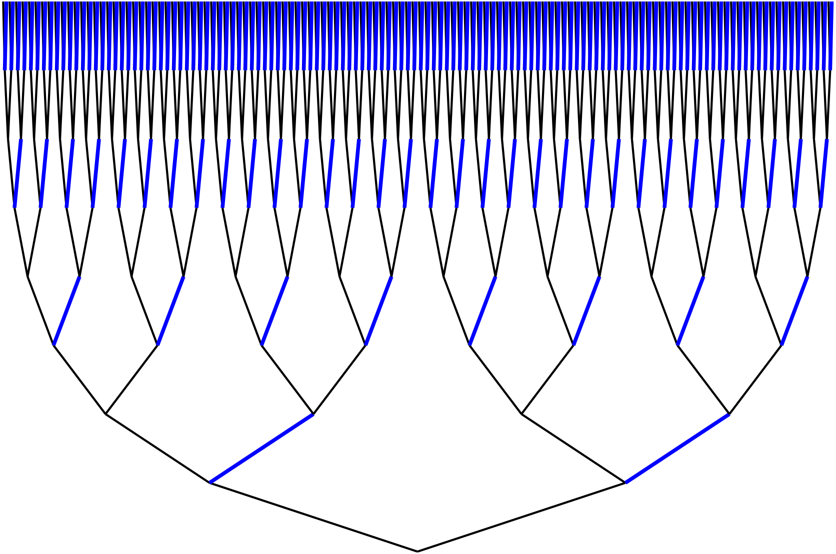
This is only one strategy. The problem is that one machine can force you to pick an entire oracle in order to defeat it. How to continue the construction and defeat other requirements ?

One solution : The perfect treesh-bone !

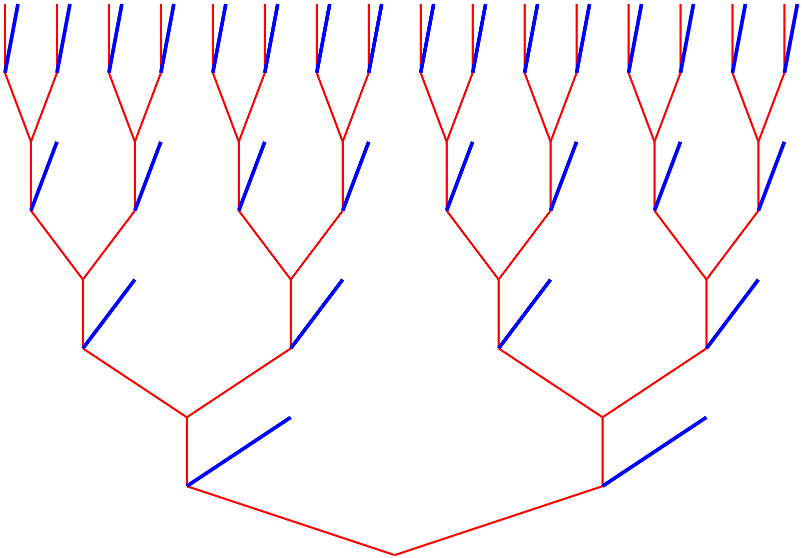
The treesh-bone (1)



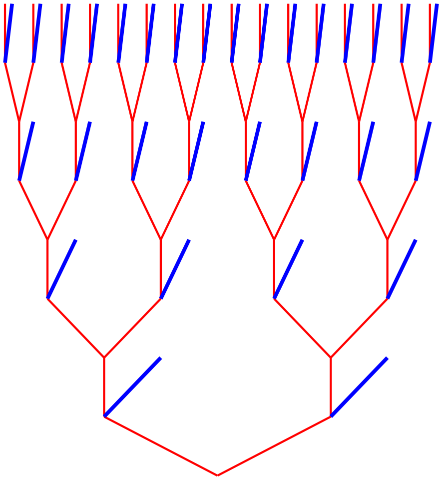
The treesh-bone (2)



The treesh-bone (3)



The treesh-bone (4)



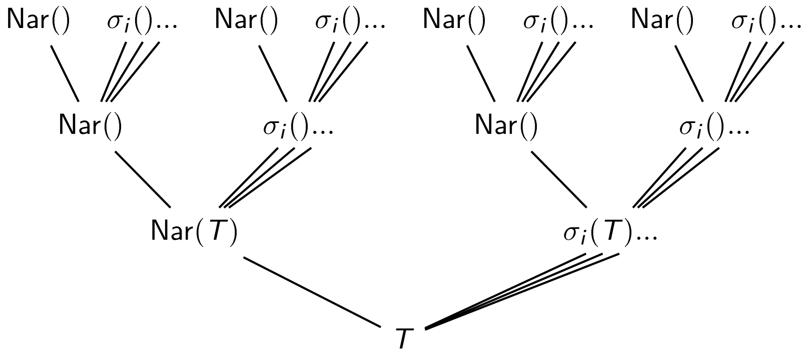
Put $\sigma 0$
along all the blue strings

Even if we
are forced to stay along the red part of the tree, we still have a prefect tree that we can continue to work with !

—:$\mathrm{Nar}(T)$,
The narrow subtree of T
—:$\sigma_i(T)$, the subtree of T extending the string σ_i

The tree of trees (1)

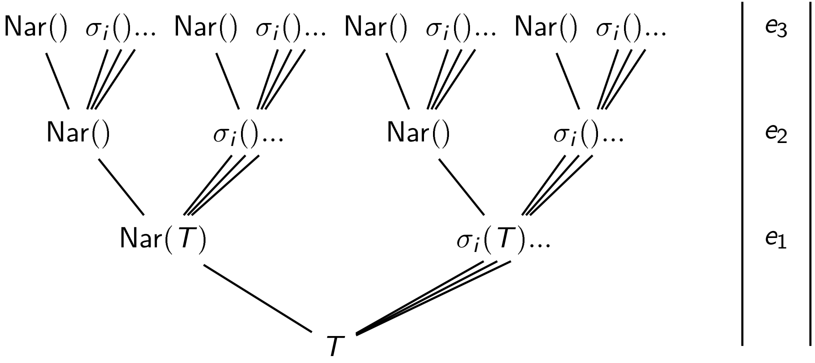
We can imagine that working in a tree of trees.



- The left node of T correspond to $Nar(T)$
- There is infinitely many right node $\sigma_i(T)$

The tree of trees (2)

We now order the requirement to do a higher finite injury argument:



The Higher Turing reduction

So some consistent map of strings cannot be made equivalent to some consistent map of strings whose domain is closed by prefixes.

Similarly we can prove that if a map of strings, not consistent everywhere, sends X to Y, there is not necessarily a consistent map of strings sending X to Y.

These brings the new definition:

Definition

We say that $A \geqslant_T B$ if there is a Π^1_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent on prefixes of** A, such that
$\forall n \, \exists \tau \prec X \quad \exists \sigma \prec A \quad |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R.$

Oracles for which reductions collapses

For a large class of oracles, in a measure theoretic sense, the three notions of reductions are the same:

Fact

If $\omega_1^A = \omega_1^{\mathrm{ck}}$ and $A \geqslant_T X$ then $A \geqslant_{fin-h} X$.

Thank you