Higher randomness and forcing with closed sets

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Overview

The goal of this work is to get a better understanding of two notions of randomness :

1/ The first one is called weakly-Π^1_1-randomness (Notion introduced by Nies, in 'Computability and randomness', 2009)

2/ The second one is called Π^1_1-randomness (Notion introduced by Sacks, in 'Higher recursion theory', 1990)

The goal : Show that 2/ is stronger than 1/

How : Yu and Chong proposed to show that the two notions have different Borel complexity.

We showed that this is not possible, as both sets are $\mathbf{\Sigma^0_3}$.

Complexity of sets

Section 1

Complexity of sets

The Cantor space

What do we work with ?

Our playground	The Cantor space
Denoted by	2^ω
Topology	The one generated by the cylinders $[\sigma]$, the set of sequences exntended σ, for every string σ
An open set $\mathcal{U}$ is	A union of cylinders

Arithmetical complexity of sets

The Borel sets on the Cantor space

$\mathbf{\Sigma}^0_1$ sets are	Open sets
$\mathbf{\Pi}^0_1$ sets are	Closed sets
$\mathbf{\Sigma}^0_{n+1}$ sets are	Countable unions of $\mathbf{\Pi}^0_n$ sets
$\mathbf{\Pi}^0_{n+1}$ sets are	Complements of $\mathbf{\Sigma}^0_{n+1}$ sets

What about a union of $\mathbf{\Pi}^0_n$ sets for n unbounded in ω ?

Arithmetical complexity of sets

We define Borel sets by induction over the ordinals

$\mathbf{\Sigma}^0_1$ sets are	Open sets
$\mathbf{\Pi}^0_1$ sets are	Closed sets
$\mathbf{\Sigma}^0_{\alpha+1}$ sets are	Countable unions of $\mathbf{\Pi}^0_\alpha$ sets
$\mathbf{\Pi}^0_{\alpha+1}$ sets are	Complements of $\mathbf{\Sigma}^0_{\alpha+1}$ sets
$\mathbf{\Sigma}^0_{\sup_n \alpha_n}$ sets are	Countable unions of $\mathbf{\Pi}^0_\beta$ sets for $\beta < \sup_n \alpha_n$

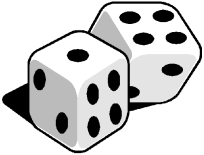
Effectivize the arithmetical complexity of sets

$$\text{Notation : } [W_e] = \bigcup_{\sigma \in W_e} [\sigma]$$

Σ^0_1 sets are	of the form $[W_e]$
Π^0_1 sets are	of the form $[W_e]^c$
Σ^0_2 sets are	of the form $\bigcup_{n \in W_e} [W_n]^c$
Π^0_2 sets are	of the form $\bigcap_{n \in W_e} [W_n]$
...	...

Effectivize Σ^0_α for α not 'too big' is also doable.

Randomness and genericity



Section 2

Randomness and genericity

Algorithmic randomness

The general idea of randomness :

1. Fix a countable class of sets of measure 1
2. Say that something is random if it belongs to all of them

1-random	every Σ^0_2 sets 'effectively of measure 1'
weakly-2-random	every Σ^0_2 sets of measure 1
2-random	every Σ^0_3 sets 'effectively of measure 1'
...	...

Algorithmic randomness

Effectively of measure 1 sets

A union of set $\bigcup \mathcal{A}_n$ is said to be effectively of measure 1 if there is a computable function $f : \omega \mapsto [0, 1]$ going to 1, such that $\lambda(\mathcal{A}_n) > f(n)$.

We have :

1-random $\leftarrow$ w2-random $\leftarrow$ 2-random $\leftarrow$ w3-random $\leftarrow \ldots$

All implications are strict

Genericity

The general idea of genericity :

1. Fix a countable class of dense open sets of 2^ω
2. Say that something is generic if it belongs to all of them

weakly-1-generic	every dense Σ^0_1 set
1-generic	every set $\mathcal{U} \cup (\overbrace{2^\omega - \mathcal{U}}^{\circ})$ for $\mathcal{U}$ a Σ^0_1 set
weakly-2-generic	every dense $\Sigma^0_1(\emptyset')$ set

weakly-1-generic $\leftarrow$ 1-generic $\leftarrow$ weakly-2-generic $\leftarrow$ 2-generic $\leftarrow$ ...

All implications are strict

Another type of genericity : Π^0_1-Genericity

We consider genericity for the **topology generated by Π^0_1 sets**

Dense open sets

An open set is a countable union of Π^0_1 set. It is **dense** if it intersects any non empty Π^0_1 sets.

weakly-Π^0_1-generic	every dense Σ^0_2 set
Π^0_1-generic	every set of the form $\mathcal{F} \cup (\overbrace{2^\omega - \mathcal{U}}^{\circ})$ for $\mathcal{F}$ a Σ^0_2 set

Mix genericity and randomness : Π^0_1-Solovay-Genericity

We still consider genericity for the **topology generated by** Π^0_1 **sets**

Open sets

An open set is now **'dense'** if it intersects with positive measure any Π^0_1 sets of positive measure.

weakly-Π^0_1-Solovay-generic	every 'dense' Σ^0_2 set
Π^0_1-Solovay-generic	every Σ^0_2 set $\mathcal{F}$ together with all Π^0_1 sets of positive measure, included in $2^\omega - \mathcal{F}$.

Results on Π^0_1-Solovay-Genericity

Definition

A sequence X is **computably dominated** if any functions $f : \omega \to \omega$ that X can compute is dominated by a computable function $g : \omega \to \omega$ (that is we have for all n that $f(n) \leqslant g(n)$).

Theorem

A sequence X is Π^0_1-generic iff it is computably dominated.

Theorem

A sequence X is Π^0_1-Solovay generic iff it is computably dominated and weakly-2-random.

Beyond arithmetic

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Section 3

Beyond arithmetic

Computable ordinals

Computabe ordinals

An ordinal α is computable if there is a c.e. well-order $R \subseteq \omega \times \omega$ so that $|R| = \alpha$.

$\boldsymbol{W}$ and $\boldsymbol{W_\alpha}$

We denote by $\boldsymbol{W}$ the set of codes for computable ordinals, and we denote by $\boldsymbol{W_\alpha}$ the set of codes for computable ordinals strictly smaller than α.

Strict initial segment

The computable ordinals forms a strict initial segment of the countable ordinals.

Analytic and co-analytical sets

Π^1_1 sets

A set of integers/sequence is Π^1_1 if it definable by a formula of arithmetic whose quantifiers over sets are only universal.

Σ^1_1 sets

A set of integers/sequence is Σ^1_1 if it definable by a formula of arithmetic whose quantifiers over sets are only existential.

Δ^1_1 sets

A set of integers/sequence is Δ^1_1 if it both Π^1_1 and Σ^1_1.

Analytic and co-analytical sets

Theorem

A set of integers A is Π^1_1 iff there is a computable function $f : \omega \mapsto \omega$ so that $n \in A$ iff $f(n) \in \boldsymbol{W}$.

A set of sequences $\mathcal{A}$ is Π^1_1 iff there is a code e so that $X \in A$ iff $e \in \boldsymbol{W}^X$.

Theorem

A set of integers A is Δ^1_1 iff there is a computable function $f : \omega \mapsto \omega$ and a computable ordinal α so that $n \in A$ iff $f(n) \in \boldsymbol{W}_\alpha$.

A set of sequences $\mathcal{A}$ is Δ^1_1 iff there is a code e and a computable ordinal α so that $X \in \mathcal{A}$ iff $e \in \boldsymbol{W}^X_\alpha$.

Example of Π^1_1 set of sequence

ω_1^{ck}

We denote by ω_1^{ck} the smallest non-computable ordinal.

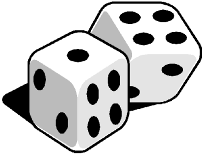
ω_1^X

We denote by ω_1^X the smallest non-computable-in-X ordinal.

The set $\{X \ : \ \omega_1^X > \omega_1^{ck}\}$ is :

- A Π^1_1 set
- of measure 0
- meager

Higher randomness/genericity



Section 4

Higher randomness & genericity

Another hierarchy

Π^1_1 open sets & Σ^1_1 closed sets

An open set is a Π^1_1 open set if it can be described with a Π^1_1 set of strings. The Σ^1_1 closed sets are complements of Π^1_1 open set.

We can have a 'higher Borel hierarchy' starting from Π^1_1 open sets and Σ^1_1 closed sets :

1. A set is $\Pi^0_2(\Pi^1_1)$ is it is a uniform intersection of Π^1_1 open sets.
2. A set is $\Sigma^0_2(\Sigma^1_1)$ is it is a uniform union of Σ^1_1 closed sets.
3. A set is $\Pi^0_3(\Sigma^1_1)$ is it is a uniform intersection of a uniform union of Σ^1_1 closed sets.
4. A set is $\Sigma^0_3(\Pi^1_1)$ is it is a uniform union of a uniform intersection of Π^1_1 open sets.

Higher randomness

Δ^1_1-random

A sequence is weakly-Δ^1_1-random if it belongs to no Δ^1_1 nullset

weakly-Π^1_1-random

A sequence is weakly-Π^1_1-random if it belongs to no $\Pi^0_2(\Pi^1_1)$ nullset

Π^1_1-random

A sequence is Π^1_1-random if it belongs to no Π^1_1 nullset

We have

$$\Delta^1_1\text{-random} \leftarrow \text{weakly-}\Pi^1_1\text{-random} \leftarrow \Pi^1_1\text{-random}.$$

Higher randomness

Is Π^1_1-random different from weakly-Π^1_1-random ?

An approach to solve this question is to show that the set of Π^1_1-randoms and the set of weakly-Π^1_1-randoms have different Borel complexity. We showed that is is not possible, as both sets are $\mathbf{\Sigma^0_3}$.

Higher randomness with genericity

We give an analogue of the notions of Solovay genericity with Π^0_1 sets.

weakly-Σ^1_1-Solovay-generic

A sequence is weakly-Σ^1_1-Solovay-generic if it is in every $\Sigma^0_2(\Sigma^1_1)$ sets intersecting with positive measure every Σ^1_1 closed set.

Σ^1_1-Solovay-generic

A sequence X is Σ^1_1-Solovay-generic if for every $\Sigma^0_2(\Sigma^1_1)$ sets $\mathcal{F}$, either X is in it or X belongs to a Σ^1_1 set of positive measure, included in the complement of $\mathcal{F}$.

Equivalence 1

Fact

The set of weakly-Π^1_1-randoms is obvisouly $\mathbf{\Sigma^0_3}$.

Theorem 1

A sequence is weakly-Σ^1_1-Solovay-generic iff it is weakly-Π^1_1-random.

→ If X is weakly-Σ^1_1-Solovay-generic then it is surely in every $\Sigma^0_2(\Sigma^1_1)$ sets of measure 1.

← If X is not weakly-Σ^1_1-Solovay-generic then it is in a $\Pi^0_2(\Pi^1_1)$ containing no Σ^1_1 closed set of positive measure. But such a $\Pi^0_2(\Pi^1_1)$ is necessarily of measure 0 (Not trivial).

Equivalence 2

Theorem 2

A sequence is Σ^1_1-Solovay-generic iff it is Π^1_1-random

Theorem Chong, Yu, Nies

A sequence is Π^1_1-random iff it is Δ^1_1-random and $\omega_1^X = \omega_1^{ck}$.

Proof Idea of theorem 2 :

- $\leftarrow$ If X is not Σ^1_1-Solovay-generic then it is in a $\Pi^0_2(\Pi^1_1)$ set $\mathcal{G}$ so that for no Σ^1_1 closed set of positive measure $\mathcal{F}$ we have $X \in \mathcal{F} \subseteq \mathcal{G}$. If $\omega_1^X = \omega_1^{ck}$ we can deduce from this that X is in a Δ^1_1 set of measure 0 (not trivial).

- $\rightarrow$ If X is Σ^1_1-Solovay-generic then surely X is Δ^1_1-random. All that need to be proved is that $\omega_1^X = \omega_1^{ck}$. This is the tricky part.

Consequences

Corollary

The set of Π^1_1-randoms has the same Borel complexity than the weakly-Π^1_1-randomns : $\mathbf{\Sigma^0_3}$. Using a result of Liang Yu, this complexity is strict (it is not $\mathbf{\Pi^0_3}$).

However, using a another technique, we recently proved that :

Last results

There is some X which is weakly-Π^1_1-random but not Π^1_1-random.

Thank you