A COMPUTABLE ANALYSIS OF VARIABLE WORDS THEOREMS

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Abstract. The Carlson-Simpson lemma is a combinatorial statement occurring in the proof of the Dual Ramsey theorem. Formulated in terms of variable words, it informally asserts that given any finite coloring of the strings, there is an infinite sequence with infinitely many variables such that for every valuation, some specific set of initial segments is homogeneous. Friedman, Simpson, and Montalban asked about its reverse mathematical strength. We study the computability-theoretic properties and the reverse mathematics of this statement, and relate it to the finite union theorem. In particular, we prove the Ordered Variable word for binary strings in ACA_0.

1. Introduction

Let $(\mathbb{N})^k$ and $(\mathbb{N})^\infty$ denote the set of partitions of $\mathbb{N}$ into exactly k and infinitely many non-empty pieces, respectively. For $X \in (\mathbb{N})^\infty$, $(X)^k$ is the set of all $Y \in (\mathbb{N})^k$ which are coarser than X.

Statement 1.1 (Dual Ramsey theorem)**.** DRT^k is the statement "If $(\mathbb{N})^k$ is colored with finitely many Borel colors, then there is some $X \in (\mathbb{N})^\infty$ such that $(X)^k$ is monochromatic".

The Dual Ramsey theorem was proven by Carlson and Simpson [1], and studied from a reverse mathematical viewpoint by Slaman [10], Miller and Solomon [6] and Dzhafarov et al. [3]. In this paper, we shall focus on a combinatorial lemma used by Carlson and Simpson to prove the Dual Ramsey theorem. This lemma can be formulated in terms of *variable words*.

Definition 1.2 (Variable word)**.** An *infinite variable word* on a finite alphabet A is an ω-sequence W of elements of $A \cup \{x_i : i \in \mathbb{N}\}$ in which all variables occur at least once, and finitely often. Moreover, the first occurrence of x_i comes before the first occurrence of x_{i+1}. A *finite variable word* is an initial segment of an infinite variable word. A finite or infinite variable word is *ordered* if moreover all occurences of x_i come before any occurrence of x_{i+1}. Given $\bar{a} = a_0 a_1 \dots a_{k-1} \in A^{<\omega}$, we let $W(\bar{a})$ denote the finite A-string obtained by replacing x_i with a_i in W and then truncating the result just before the first occcurence of x_k.

Statement 1.3 (Variable word theorem)**.** $\mathsf{VW(n,r)}$ is the statement "If $A^{<\omega}$ is colored with r colors for some alphabet A of cardinality n, there exists an infinite variable word W such that $\{W(\bar{a}) : \bar{a} \in A^{<\omega}\}$ is monochromatic. $\mathsf{OVW(n,r)}$ is the same statement as $\mathsf{VW(n,r)}$ but for ordered variable words.

In this paper, we study the computability-theoretic properties of the variable word theorems using the framework of reverse mathematics.[1]

1.1. **Reverse mathematics.** Reverse mathematics is a vast foundational program aiming to determine the optimal axioms to prove ordinary theorems. It uses the framework of second-order arithmetic, with a base theory RCA_0 consisting of the axioms of Robinson arithmetic, the Σ^0_1 induction scheme and the Δ^0_1 comprehension scheme. The system RCA_0 arguably captures *computable mathematics*. Starting from a proof-theoretic perspective, modern reverse mathematics tends to be seen as a framework to analyse the computability-theoretic features of theorems. Among the distinguished statements, let us mention weak König's lemma (WKL), asserting that every infinite binary tree has an infinite path, the arithmetic comprehension axiom (ACA), and the Π^1_1 comprehension axiom ($\Pi^1_1\mathsf{CA}$), consisting of the comprehension scheme restricted to arithmetic and Π^1_1 formulas, respectively. See Simpson [9] for reference book on classical reverse mathematics.

Liu Lu is supported by NSF Grant of Hunan Province (2018JJ3623). Authors would like to thank the support of Institute for Mathematical Science of National University of Singapore.

[1]The authors thank Damir Dzhafarov, Stephen Flood, Reed Solomon and Linda Brown Westrick for bringing the attention of the authors to the Carlson-Simpson lemma, and for numerous discussions. The authors are also thankful to Denis Hirschfeldt and Barbara Csima for showing them how to use Lovasz Local Lemma to prove lower bounds to combinatorial theorems.

The statements studied within this framework are mainly of the form $(\forall X)[\Phi(X) \to (\exists Y)\Psi(X,Y)]$, where Φ and Ψ are arithmetic formulas with set parameters, and can be considered as *problems*. Given a statement P of this form, a set X such that $\Phi(X)$ holds is an *instance* of P, and a set Y such that $\Psi(X,Y)$ holds is a *solution* to the P-instance X. In this paper, we shall consider exclusively statements of this kind.

Friedman and Simpson [4], and later Montalban [7], asked about the reverse mathematical strength of the ordered variable word. The statement $\mathsf{OVW}(\mathsf{k},\ell)$ is known to be provable in $\mathsf{RCA}_0 + \Pi^1_1\mathsf{CA}$. Our main result is a direct combinatorial proof of $\mathsf{OVW}(2,\ell)$ in $\mathsf{RCA}_0 + \mathsf{ACA}$.

Theorem 1.4. *For every $\ell \geq 2$, $\mathsf{RCA}_0 + \mathsf{ACA} \vdash \mathsf{OVW}(2,\ell)$.*

On the lower bound hand, Miller and Solomon [6] constructed a computable instance c of $\mathsf{OVW}(2,2)$ with no Δ^0_2 solution, and deduced that $\mathsf{RCA}_0 + \mathsf{WKL}$ does not prove $\mathsf{VW}(2,2)$. Indeed, seeing the instance c of $\mathsf{OVW}(2,2)$ as an instance of $\mathsf{VW}(2,2)$, and noticing that the jump of a solution to $\mathsf{VW}(2,2)$ gives a solution to $\mathsf{OVW}(2,2)$, one can deduce that c has no low $\mathsf{VW}(2,2)$-solution. In this paper, we improve their lower bound by constructing a computable instance of $\mathsf{OVW}(2,2)$ whose solutions are of DNC degree relative to $\emptyset'$.

1.2. **Organization of the paper.** In section 2, we shall give a simple proof of the ordered variable word for binary strings ($\mathsf{OVW}(2,\ell)$) using the finite union theorem. Then, in section 3, we provide a direct combinatorial proof of the same statement over $\mathsf{RCA}_0 + \mathsf{ACA}$. Finally, in section 4, we give a new lower bound on the strength of $\mathsf{OVW}(2,\ell)$ using a computable version of Lovasz Local Lemma.

1.3. **Notation.** Given two sets A and B, we write $A < B$ for the formula $(\forall x \in A)(\forall y \in B) x < y$. Given a set A, we write $A^{<\omega}$ for the set of finite A-valued strings. In particular, $2^{<\omega}$ is the set of binary strings. We denote by $\mathcal{P}_{fin}(\mathbb{N})$ the collection of finite *non-empty* subsets of $\mathbb{N}$. Given two strings $\sigma, \tau \in A^{<\omega}$, $\sigma * \tau$ denotes their concatenation. We may also write $\sigma\tau$ when there is no ambiguity. Given a string or a sequence X and some $n \in \omega$, we write $X{\restriction}n$ for the initial segment of X of length n. In particular, $X{\restriction}0$ is the empty string, written ε.

2. A simple proof of the Ordered Variable Word theorem from the Finite Union Theorem

Simpson first noted a relation between Hindman's theorem and the Carlson-Simpson lemma [1]. In this section, we give a formal counterpart to his observation by giving a simple proof of $\mathsf{OVW}(2,\ell)$ using the Finite Union Theorem, a statement known to be equivalent to Hindman's theorem. A variation of the proof below was used by Dzhafarov et al. [3] to give an upper bound to the Open Dual Ramsey's theorem. A direct combinatorial proof of $\mathsf{OVW}(2,\ell)$ in $\mathsf{RCA}_0 + \mathsf{ACA}$ will be given in the next section.

Definition 2.1. An *IP collection* is an infinite collection of finite sets $\mathcal{I} \subseteq \mathcal{P}_{fin}(\mathbb{N})$ which is closed under *non-empty* finite unions and contains an infinite subcollection of pairwise disjoint sets.

Note that any IP collection $\mathcal{I}$ necessarily contains an infinite $\mathcal{I}$-computable sequence $S_0 < S_1 < \dots$.

Statement 2.2 (Finite union theorem)**.** For every $\ell \in \mathbb{N}$, FUT_ℓ is the statement "For every coloring $c : \mathcal{P}_{fin}(\mathbb{N}) \to \ell$, there is a monochromatic IP collection". wFUT^2_ℓ is the statement "For every coloring $c : \mathcal{P}_{fin}(\mathbb{N}) \times \mathbb{N} \to \ell$, there is an IP collection $\mathcal{I}$ and a color $i < \ell$ such that $c(S, \min T) = i$ for every $S < T \in \mathcal{I}$."

Theorem 2.3. $\mathsf{RCA}_0 \vdash \forall \ell(\mathsf{FUT}_\ell \to \mathsf{wFUT}^2_\ell)$.

Proof. Assume $\ell \geq 2$, the other case being trivial. Let $f : \mathcal{P}_{fin}(\mathbb{N}) \times \mathbb{N} \to \ell$ be an instance of wFUT^2_ℓ. Note that over RCA_0, $\mathsf{FUT}_\ell \to \mathsf{ACA}$ and $\mathsf{ACA} \to \mathsf{COH}$. Let $\vec{R}$ be a sequence of set defined for every $S \in \mathcal{P}_{fin}(\mathbb{N})$ and $i < \ell$ by $R_{S,i} = \{n \in \mathbb{N} : f(S,n) = i\}$. Apply COH to $\vec{R}$ to obtain an infinite $\vec{R}$-cohesive set C. In particular, for every $S \in \mathcal{P}_{fin}(\mathbb{N})$, $\lim_{n \in C} f(S,n)$ exists.

Let $h : \omega \to C$ be a C-computable bijection. Let $\tilde{f} : \mathcal{P}_{fin}(\mathbb{N}) \to \ell$ be defined by $\tilde{f}(S) = \lim_{n \in C} f(h[S], n)$. $\tilde{f}$ is a $\Delta^{0,f\oplus C}_2$ instance of FUT_ℓ, so by the finite union theorem, there is an IP collection $\mathcal{I} \subseteq \mathcal{P}_{fin}(\mathbb{N})$. and a color $i < \ell$ such that for every $S \in \mathcal{I}$, $\tilde{f}(S) = \lim_{n \in C} f(h[S], n) = i$. Note that for every $S \in \mathcal{I}$, $\min h[S] \in C$. Therefore, by f-computably thinning-out the set $\mathcal{I}$, we obtain an IP collection $\mathcal{J} \subseteq \mathcal{I}$ such that for every $S < T \in \mathcal{J}$, $f(h[S], \min h[T]) = i$. The set $\{h[S] : S \in \mathcal{J}\}$ is a solution to f. $\square$

Theorem 2.4. $\mathsf{RCA}_0 \vdash \forall \ell(\mathsf{wFUT}^2_\ell \to \mathsf{OVW}(2,\ell))$.

Proof. Let $f : 2^{<\omega} \to \ell$ be an instance of $\mathsf{OVW}(2,\ell)$. Define an instance $g : \mathcal{P}_{fin}(\mathbb{N}) \times \mathbb{N} \to \ell$ of wFUT^2_ℓ as follows: Given some $S \in \mathcal{P}_{fin}(\mathbb{N})$ and $n \in \mathbb{N}$, if $\max S < n$, then set $g(S,n) = f(\sigma)$, where σ is the binary string of length n defined by $\sigma(i) = 1$ iff $i \in S$. If $n \leq \max S$, set $g(S,n) = 0$. By wFUT^2_ℓ, there is an IP collection $\mathcal{I}$ and a color $i < \ell$ such that $g(S, \min T) = i$ for every $S < T \in \mathcal{I}$. Compute from $\mathcal{I}$ an infinite increasing sequence of pairwise disjoint finite sets $F_0 < F_1 < \dots$ Let W be the infinite variable word defined by

$$W(n) = \begin{cases} 1 & \text{if } n \in F_0 \\ x_i & \text{if } n \in F_i \text{ for some } i \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

The variable word W and the sequence of the F's is a solution to the instance f of $\mathsf{OVW}(2,\ell)$. □

Corollary 2.5. $\mathsf{RCA}_0 \vdash \mathsf{ACA}^+ \to \forall \ell \mathsf{OVW}(2,\ell)$.

Proof. Immediate since $\mathsf{ACA}^+ \to \forall \ell\, \mathsf{FUT}_\ell \to \forall \ell\, \mathsf{wFUT}^2_\ell \to \forall \ell \mathsf{OVW}(2,\ell)$ over RCA_0. □

3. A proof of the Ordered Variable Word theorem in ACA

The proof of the previous section gave a very coarse computability-theoretic upper bound of the Ordered Variable Word theorem in terms of ω-jumps. In this section, we give a direct combinatorial proof of $\mathsf{OVW}(2,\ell)$ in $\mathsf{RCA}_0 + \mathsf{ACA}$. Actually, every PA degree relative to $\emptyset'$ is sufficient to compute a solution of a computable instance of $\mathsf{OVW}(2,\ell)$. We thereby answer a question of Miller and Solomon [6].

Theorem 3.1. *For every $\ell \in \omega$, every computable instance c of $\mathsf{OVW}(2,\ell)$, every PA degree over $\emptyset'$ computes a solution to c.*

A formalization of Theorem 3.1 yields a proof of Theorem 1.4.

Proof of Theorem 1.4. The proof of Theorem 3.1 can be formalized within $\mathsf{RCA}_0 + \mathsf{ACA}$. Indeed, the arguments require only arithmetical induction to be carried out, and every model of $\mathsf{RCA}_0 + \mathsf{ACA}$ is a model of the statement "For every set X, there is a set of PA degree over the jump of X." □

Let us first introduce some notation. For a finite set F and a string $\sigma \in 2^{<\omega}$ let σ_F be the binary string of length $|\sigma|$ defined by $\sigma_F(i) = \sigma(i)$ if $i \notin F$, and $\sigma_F(i) = 1 - \sigma(i)$ otherwise. Let $\leq_{lex}$ denote the shortlex order on $\omega^{<\omega}$, that is, the order with the shortest length first, and with the strings of same length sorted lexicographically.

In what follows, fix a coloring $\tilde{c} : 2^{<\omega} \to \ell$, and a string $\rho \in 2^{<\omega}$.

The main combinatorial lemma we use is Lemma 3.4. As a warm up, we first prove the following lemma 3.2, which is a consequence of Lemma 3.4 and the proof is somehow similar but much simpler. In the following lemma, one may think of $\rho_{P'}$ as a finite variable word, where the positions at $\tilde{P}$ are replaced by a same variable kind.

Lemma 3.2. *For any $P \subseteq \{0, \cdots, |\rho|-1\}$ with $(\forall n \in P)[\rho(n) = 0] \wedge |P| \geq \ell$, there exist two subsets $P' < \tilde{P}$ of P with $\tilde{P} \neq \emptyset$ such that $\tilde{c}(\alpha) = \tilde{c}(\alpha_{\tilde{P}})$ where $\alpha = \rho_{P'}$.*

Proof. Suppose $P = \{p_0 < \cdots < p_{m-1}\}$. Let $\ell_0, \dots, \ell_m$ be defined by $\ell_i = \tilde{c}(\rho_{\{p_0,\dots,p_{i-1}\}})$. In particular, $\ell_0 = \tilde{c}(\rho)$. Since $|P| = m \geq \ell$, so among $\ell_0, \cdots, \ell_m$, there must exists $i < j$ such that $\ell_i = \ell_j$. Let $P' = \{p_0, \cdots, p_{i-1}\}$ (if $i = 0$ then $P' = \emptyset$), and $\tilde{P} = \{p_i, \cdots, p_{j-1}\}$, let $\alpha = \rho_{P'}$. Clearly $P' < \tilde{P}$ and $\tilde{P} \neq \emptyset$. It is also easy to see that $\tilde{c}(\alpha) = \ell_i = \ell_j = \tilde{c}(\alpha_{\tilde{P}})$. □

We now prove a technical lemma used in the proof of our main combinatorial lemma (Lemma 3.4). The sequence in the following lemma is obtained by a simple greedy algorithm, with finitely many resets.

Lemma 3.3. *There exists a nonempty set of colors $L \subseteq \{0, 1, \cdots, \ell-1\}$, $|L|+1$ many sets of binary strings $\Gamma_0 = \{\tau^\eta\}_{\eta \in L}, \Gamma_1 = \{\tau^\eta\}_{\eta \in L^2}, \cdots, \Gamma_{|L|} = \{\tau^\eta\}_{\eta \in L^{|L|+1}}$, such that, letting*

$$\tilde{\eta} = \underbrace{\max L * \max L * \cdots * \max L}_{|L|+1 \text{ many}}$$

and letting $\tilde{\rho} = \tau^{\tilde{\eta}} * 0$, the following holds:

(1) $\rho \prec \Gamma_0$ *and* $\tau^\eta \prec \tau^\beta \Leftrightarrow \eta <_{lex} \beta$*;*
(2) $\tilde{\rho}(|\tau|) = 0$ *for all* $\tau \in \Gamma_i, i \leq |L|$*;*
(3) for all $i \leq |L|$*,* $\eta \in L^{i+1}$*, let* $\eta_0 \prec \eta_1 \prec \cdots \prec \eta_{i-1}$ *denote all nonempty predecessors of* η*, let* $Q = \{|\tau^{\eta_0}|, |\tau^{\eta_1}|, \cdots, |\tau^{\eta_{i-1}}|\}$ *(if* $i = 0$ *then* $Q = \emptyset$*), then* $\tilde{c}(\tau^\eta_Q) = \eta(i)$*;*
(4) let $P = \{|\tau^\eta|\}_{\eta \in L^{\leq |L|+1}}$*, for all subset* Q *of* P*, all* $\tau \succeq \tilde{\rho}$*,* $\tilde{c}(\tau_Q) \in L$*.*

Moreover, $\Gamma_i, i \leq |L|$ *is computable in the jump of* $\tilde{c}$*, uniformly in* ρ*.*

Proof. We firstly show how to find Γ_0. Start with $L = \{0, 1, \cdots, \ell - 1\}$. At step 1, try to find a string $\tau \in 2^{<\omega}$ such that $\tilde{c}(\rho\tau) = 0$ and let $\tau^0 = \rho\tau$. Then try to find a τ such that $\tilde{c}(\tau^0 0\tau) = 1$ and let $\tau^1 = \tau^0 0\tau$. Generally, after τ^j is found, try to find τ such that $\tilde{c}(\tau^j 0\tau) = j + 1$ and let $\tau^{j+1} = \tau^j 0\tau$ if τ is found. If during the above process, after τ^{j-1} is defined ($\tau^{-1} = \rho$), there is no τ such that $\tilde{c}(\tau^{j-1}0\tau) = j$, then we start all over again with ρ replaced by $\rho_1 = \tau^j 0$ and with L replaced by $L \setminus \{j\}$.

Generally, given a set of colors L and after τ^β is found, let η be the immediate successor (with respect to $\leq_{lex}$ order restricted to L-strings) of β, let $\eta_0 \prec \eta_1 \prec \cdots \prec \eta_{i-1}$ denote all nonempty predecessors of η, let $Q = \{|\tau^{\eta_0}|, |\tau^{\eta_1}|, \cdots, |\tau^{\eta_{i-1}}|\}$ (if $i = 0$ then $Q = \emptyset$), we try to find τ such that $\tilde{c}((\tau^\beta 0\tau)_Q) = \eta(|\eta| - 1)$. If such a string τ does not exists then we start all over again with ρ replaced by $\tau^\beta 0_Q$ and L replaced by $L \setminus \{\eta(|\eta| - 1)\}$. If such τ exists then let $\tau^\eta = \tau^\beta 0\tau$.

Note that we have to start over for at most $\ell - 1$ times before we ultimately succeed since there are ℓ colors in total. It is plain to check all the four items. Also note that the sequence $\Gamma_0, \cdots, \Gamma_{|L|}$ is $\tilde{c}'$-computable since we only need to use the jump of $\tilde{c}$ to know whether the next τ^η can be found.

□

Lemma 3.4. *There exists a string* $\tilde{\rho} \succ \rho$ *and a finite set* $P \subseteq \{|\rho|, \cdots, |\tilde{\rho}| - 1\}$ *with* $(\forall i \in P)[\tilde{\rho}(i) = 0]$ *such that for all* $\sigma \succeq \tilde{\rho}$ *there exists two subsets* $P' < \tilde{P}$ *of* P *with* $\tilde{P} \neq \emptyset$ *such that, letting* $\alpha = \sigma_{P'}$*,* $\tilde{c}(\alpha) = \tilde{c}(\alpha_{\tilde{P}}) = \tilde{c}(\alpha{\upharpoonright}\min \tilde{P})$*. Moreover,* $|P| < \ell^{\ell+2}$*, and* $\tilde{\rho}, P$*, are computable in the jump of* $\tilde{c}$*, uniformly in* ρ*.*

Proof. Let L and $\tilde{\rho}$ satisfy Lemma 3.3. We claim that $\tilde{\rho}$ and $P = \{|\tau^\eta|\}_{\eta \in L^{\leq |L|+1}}$ satisfy the current lemma. It is clear by item 1 of Lemma 3.3 that $\tilde{\rho} \succ \rho$ and by item 2 of Lemma 3.3 that $(\forall i \in P)[\tilde{\rho}(i) = 0]$.

Fix an arbitrary $\sigma \succeq \tilde{\rho}$. We now describe how to construct P' and $\tilde{P}$. Define $\ell_0, \ldots, \ell_{|L|}$ and $p_0, \ldots, p_{|L|}$ inductively by $\ell_0 = \tilde{c}(\sigma)$, $\ell_{i+1} = \tilde{c}(\sigma_{\{p_0, p_1, \cdots, p_i\}})$, and $p_i = |\tau^{\ell_0 \cdots \ell_i}|$ (where $\tau^{\ell_0 \cdots \ell_i} \in \Gamma_i$). Since $\ell_0, \cdots, \ell_{|L|} \in L$ (by item 4 of Lemma 3.3), there is some $i < j \leq |L|$ such that $\ell_i = \ell_j$. Let $P' = \{p_0, \cdots, p_{i-1}\}$ (if $i = 0$ then $P' = \emptyset$), $\tilde{P} = \{p_i, \cdots, p_{j-1}\}$, and let $\alpha = \sigma_{P'}$. We claim that $\tilde{c}(\alpha) = \tilde{c}(\alpha_{\tilde{P}}) = \tilde{c}(\alpha{\upharpoonright}\min \tilde{P})$. Note that $\min \tilde{P} = p_i = |\tau^{\ell_0 \cdots \ell_i}|$. Therefore $\alpha{\upharpoonright}\min \tilde{P} = \tau^{\ell_0 \cdots \ell_i}_{P'}$. By item 3 of Lemma 3.3, we have $\tilde{c}(\tau^{\ell_0 \cdots \ell_i}_{P'}) = \ell_i$. Meanwhile, by definition of ℓ_i, $\tilde{c}(\sigma_{P'}) = \tilde{c}(\alpha) = \ell_i$. By definition of ℓ_j, $\tilde{c}(\sigma_{P' \cup \tilde{P}}) = \tilde{c}(\alpha_{\tilde{P}}) = \ell_j$. Thus, $\tilde{c}(\alpha) = \tilde{c}(\alpha_{\tilde{P}}) = \tilde{c}(\alpha{\upharpoonright}\min \tilde{P})$.

□

We say that $(\tilde{\rho}, P)$ is *$\tilde{c}$-valid* if P and $\tilde{\rho}$ satisfy Lemma 3.4. We say that $(P', \tilde{P})$ *witnesses $\tilde{c}$-validity of $(\tilde{\rho}, P)$ for $\sigma \succeq \tilde{\rho}$* if $P' < \tilde{P} \subseteq P$, and letting $\alpha = \sigma_{P'}$, $\tilde{c}(\alpha) = \tilde{c}(\alpha_{\tilde{P}}) = \tilde{c}(\alpha{\upharpoonright}\min \tilde{P})$. Before proving Theorem 3.1, we start with the following simpler version.

Theorem 3.5. *For every* $\ell \in \omega$*, every computable instance* $c : 2^{<\omega} \to \ell$ *of* $\mathsf{OVW}(2, \ell)$*, every PA degree over* $\emptyset''$ *computes a solution to* c*.*

Proof. It suffices to compute, given a PA degree relative to $\emptyset''$, an infinite binary sequence $Y \in 2^\omega$ together with a sequence of finite sets $\tilde{P}_0 < \tilde{P}_1 < \cdots$ with $(\forall i \in \omega)(\forall n \in \tilde{P}_i)[Y(n) = 0]$ such that the following holds:

Let $Position = \{\min \tilde{P}_i : i \geq 1\}$. There is some $\tilde{\ell} < \ell$ such that for all subset J of ω, letting $\tilde{P}_J = \bigcup_{i \in J} \tilde{P}_i$, then we have, $(\forall p \in Position)[c(Y_{\tilde{P}_J}{\upharpoonright}p) = \tilde{\ell}\ ]$.

Using Lemma 3.4, we first construct a $\emptyset'$-computable sequence of strings $\tilde{\rho}_0 \prec \tilde{\rho}_1 \prec \cdots$, a sequence of finite sets $P_i \subseteq \{|\tilde{\rho}_{i-1}|, \cdots, |\tilde{\rho}_i| - 1\}$ and a sequence of colorings $c_i : [\tilde{\rho}_i]^{\preceq} \to L_i$ inductively as follows. $\tilde{\rho}_0 = \varepsilon$ and $c_0 = c$. Given $\tilde{\rho}_i$ and $c_i : [\tilde{\rho}_i]^{\preceq} \to L_i$, let $\tilde{\rho}_{i+1} \succeq \tilde{\rho}_i$ and $P_i \subseteq \{|\tilde{\rho}_i|, \cdots, |\tilde{\rho}_{i+1}| - 1\}$ be such that $(\tilde{\rho}_{i+1}, P_i)$-is c_i-valid, and let c_{i+1} be the coloring of $[\tilde{\rho}_{i+1}]^{\preceq}$ which on $\sigma \succeq \tilde{\rho}_{i+1}$ associates $\langle P', \tilde{P}, j \rangle$ such that $(P', \tilde{P})$ witnesses c_i-validity of $(\tilde{\rho}_{i+1}, P_i)$ for σ, and $c_i(\sigma_{P'}) = j$. If there are multiple such tuples, take the least one, in some arbitrary order. Note that the range of c_i is some finite set L_i.

We now analyze for $\sigma \succeq \tilde{\rho}_i$ what $c_i(\sigma) = \langle P', \tilde{P}, j\rangle$ means. Note that elements of $L_i, i \in \omega$ admit a natural partial order $\lhd$ as follows: for $\langle P_0', \tilde{P}_0, j_0\rangle \in L_i, \langle P_1', \tilde{P}_1, j_1\rangle \in L_{i+1}$, $\langle P_1', \tilde{P}_1, j_1\rangle$ is an immediate successor of $\langle P_0', \tilde{P}_0, j_0\rangle$ if and only if $j_1 = \langle P_0', \tilde{P}_0, j_0\rangle$. Clearly every $j \in L_i$ admit a unique immediate predecessor.

Claim 3.6. *Fix some $n \geq 1$, and let $\tilde{\ell} \lhd \langle P_0', \tilde{P}_0, j_0\rangle \lhd \cdots \lhd \langle P_{n-1}', \tilde{P}_{n-1}, j_{n-1}\rangle = c_n(\sigma)$, Let $P' = \bigcup_{i\leq n-1} P_i'$ and $\alpha = \sigma_{P'}$. Then for any subset J of $\{0, \cdots, n-1\}$,*

$$(\forall p \in \big\{ \min \tilde{P}_j : 1 \leq j \leq n-1\big\} \cup \{|\alpha|\})\big[\ c(\alpha_{\tilde{P}_J} {\restriction} p) = \tilde{\ell}\ \big].$$

Proof. First we prove the claim for $n = 1$. By definition of $c_1(\sigma) = \langle P_0', \tilde{P}_0, j_0\rangle$, letting $\beta = \sigma_{P_0'}$, $c_0(\beta) = c_0(\beta_{\tilde{P}_0}) = j_0 = \tilde{\ell}$. In other words, for any subset $J \subseteq \{0\}$,

$$(\forall p \in \big\{ \min\{\tilde{P}_j : 1 \leq j \leq 0\}\big\} \cup \{|\beta|\})\big[\ c(\beta_{\tilde{P}_J} {\restriction} p) = \tilde{\ell}\ \big].$$

So the claim holds for $n = 1$. Suppose now the claim holds for $n - 1$.

Suppose $c_n(\sigma) = \langle P_{n-1}', \tilde{P}_{n-1}, j_{n-1}\rangle$. Let $\beta = \sigma_{P_{n-1}'}$. We have $c_{n-1}(\beta) = c_{n-1}(\beta_{\tilde{P}_{n-1}}) = c_{n-1}(\beta {\restriction} \min \tilde{P}_{n-1}) = j_{n-1} = \langle P_{n-2}', \tilde{P}_{n-2}, j_{n-2}\rangle$. As $c_{n-1}(\beta) = \langle P_{n-2}', \tilde{P}_{n-2}, j_{n-2}\rangle$ and as $\tilde{\ell} \lhd \langle P_{n-2}', \tilde{P}_{n-2}, j_{n-2}\rangle$, by induction hypothesis, for any subset J of $\{0, \cdots, n-2\}$ we have:

$$c(\beta_{(\cup_{i\leq n-2} P_i')\cup \tilde{P}_J}) = \tilde{\ell}. \tag{3.1}$$

Let $\beta' = \beta_{\tilde{P}_{n-1}}$. As $c_{n-1}(\beta') = \langle P_{n-2}', \tilde{P}_{n-2}, j_{n-2}\rangle$ and as $\tilde{\ell} \lhd \langle P_{n-2}', \tilde{P}_{n-2}, j_{n-2}\rangle$, by induction hypothesis, for any subset J of $\{0, \cdots, n-2\}$ we have:

$$c(\beta'_{(\cup_{i\leq n-2} P_i')\cup \tilde{P}_J}) = \tilde{\ell}. \tag{3.2}$$

As $c_{n-1}(\beta {\restriction} \min \tilde{P}_{n-1}) = \langle P_{n-2}', \tilde{P}_{n-2}, j_{n-2}\rangle$ and as $\tilde{\ell} \lhd \langle P_{n-2}', \tilde{P}_{n-2}, j_{n-2}\rangle$, by induction hypothesis, for any subset J of $\{0, \cdots, n-2\}$ we have:

$$(\forall p \in \big\{ \min \tilde{P}_j : 1 \leq j \leq n-2\big\} \cup \big\{|\beta {\restriction} \min \tilde{P}_{n-1}|\big\})\big[\ c(\beta_{(\cup_{i\leq n-2} P_i')\cup \tilde{P}_J} {\restriction} p) = \tilde{\ell}\ \big]. \tag{3.3}$$

But $|\beta {\restriction} \min \tilde{P}_{n-1}| = \min \tilde{P}_{n-1}$. So (3.3) means for any subset J of $\{0, \cdots, n-2\}$ we have:

$$(\forall p \in \big\{ \min \tilde{P}_j : 1 \leq j \leq n-1\big\})\big[\ c(\beta_{(\cup_{i\leq n-2} P_i')\cup \tilde{P}_J} {\restriction} p) = \tilde{\ell}\ \big].$$

Or equivalently, for any subset J of $\{0, \cdots, n-1\}$ we have:

$$(\forall p \in \big\{ \min \tilde{P}_j : 1 \leq j \leq n-1\big\})\big[\ c(\beta_{(\cup_{i\leq n-2} P_i')\cup \tilde{P}_J} {\restriction} p) = \tilde{\ell}\ \big]. \tag{3.4}$$

Now from 3.1, 3.2 and 3.4 we deduce that for any subset J of $\{0, \cdots, n-1\}$ we have:

$$(\forall p \in \big\{ \min \tilde{P}_j : 1 \leq j \leq n-1\big\} \cup \{|\beta|\})\big[\ c(\beta_{(\cup_{i\leq n-2} P_i')\cup \tilde{P}_J} {\restriction} p) = \tilde{\ell}\ \big]$$

which completes the proof of the claim since $\beta_{\cup_{i\leq n-2} P_i'} = \alpha$.

□

Let $\mathcal{T}_0$ be the $\emptyset'$-computable set of all γ such that $(\forall i \leq |\gamma|)[\gamma(i) \in L_i]$, $\gamma(i) \lhd \gamma(i+1)$ and $\gamma(|\gamma|-1) = c_{|\gamma|-1}(\tilde{\rho}_{|\gamma|})$. Then, let $\mathcal{T}$ be the downward closure of the set $\mathcal{T}_0$ by the prefix relation. The tree $\mathcal{T}$ is infinite by construction of the strings $\tilde{\rho}_i$, the colors c_i and the sets P_i : a witness for the c_i-validity of $(\tilde{\rho}_{i+1}, P_{i+1})$ for ρ_{i+1} yields a node of $\mathcal{T}_0$ of length $i+2$. The tree $\mathcal{T}$ is also $\emptyset'$-computably bounded, and $\emptyset''$-computable. Let $j_0 * \langle P_0', \tilde{P}_0, j_0\rangle * \langle P_1', \tilde{P}_1, j_1\rangle * \cdots$ be an infinite path through $\mathcal{T}$ computed by any PA degree over $\emptyset''$. By construction, $\langle P_i', \tilde{P}_i, j_i\rangle \lhd \langle P_{i+1}', \tilde{P}_{i+1}, j_{i+1}\rangle$. Let $X = \bigcup_{i\in\omega} \tilde{\rho}_i$, $P' = \bigcup_{i\in\omega} P_i'$ and let $Y = X_{P'}$. Clearly $(\forall i \forall n \in \tilde{P}_i)[Y(n) = 0]$ and Y is computable in the given PA degree relative to $\emptyset''$. Therefore, letting $Position = \big\{ \min \tilde{P}_i : i \geq 1\big\}$, it suffices to show that for all subsets J of ω,

$$(\forall p \in Position)\big[c(Y_{\tilde{P}_J} {\restriction} p) = j_0\ \big].$$

Without loss of generality, suppose $p = \min \tilde{P}_n$ and $J \subseteq \{0, \cdots, n-1\}$. Since $j_0 * \langle P_0', \tilde{P}_0, j_0\rangle * \langle P_1', \tilde{P}_1, j_1\rangle * \cdots \langle P_n', \tilde{P}_n, j_n\rangle$ is an initial segment of some element in $\mathcal{T}_0$, there must exist some $N > n$ such that $c_N(\tilde{\rho}_{N+1}) = \langle P_{N-1}', \tilde{P}_{N-1}, j_{N-1}\rangle$. Let $\sigma = \tilde{\rho}_{N+1}, \alpha = \sigma_{P'}$. Clearly $\alpha \prec Y \wedge |\alpha| > p$. Moreover, by Claim 3.6, $c(\alpha_{\tilde{P}_J} {\restriction} p) = j_0$. Thus $c(Y_{\tilde{P}_J} {\restriction} p) = j_0$.

□

Finally, we slightly modify the proof of Theorem 3.5 to derive Theorem 3.1.

Proof of Theorem 3.1. The main point is to make the tree $\mathcal{T}$ $\emptyset'$-computable. To ensure this, after we obtain $\tilde{\rho}_i, c_i$, we do not directly go to $\tilde{\rho}_{i+1}$. Instead, we $\emptyset'$-compute $\tilde{\rho}_i^0 \prec \tilde{\rho}_i^1 \prec \cdots \prec \tilde{\rho}_i^{r_i}$ such that $\tilde{\rho}_i^0 \succ \tilde{\rho}_i$ and $c_i\big(\{\tau : \tau \succeq \tilde{\rho}_i^{r_i}\}\big) \subseteq c_i\big(\{\tilde{\rho}_i^0, \cdots, \tilde{\rho}_i^{r_i}\}\big)$. Then we $\emptyset'$-compute $\tilde{\rho}_{i+1} \succ \tilde{\rho}_i^{r_i}$ as in the proof of Theorem 3.5. Note that this indeed can be achieved using $\emptyset'$ since c_i is computable. Define $\mathcal{T}$ to be the set of all γ such that $(\forall i \leq |\gamma|)[\gamma(i) \in L_i]$, $\gamma(i) \lhd \gamma(i+1)$, and either $|\gamma| = 1 \wedge \gamma \in L_0$ or there exists $\tilde{\rho}^u_{|\gamma|-1}$ with $c_{|\gamma|-1}(\tilde{\rho}^u_{|\gamma|-1}) = \gamma(|\gamma|-1)$. It is easy to see that $\mathcal{T}$ is $\emptyset'$-computable since c_i is computable for all i and the sequences $\langle c_i : i \in \omega\rangle$ and $\langle \tilde{\rho}_i^v : i \in \omega, v \leq r_i\rangle$ are $\emptyset'$-computable.

Now we show that $\mathcal{T}$ is a tree. Suppose $\gamma \in \mathcal{T}$, $|\gamma| = n+1$ with $n \geq 1$, and $c_n(\tilde{\rho}_n^u) = \gamma(n) = \langle P', \tilde{P}, j\rangle \in L_n$. We claim that $\gamma{\restriction}n \in \mathcal{T}$. If $n = 1$, then $\gamma{\restriction}1 \in L_0 \subseteq \mathcal{T}$. Otherwise, let $\langle Q', \tilde{Q}, k\rangle \in L_{n-1}$ be the predecessor of $\langle P', \tilde{P}, j\rangle$. We need to show that there exists $\tilde{\rho}^v_{n-1}$ such that $c_{n-1}(\tilde{\rho}^v_{n-1}) = \langle Q', \tilde{Q}, k\rangle$. $c_n(\sigma) = \langle P', \tilde{P}, j\rangle$ implies that, letting $\alpha = \sigma_{P'}$, $c_{n-1}(\alpha) = c_{n-1}(\alpha{\restriction}\min \tilde{P}) = j = \langle Q', \tilde{Q}, k\rangle$. Note that $\alpha \succeq \tilde{\rho}^{r_{n-1}}_{n-1}$ since $P' > |\tilde{\rho}^{r_{n-1}}_{n-1}|$. But $c_{n-1}\big(\{\tau : \tau \succeq \tilde{\rho}^{r_{n-1}}_{n-1}\}\big) \subseteq c_{n-1}\big(\{\tilde{\rho}^0_{n-1}, \cdots, \tilde{\rho}^{r_{n-1}}_{n-1}\}\big)$. Therefore there exists $\tilde{\rho}^u_{n-1}$ such that $c_{n-1}(\tilde{\rho}^u_{n-1}) = \langle Q', \tilde{Q}, k\rangle$. It follows that $\gamma{\restriction}n \in \mathcal{T}$ and that $\mathcal{T}$ is a tree. Any PA degree relative to $\emptyset'$ computes an infinite path through $\mathcal{T}$. The rest of the proof goes exactly the same as Theorem 3.5. □

We now give an alternative proof of Theorem 3.1 based on the definitional complexity of the solutions of c.

Second proof of Theorem 3.1. Let $P_0, P_1, \dots$ be the $\emptyset'$-computable sequence defined in the proof of Theorem 3.5. We have seen that there exists an infinite ordered variable word such that the nth variable kind appears before the position $\max P_n$. Let $\mathcal{T}$ be the tree of all finite ordered variable words which are finite solutions to c and such that the nth variable appears before the position $\max P_n$. By the previous observation, the tree is infinite, $\emptyset'$-computable, and $\emptyset'$-computably bounded. Any PA degree relative to $\emptyset'$ computes an infinite variable word which, by construction of $\mathcal{T}$, is a solution to c. This completes the proof of Theorem 3.1. □

Note that the above proof can be slightly modified to obtain a proof of a sequential version of the ordered variable word.

Statement 3.7. $\mathsf{SeqOVW}(\mathsf{n}, \ell)$ is the statement "If $c_0, c_1, \dots$ is a sequence of ℓ-colorings of a fixed alphabet A of cardinality n, there exists a variable word W such that for every $i \in \omega$ and every $\bar{b} \in A^i$, $\{W(\bar{b}\bar{a}) : \bar{a} \in A^{<\infty}\}$ is monochromatic for c_i."

Theorem 3.8. *For every computable instance $c_0, c_1, \dots$ of $\mathsf{SeqOVW}(2, \ell)$, every PA degree relative to $\emptyset'$ computes a solution to $\bar{c}$.*

Proof. The proof is similar to Theorem 3.1. Using Lemma 3.4, we first construct a $\emptyset'$-computable sequence of strings $\tilde{\rho}_0 \prec \tilde{\rho}_1 \prec \cdots$, a sequence of finite sets $P_i \subseteq \big\{|\tilde{\rho}_{i-1}|, \cdots, |\tilde{\rho}_i| - 1\big\}$ and a sequence of colorings $d_i : [\tilde{\rho}_i]^{\preceq} \to L_i$ inductively as follows. $\tilde{\rho}_0 = \varepsilon$ and $d_0 = c_0$. Given $\tilde{\rho}_i$ and $d_i : [\tilde{\rho}_i]^{\preceq} \to L_i$, let $\tilde{\rho}_{i+1} \succeq \tilde{\rho}_i$ and $P_i \subseteq \big\{|\tilde{\rho}_i|, \cdots, |\tilde{\rho}_{i+1}| - 1\big\}$ be such that $(\tilde{\rho}_{i+1}, P_i)$- is d_i-valid, and let d_{i+1} be the coloring of $[\tilde{\rho}_{i+1}]^{\preceq}$ which on $\sigma \succeq \tilde{\rho}_{i+1}$ associates $\langle P', \tilde{P}, j, k\rangle$ such that $(P', \tilde{P})$ witnesses d_i-validity of $(\tilde{\rho}_{i+1}, P_i)$ for σ, $d_i(\sigma_{P'}) = j$ and $c_{i+1}(\sigma_{P'}) = k$. Note that the main difference with the previous construction is that we handle more and more colorings among $c_0, c_1, \dots$ at each level. The remainder of the proof is the same as in Theorem 3.1. □

The theorem above is optimal, in that we can obtain the following reversal.

Theorem 3.9. *There is a computable instance $c_0, c_1, \dots$ of $\mathsf{SeqOVW}(2, 2)$, such that every solution is of PA degree relative to $\emptyset'$.*

Proof. Let $R_0, R_1, \dots$ be a uniformly computable sequence of sets such that for every e, if $\Phi_e^{\emptyset'}(e) \downarrow = 0$ then R_e is finite, and if $\Phi_e^{\emptyset'}(e) \downarrow = 1$ then R_e is cofinite. In particular, any function $f : \omega \to 2$ such that $f(e)$ gives a side of R_e which is infinite, is DNC_2 relative to $\emptyset'$, hence of PA degree relative to $\emptyset'$. Let $c_i : 2^{<\infty} \to 2$ be defined by $c_i(\sigma) = 1$ iff $|\sigma| \in R_i$, and let W be a solution to $\bar{c}$, that is, a variable word W such that for every $i \in \omega$ and every $\bar{b} \in A^i$, $\{W(\bar{b}\bar{a}) : \bar{a} \in A^{<\infty}\}$ is monochromatic for c_i. We claim that W computes such a function f. Given e, let $f(e) = c_e(W(\bar{b}))$, where $\bar{b} \in 2^e$ is arbitrary (this is well-defined, since $c_e(\bar{b})$ depends only on the length of $\bar{b}$). By definition of W, $\{W(\bar{b}\bar{a}) : \bar{a} \in A^{<\infty}\}$ is monochromatic for c_e, the color of $c_e(W(\bar{b}))$ appears infinitely often in R_e. Therefore, W is of PA degree relative to $\emptyset'$. This completes the proof. □

4. A difficult instance of the Ordered Variable Word theorem

Miller and Solomon [6] constructed a computable instance of $\mathsf{OVW}(2,2)$ with no Δ^0_2 solution. In this section, we strengthen their proof by constructing a computable instance of $\mathsf{OVW}(2,2)$ such that every solution is of DNC degree relative to $\emptyset'$, using a significantly simpler argument.

The proof makes an essential use of a computable version of Lovasz Local Lemma proven by Rumyantsev and Shen [8]. The idea of using Lovasz Local Lemma to analyse the computability-theoretic strength of problems in reverse mathematics comes from Csima and Dzhafarov, Hirschfeldt, Jockusch, Solomon and Westrick [**?**], who proved that a version of Hindman's theorem for subtractions is not computably true.

Definition 4.1. Fix a countable set of variables $x_0, x_1, \dots$ A *(disjunctive) clause* C is a tuple of the form $(x_{n_1} = i_1 \vee \dots \vee x_{n_k} = i_k)$, with $i_1, \dots, i_k < 2$. The *length* of C is the integer k. An *infinite CNF formula* is an infinite conjunction of disjunctive clauses. An infinite CNF formula $\bigwedge_n C_n$ is *computable* if the function which given n outputs a code for C_n is computable, and the set of n such that C_n contains the variable x_j is uniformly computable in j.

Theorem 4.2 (Rumyantsev and Shen [8])**.** *For every $\alpha \in (0,1)$, there exists some $N \in \omega$ such that every computable infinite CNF where each variable appears in at most $2^{\alpha n}$ clauses of size n (for every n) and all clauses have size at least N, has a computable satisfying assignment.*

Theorem 4.3. *There is a computable instance c of $\mathsf{OVW}(2,2)$ and a computable function $h : \omega \to \omega$ such that if $\Phi^{\emptyset'}_e$ outputs a finite variable word in which the first $h(e)$ variable kinds occur, then $\Phi^{\emptyset'}_e$ is not extendible into an infinite solution to c.*

Proof. Fix $\alpha = 0.5$, and let N be the threshold of Theorem 4.2. For every index e and stage s, we interpret $\Phi^{\emptyset'}_e[s]$ as a finite variable word $W_{e,s}$ with exactly $N+e$ variable kinds, and where a new variable occurs right after $W_{e,s}$. Such a variable word induces a binary tree $T_{e,s}$ with 2^{N+e} leaves. Let $L_{e,s}$ be the set of leaves of $T_{e,s}$, that is, the set of all instantiations of the variable word $W_{e,s}$. Moreover, all the leaves of $T_{e,s}$ have the same length $n_{e,s}$.

The idea is the following: since the variable word is ordered and a new variable kind occurs right after $W_{e,s}$, no variable among the first $N+e$ variables can occur after $W_{e,s}$. If W is a solution to c with initial segment $W_e = \lim_s W_{e,s}$ for some color i, then W must be homogeneous for c for every instance of the variables, so in particular when setting all the variables after the $N+e$ first ones to 0. Hence, there must be infinitely many strings τ such that for every $\sigma \in \lim_s L_{e,s}$, $c(\sigma\tau) = i$. By ensuring that for cofinitely many τ, there is some $\sigma \in L_{e,|\tau|}$ such that $c(\sigma\tau) \neq i$, we force W_e not to be a solution to c for color i.

Fix a countable collection of variables $(x_\rho : \rho \in 2^{<\omega})$. Each variable x_ρ corresponds to the color of the string ρ. Given some $s \in \omega, \tau \in 2^{<\omega}$ and some $i < 2$, if $n_{e,s} + |\tau| = s$, then let $C_{e,s,\tau,i}$ be the disjunctive 2^{N+e}-clause

$$\bigvee \{x_{\sigma\tau} = i : \sigma \in L_{e,s}\}.$$

And let C be the conjunction

$$\bigwedge_{n_{e,s}+|\tau|=s} \{C_{e,s,\tau,i} : e \in \omega, \tau \in 2^{<\omega}, i < 2\}.$$

This infinite CNF formula is clearly computable. Clearly $C_{e,s,\tau,i}$ has length 2^{N+e}. Note that for every ρ, e, there exists at most one τ such that $(\exists \sigma \in L_{e,|\rho|})[\sigma\tau = \rho]$. Therefore, each variable x_ρ appears in at most 2 clauses of length 2^{N+e}, namely, $C_{e,|\rho|,\tau,0}$ and $C_{e,|\rho|,\tau,1}$, where τ is such that $(\exists \sigma \in L_{e,|\rho|})[\sigma\tau = \rho]$. Therefore, this formula satisfies the conditions of Theorem 4.2, and has a computable assignment $c : 2^{<\omega} \to 2$. By construction, letting $h(e) = N + e + 1$, the formula ensures that if $\Phi^{\emptyset'}_e$ outputs a finite variable word in which the first $h(e)$ variables kinds occur, then $\Phi^{\emptyset'}_e$ is not extendible into an infinite solution to c. □

Definition 4.4. A function $f : \omega \to \omega$ is *diagonally non-computable relative to X* (or X-dnc) if for every e, $f(e) \neq \Phi^X_e(e)$.

Corollary 4.5. *There is a computable instance c of $\mathsf{OVW}(2,2)$ such that every solution is of $\emptyset'$-dnc degree.*

Proof. Let c and h be as in Theorem 4.3. For every e, let α_e be a computable bijection from the finite variable words in which the first $h(e)$ variable kinds occur, to the set of the integers. By Kleene's fixpoint theorem, there is a computable function $g : \omega \to \omega$ such that for every e, $\Phi^{\emptyset'}_{g(e)} = \alpha^{-1}_{g(e)}(\Phi^{\emptyset'}_e(e))$.

Let W be a solution to c, that is, an infinite variable word. Let f be the W-computable function defined by $f(e) = \alpha_{g(e)}(w_e)$, where w_e is the first initial segment of W in which the first $h(g(e))$ variable kinds occur. We claim that f is $\emptyset'$-dnc. Indeed, given $e \in \omega$, $w_e \neq \Phi^{\emptyset'}_{g(e)}$, so

$$f(e) = \alpha_{g(e)}(w_e) \neq \alpha_{g(e)}(\Phi^{\emptyset'}_{g(e)}) = \Phi^{\emptyset'}_e(e)$$

This completes our proof. □

We conclude this section with a small computational observation about $\mathsf{VW}(2,2)$ based on the syntactical form of the statement.

Definition 4.6. A function $g : \omega \to \omega$ *dominates* $f : \omega \to \omega$ if $(\forall x)f(x) < g(x)$. A function $f : \omega \to \omega$ is *hyperimmune* if it is not dominated by any computable function. A Turing degree is *hyperimmune-free* if it does not contain any hyperimmune function.

Lemma 4.7 (Folklore)**.** *Let* P *be a statement of the form* $(\forall X)[\Phi(X) \to (\exists Y)\Psi(X,Y)]$ *where* Φ *is an arbitrary predicate, and* Ψ *is a* Π^0_2 *predicate. For every computable instance* I *of* P*, if* I *has a solution of hyperimmune-free degree, then every PA degree computes a solution to* I*.*

Proof. Say $\Psi(X,Y) \equiv (\forall x)(\exists y)\Theta(X{\upharpoonright}y, Y{\upharpoonright}y, x, y)$, where Θ is a decidable predicate. Let I be a computable P-instance with a solution S of hyperimmune-free degree. Let $h : \omega \to \omega$ be the S-computable function such that for every x, $\Theta(I, S, x, h(x))$ holds. In particular, there is a computable function $g : \omega \to \omega$ such that $(\forall x)\max(h(x), S(x)) < g(x)$. Let $T \subseteq \omega^{<\omega}$ be the computably bounded tree defined by

$$T = \left\{ \sigma \in \omega^{<\omega} : \begin{array}{l} (\forall x < |\sigma|)\sigma(x) < g(x)) \wedge \\ (\forall x < |\sigma|)[g(x) < |\sigma| \to (\exists y < |\sigma|)\Theta(I{\upharpoonright}y, \sigma{\upharpoonright}y, x, y)] \end{array} \right\}$$

In particular, $S \in [T]$, so the tree is infinite. Moreover, any $R \in [T]$ is a solution to I, and any PA degree computes a member of $[T]$. This completes the proof. □

Corollary 4.8. *There is a computable instance of* $\mathsf{VW}(2,2)$ *such that every solution is of hyperimmune degree.*

Proof. First, note that the statement $\mathsf{VW}(2,2)$ is of the form of Lemma 4.7. Let $c : 2^{<\omega} \to 2$ be the computable instance of $\mathsf{VW}(2,2)$ with no low solution constructed by Miller and Solomon [6] or by Theorem 4.3. Letting $\mathbf{d}$ be a low PA degree, $\mathbf{d}$ computes no solution to c, hence by Lemma 4.7, every solution to c is of hyperimmune degree. □

It is still unknown whether there is a computable instance of $\mathsf{OVW}(2,2)$ such that every solution is PA over $\emptyset'$, or even just computes $\emptyset'$. In particular the following questions remain open:

Question 4.9. Does $\mathsf{VW}(2,2)$ or $\mathsf{OVW}(2,2)$ imply ACA over RCA_0?

Question 4.10. Is there a computable instance of $\mathsf{VW}(2,2)$ or $\mathsf{OVW}(2,2)$ such that the measure of oracles computing a solution to it is null?

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