THE WEAKNESS OF THE PIGEONHOLE PRINCIPLE UNDER HYPERARITHMETICAL REDUCTIONS

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Abstract. The infinite pigeonhole principle for 2-partitions (RT^1_2) asserts the existence, for every set A, of an infinite subset of A or of its complement. In this paper, we study the infinite pigeonhole principle from a computability-theoretic viewpoint. We prove in particular that RT^1_2 admits strong cone avoidance for arithmetical and hyperarithmetical reductions. We also prove the existence, for every Δ^0_n set, of an infinite low$_n$ subset of it or its complement. This answers a question of Wang. For this, we design a new notion of forcing which generalizes the first and second-jump control of Cholak, Jockusch and Slaman.

1. Introduction

In this paper, we study the infinite pigeonhole principle (RT^1_k) from a computability-theoretic viewpoint. The infinite pigeonhole principle asserts that every finite partition of ω admits an infinite part. More formally, RT^1_k is the problem whose instances are colorings $f : \omega \to k$. An RT^1_k-solution to f is an infinite set $H \subseteq \omega$ such that $|f[H]| = 1$. The general question we aim to address is the following:

Question 1.1. Does every instance of RT^1_k admit a "weak" solution?

We consider various notions of weakness, among which the inability to bound a fixed non-zero degree for the Turing, arithmetical and hyperarithmetical reduction. This property is known as *strong cone avoidance*. We also study restrictions of the infinite pigeonhole principle to Δ^0_n instances. Our main theorems are:

Theorem 1.2 (Main theorem 1) Let B be non (hyper)arithmetical. Every set A has an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that B is not (hyper)arithmetical in H.

Theorem 1.3 (Main theorem 2) Fix $n \geq 1$. Every Δ^0_n set A has an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ of low$_n$ degree.

Our motivation comes from *reverse mathematics*. Reverse mathematics is a foundational program which aims to find the weakest axioms needed to prove ordinary theorems. The early reverse mathematics showed the existence of an empirical structural phenomenon, in that most theorems are provably equivalent to one among five main systems of axioms, linearly ordered by the logical implication. See Simpson's book [25] for a reference on reverse mathematics. However, some natural statements escape this structural phenomenon, the most famous one being *Ramsey's theorem for pairs* (RT^2_2). Given a set X, let $[X]^n$ denote the set of unordered n-tuples over X. Ramsey's theorem for n-tuples and k-colors (RT^n_k) asserts the existence, for every coloring $f : [\omega]^n \to k$, of an infinite set $H \subseteq \omega$ such that $|f[\omega]^n| = 1$. In particular, RT^1_k is the infinite pigeonhole principle.

Ramsey's theorem for pairs and two colors received a lot of attention from the computability community as it was historically the first example of statement escaping the structural phenomenon of reverse mathematics. The study of RT^2_k revealed a deep connection between the computability-theoretic features of RT^2_k and the combinatorial features of RT^1_k. More precisely, almost every proof of a statement of the form "Every computable instance of RT^2_k admits a weak solution" can be obtained by a proof of the statement "every (arbitrary) instance of RT^1_k admits a weak solution", with the help of very weak computability-theoretic notion called *cohesiveness*. This is in particular the case for cone avoidance [24, 6], PA avoidance [12], constant-bound trace avoidance [13], preservation of hyperimmunity [20], and preservation of non-c.e. definitions [31, 19], among others. In many cases, the combinatorial features of RT^1_k and the computability-theoretic features of RT^2_k can be proven to be equivalent. See Cholak and Patey [3, Theorem 1.5] for an equivalence in the case of cone avoidance. It therefore seems essential to obtain a good understanding of the infinite pigeonhole principle in order to better understand why Ramsey's theorem for pairs escapes the structural phenomenon of reverse mathematics.

1.1. **Strong cone avoidance**

Given a partial order $\leq_r$ on 2^ω and a set X, we let $\deg_r(X) = \{Y : X \equiv_r Y\}$ be the *degree* of X, where $X \equiv_r Y$ if $X \leq_r Y$ and $Y \leq_r X$. We are in particular interested in the case where $\leq_r$ is among the Turing reduction $\leq_T$, the arithmetical reduction $\leq_{arith}$ and the hyperarithmetical reduction $\leq_{hyp}$. Given a mathematical problem P formulated in terms of instances and solutions, it is natural to ask which sets are P-*encodable*. Here, we say that a set X is P-encodable if there is an instance I of P such that for every P-solution Y to I, $X \leq_r Y$. Some problems are very weak with respect to the order $\leq_r$, and satisfy the following property:

Definition 1.4 (Strong cone avoidance)**.** A problem P *admits strong cone avoidance* for $\leq_r$ if for every pair of sets Z and C such that $C \not\leq_r Z$, every instance X of P admits a solution Y such that $C \not\leq_r Z \oplus Y$.

Dzhafarov and Jockusch [6] proved that RT^1_2 admits strong cone avoidance of the Turing reduction. Their theorem has practical applications, and yield a simpler proof of Seetapun's theorem [24]. We prove a similar result for arithmetical and hyperarithmetical reductions.

Theorem 1.5 (Main theorem 1) RT^1_2 admits strong cone avoidance for arithmetical and hyperarithmetical reductions.

This weakness also holds layer-wise in the arithmetical hierarchy, in the following sense.

Theorem 1.6 Fix $n \geq 1$ and let B be a non-Σ^0_n set. For every set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that B is not $\Sigma^{0,H}_n$.

These theorems show the combinatorial weakness of the pigeonhole principle with respect RT^1_2-encodability. To prove this, we designed a new notion of forcing with an iterated jump control generalizing the first and second jump control of Cholak, Jockusch and Slaman [2].

1.2. **Lowness and hierarchies**

The computability-theoretic study of the pigeonhole principle is also motivated by questions on the strictness of hierarchies in reverse mathematics. Some consequences of Ramsey's theorem form hierarchies of statements, parameterized by the size of the colored tuples. A first example is Ramsey's theorem itself. Indeed, RT^{n+1}_k implies RT^n_k for every $n, k \geq 1$. By the work of Jockusch [9], this hierarchy collapses starting from the triples, and by Seetapun [24], Ramsey's theorem for pairs is strictly weaker than Ramsey's theorem for triples. We therefore have

$$\mathsf{RT}^1_k < \mathsf{RT}^2_k < \mathsf{RT}^3_k = \mathsf{RT}^4_k = \dots$$

Some other hierarchies have been considered in reverse mathematics. Friedman [7] introduced the free set (FS^n) and thin set theorems (TS^n), while Csima and Mileti [4] introduced and studied the rainbow Ramsey theorem (RRT^n_k). These statements are all of the form P^n: "For every coloring $f : [\omega]^n \to \omega$, there is an infinite set $H \subseteq \omega$ such that $f{\restriction}[H]^n$ avoids some set of forbidden patterns". The reverse mathematics of these statements were extensively studied in the literature [1, 4, 11, 16, 17, 19, 21, 28, 29, 30, 31, 32]. In particular, these theorems form hierarchies which are not known to be strictly increasing.

Question 1.7. Are the hierarchies of the free set, thin set, and rainbow Ramsey theorem strictly increasing?

Partial results were however obtained. All these statements admit lower bounds of the form "For every $n \geq 2$, there is a computable instance of P^n with no Σ^0_n solution", where P^n denotes any of RT^n_k (Jockusch [9]), RRT^n_k (Csima and Mileti [4]), FS^n, or TS^n (Cholak, Giusto, Hirst and Jockusch [1]). From the upper bound viewpoint, all these statements follow from Ramsey's theorem. Therefore, by Cholak, Jockusch and Slaman [2], every computable instance of P^1 admits a computable solution, and every computable instance of P^2 admits a low$_2$ solution. These results are sufficient to show that $\mathsf{P}^1 < \mathsf{P}^2 < \mathsf{P}^3$ in reverse mathematics. This upper bound becomes too coarse for triples. Wang [30] proved that every computable instance of RRT^3_k admits a low$_3$ solution. The following question is still open. A positive answer would also answer positively Question 1.7.

Question 1.8. Does every computable instance of FS^n, TS^n, and RRT^n_k admit a low$_n$ solution?

Upper bounds to FS^n, TS^n, and RRT^n_k, are usually proven inductively over n [32, 16, 20], starting with the infinite pigeonhole principle for $n = 1$. In this paper, we therefore prove the following theorem, which introduces the machinery that hopefully will serve to answer positively Question 1.8.

Theorem 1.9 (Main theorem 2) Fix $n \geq 1$. Every Δ^0_n set A has an infinite subset $H \subseteq A$ or $H \subseteq \overline{A}$ of low$_n$ degree.

In particular, we fully answer two questions of Wang [30, Questions 6.1 and 6.2], also asked by the second author [18, Question 5.4]. The cases $n = 2$ and $n = 3$ were proven by Cholak, Jockusch and Slaman [2] and by the authors [15], respectively.

1.3. Definitions and notation

A *binary string* is an ordered tuple of bits $a_0, \dots, a_{n-1} \in \{0, 1\}$. The empty string is written ϵ. A *binary sequence* (or a *real*) is an infinite listing of bits $a_0, a_1, \dots$. Given $s \in \omega$, 2^s is the set of binary strings of length s and $2^{<s}$ is the set of binary strings of length $< s$. As well, $2^{<\omega}$ is the set of binary strings and 2^ω is the set of binary sequences. Given a string $\sigma \in 2^{<\omega}$, we use $|\sigma|$ to denote its length. Given two strings $\sigma, \tau \in 2^{<\omega}$, σ is a *prefix* of τ (written $\sigma \preceq \tau$) if there exists a string $\rho \in 2^{<\omega}$ such that $\sigma^\frown\rho = \tau$. Given a sequence X, we write $\sigma \prec X$ if $\sigma = X \upharpoonright n$ for some $n \in \omega$. A binary string σ can be interpreted as a finite set $F_\sigma = \{x < |\sigma| : \sigma(x) = 1\}$. We write $\sigma \subseteq \tau$ for $F_\sigma \subseteq F_\tau$. We write $\#\sigma$ for the size of F_σ. Given two strings σ and τ, we let $\sigma \cup \tau$ be the unique string ρ of length $\max(|\sigma|, |\tau|)$ such that $F_\rho = F_\sigma \cup F_\tau$.

A *binary tree* is a set of binary strings $T \subseteq 2^{<\omega}$ which is closed downward under the prefix relation. A *path* through T is a binary sequence $P \in 2^\omega$ such that every initial segment belongs to T.

A *Turing ideal* $\mathcal{I}$ is a collection of sets which is closed downward under the Turing reduction and closed under the effective join, that is, $(\forall X \in \mathcal{I})(\forall Y \leq_T X) Y \in \mathcal{I}$ and $(\forall X, Y \in \mathcal{I}) X \oplus Y \in \mathcal{I}$, where $X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}$. A *Scott set* is a Turing ideal $\mathcal{I}$ such that every infinite binary tree $T \in \mathcal{I}$ has a path in $\mathcal{I}$. In other words, a Scott set is the second-order part of an ω-model of $\mathsf{RCA}_0 + \mathsf{WKL}$. A Turing ideal $\mathcal{M}$ is *countable coded* by a set X if $\mathcal{M} = \{X_n : n \in \omega\}$ with $X = \bigoplus_n X_n$. A formula is $\Sigma^0_1(\mathcal{M})$ (resp. $\Pi^0_1(\mathcal{M})$) if it is $\Sigma^0_1(X)$ (resp. $\Pi^0_1(X)$) for some $X \in \mathcal{M}$.

Given two sets A and B, we denote by $A < B$ the formula $(\forall x \in A)(\forall y \in B)[x < y]$. We write $A \subseteq^* B$ to mean that $A - B$ is finite, that is, $(\exists n)(\forall a \in A)(a \notin B \to a < n)$. A *$k$-cover* of a set X is a sequence of sets $Y_0, \dots, Y_{k-1}$ such that $X \subseteq Y_0 \cup \dots \cup Y_{k-1}$.

2. Generalized Mathias forcing

The notion of forcing used to build solutions to the pigeonhole principle while controlling the first jump is a variant of Mathias forcing. In this section, we extend Mathias forcing to a more general notion of forcing while controlling iterated jumps. Then, in the next section, we will design a variant of this generalized Mathias forcing to control iterated jumps of solutions to the pigeonhole principle.

Before defining the generalized Mathias forcing, we need to introduce some core machinery which will be used all over the article.

2.1. Largeness classes

The following notion of largeness class was introduced by the authors in [15] to design a notion of forcing controlling the second jump of solutions to the pigeonhole principle.

Definition 2.1. A *largeness class* is a collection of sets $\mathcal{A} \subseteq 2^\omega$ such that

(a) If $X \in \mathcal{A}$ and $Y \supseteq X$, then $Y \in \mathcal{A}$
(b) For every k-cover $Y_0, \dots, Y_{k-1}$ of ω, there is some $j < k$ such that $Y_j \in \mathcal{A}$.

For example, the collection of all the infinite sets is a largeness class. Moreover, any superclass of a largeness class is again a largeness class.

Fix an effective enumeration $\mathcal{U}_0^Z, \mathcal{U}_1^Z, \dots$ of all the $\Sigma_1^{0,Z}$ classes upward-closed under the superset relation, that is, if $X \in \mathcal{U}_e^Z$ and $Y \supseteq X$, then $Y \in \mathcal{U}_e^Z$.

Lemma 2.2 Suppose $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots$ is a decreasing sequence of largeness classes. Then $\bigcap_s \mathcal{A}_s$ is a largeness class.

Proof. If $X \in \bigcap_s \mathcal{A}_s$ and $Y \supseteq X$, then for every s, since $\mathcal{A}_s$ is a largeness class, $Y \in \mathcal{A}_s$, so $Y \in \bigcap_s \mathcal{A}_s$. Let $Y_0, \dots, Y_{k-1}$ be a k-cover of ω. For every $s \in \omega$, there is some $j < k$ such that $Y_j \in \mathcal{A}_s$. By the infinite pigeonhole principle, there is some $j < k$ such that $Y_j \in \mathcal{A}_s$ for infinitely many s. Since $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq$ is a decreasing sequence, $Y_j \in \bigcap_s \mathcal{A}_s$. □

Lemma 2.3 Let $\mathcal{A}$ be a Σ_1^0 class. The sentence "$\mathcal{A}$ is a largeness class" is Π_2^0.

Proof. Say $\mathcal{A} = \{X : (\exists \sigma \preceq X)\varphi(\sigma)\}$ where φ is a Σ_1^0 formula. By compactness, $\mathcal{A}$ is a largeness class iff for every σ and τ such that $\sigma \subseteq \tau$ and $\varphi(\sigma)$ holds, $\varphi(\tau)$ holds, and for every k, there is some $n \in \omega$ such that for every $\sigma_0 \cup \dots \cup \sigma_{k-1} = \{0, \dots, n\}$, there is some $j < k$ such that $\varphi(\sigma_j)$ holds. □

Given an infinite set X, we let $\mathcal{L}_X$ be the $\Pi_2^0(X)$ largeness class of all sets having an infinite intersection with X. In what follows, fix a Scott set $\mathcal{M} = \{X_0, X_1, \dots\}$ countable coded by a set M. Given a set $C \subseteq \omega^2$, we write

$$\mathcal{U}_C^{\mathcal{M}} = \bigcap_{\langle e,i \rangle \in C} \mathcal{U}_e^{X_i}$$

Definition 2.4. A class $\mathcal{A}$ is $\mathcal{M}$*-cohesive* if for every $X \in \mathcal{M}$, either $\mathcal{A} \subseteq \mathcal{L}_X$ or $\mathcal{A} \subseteq \mathcal{L}_{\overline{X}}$.

Lemma 2.5 Let $\mathcal{U}_C^{\mathcal{M}}$ be an $\mathcal{M}$-cohesive class. Let $\mathcal{U}_D^{\mathcal{M}}$ and $\mathcal{V}_E^{\mathcal{M}}$ be such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}}$ and $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$ are both largeness classes. Then $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$ is a largeness class.

Proof. Suppose for contradiction that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$ is not a largeness class. Then by Lemma 2.2, there is some finite $C_1 \subseteq C$, $D_1 \subseteq D$ and $E_1 \subseteq E$ such that $\mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}}$ is not a largeness class. Since $\mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}}$ is $\Sigma_1^0(\mathcal{M})$ and $\mathcal{M}$ is a Scott set, there is a partition $Y_0 \sqcup \dots \sqcup Y_{k-1} = \omega$ in $\mathcal{M}$ such that for every $i < k$, $Y_i \notin \mathcal{U}_{C_1}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}} \cap \mathcal{U}_{E_1}^{\mathcal{M}} \supseteq \mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}} \cap \mathcal{U}_E^{\mathcal{M}}$. Since $\mathcal{U}_C^{\mathcal{M}}$ is $\mathcal{M}$-cohesive, there must be some $i < k$ such that $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{Y_i}$. In particular, $Y_i \in \mathcal{U}_C^{\mathcal{M}}$, so $Y_i \notin \mathcal{U}_D^{\mathcal{M}}$ or $Y_i \notin \mathcal{U}_E^{\mathcal{M}}$. Suppose $Y_i \notin \mathcal{U}_D^{\mathcal{M}}$, as the other case is symmetric. Since $Y_j \cap Y_i = \emptyset$ for every $j \neq i$, then $Y_j \notin \mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{Y_i}$ for every $j \neq i$. It follows that $Y_0, \dots, Y_{k-1}$ witnesses that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_D^{\mathcal{M}}$ is not a largeness class. Contradiction. □

Definition 2.6. A class $\mathcal{A}$ is $\mathcal{M}$*-minimal* if for every $X \in \mathcal{M}$ and $e \in \omega$, either $\mathcal{A} \subseteq \mathcal{U}_e^X$ or $\mathcal{A} \cap \mathcal{U}_e^X$ is not a largeness class.

The following is a corollary of the previous lemma.

Lemma 2.7 Given an $\mathcal{M}$-cohesive largeness class $\mathcal{U}_C^{\mathcal{M}}$, the collection of sets

$$\langle \mathcal{U}_C^{\mathcal{M}} \rangle = \bigcap_{e \in \omega, X \in \mathcal{M}} \{\mathcal{U}_e^X : \mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^X \text{ is a largeness class}\}$$

is an $\mathcal{M}$-minimal largeness class contained in $\mathcal{U}_C^{\mathcal{M}}$.

Proof. We first prove that $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is a largeness class. Let $e_0, e_1, \dots$ and $X_0, X_1, \dots$ be an enumeration of all pairs $(e, X) \in \omega \times \mathcal{M}$ such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^X$ is a largeness class. By induction on n using Lemma 2.5, $\bigcap_{i<n} \mathcal{U}_{e_i}^{X_i}$ is a largeness class for every $n \in \omega$. Thus, by Lemma 2.2, $\langle \mathcal{U}_C^{\mathcal{M}} \rangle = \bigcap_i \mathcal{U}_{e_i}^{X_i}$ is a largeness class.

Next, we prove that $\langle \mathcal{U}_C^{\mathcal{M}} \rangle \subseteq \mathcal{U}_C^{\mathcal{M}}$. For every $\langle e, i \rangle \in C$, $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^{X_i}$ is a largeness class, thus $\langle \mathcal{U}_C^{\mathcal{M}} \rangle \subseteq \mathcal{U}_e^{X_i}$. Therefore $\langle \mathcal{U}_C^{\mathcal{M}} \rangle \subseteq \mathcal{U}_C^{\mathcal{M}}$. It follows that $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is $\mathcal{M}$-minimal. $\square$

Note that $\langle \mathcal{U}_C^{\mathcal{M}} \rangle = \mathcal{U}_D^{\mathcal{M}}$ where D is the set of all $\langle e, i \rangle$ such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^{X_i}$ is a largeness class.

Definition 2.8. A *partition regular class* is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that

(a) $\omega \in \mathcal{L}$
(b) For every $X \in \mathcal{L}$ and $Y_0 \cup \dots \cup Y_{k-1} \supseteq X$, there is some $j < k$ such that $Y_j \in \mathcal{L}$.

In particular, the class of all infinite sets is partition regular.

Definition 2.9. Let $\mathcal{A}$ be a largeness class. Define

$$\mathcal{L}(\mathcal{A}) = \{X \in 2^\omega : \forall k \forall X_0 \cup \dots \cup X_{k-1} \supseteq X \exists i < k X_i \in \mathcal{A}\}$$

Lemma 2.10 Let $\mathcal{A}$ be a largeness class. Then $\mathcal{L}(\mathcal{A})$ is the largest partition regular subclass of $\mathcal{A}$.

Proof. We first prove that $\mathcal{L}(\mathcal{A})$ is a partition regular subclass of $\mathcal{A}$. By definition of $\mathcal{A}$ being a largeness class, $\omega \in \mathcal{L}(\mathcal{A})$. Let $X \in \mathcal{L}(\mathcal{A})$ and $X_0 \cup \dots \cup X_{k-1} \supseteq X$. Suppose for the sake of absurd that $X_i \notin \mathcal{L}(\mathcal{A})$ for every $i < k$. Then for every $i < k$, there is some $k_i \in \omega$ and some $Y_i^0 \cup \dots \cup Y_i^{k_i - 1} \supseteq X_i$ such that $Y_i^j \notin \mathcal{A}$ for every $j < k_i$. Then $\{Y_i^j : i < k, j < k_i\}$ is a cover of X contradicting $X \in \mathcal{L}(\mathcal{A})$. Therefore $\mathcal{L}(\mathcal{A})$ is a partition regular class. Moreover, $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$ as witnessed by taking the trivial cover of X by X itself.

We now prove that $\mathcal{L}(\mathcal{A})$ is the largest partition regular subclass of $\mathcal{A}$. Indeed, let $\mathcal{B}$ be a partition regular subclass of $\mathcal{A}$. Then for every $X \in \mathcal{B}$, every $X_0 \cup \dots \cup X_{k-1} \supseteq X$, there is some $j < k$ such that $X_j \in \mathcal{B} \subseteq \mathcal{A}$. Thus $X \in \mathcal{L}(\mathcal{A})$, so $\mathcal{B} \subseteq \mathcal{L}(\mathcal{A})$. $\square$

Also note that $\mathcal{L}(\mathcal{U}_C^{\mathcal{M}}) = \mathcal{U}_D^{\mathcal{M}}$ for some $D \subseteq \omega^2$.

Corollary 2.11 Suppose $\mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-minimal largeness class. Then $\mathcal{U}_C^{\mathcal{M}}$ is partition regular.

Proof. Let D be such that $\mathcal{U}_D^{\mathcal{M}} = \mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$. By Lemma 2.10, $\mathcal{U}_D^{\mathcal{M}} \subseteq \mathcal{U}_C^{\mathcal{M}}$. By $\mathcal{M}$-minimality of $\mathcal{U}_C^{\mathcal{M}}$, $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{U}_D^{\mathcal{M}}$. It follows that $\mathcal{U}_C^{\mathcal{M}} = \mathcal{U}_D^{\mathcal{M}}$. Since $\mathcal{U}_D^{\mathcal{M}}$ is partition regular, then so is $\mathcal{U}_C^{\mathcal{M}}$. $\square$

It follows that if $\mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive largeness class, then $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ is an $\mathcal{M}$-minimal partition regular class.

Lemma 2.12 Every PA degree relative to M' computes a set $C \subseteq \omega^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive largeness class.

Proof. Let $\{X_0, X_1, \dots\}$ be an M-computable sequence of sets containing all the sets of $\mathcal{M}$ (and possibly more). Let T be the tree of all $\sigma \in 2^{<\omega}$ such that $\bigcap_{i \in \sigma} \mathcal{L}_{X_i} \bigcap_{i \notin \sigma} \mathcal{L}_{\overline{X}_i}$ is non-empty. The tree is M'-computable, thus every PA degree relative to M' computes a path $P \in [T]$. Let e_0 and e_1 be such that $\mathcal{U}_{e_1}^X = \mathcal{L}_X$ and $\mathcal{U}_{e_0}^X = \mathcal{L}_{\overline{X}}$, respectively. Then letting $C = \{\langle e_{P(i)}, i \rangle : i \in \omega\}$ is such that $\mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive largeness class. $\square$

Corollary 2.13 There exists a set $C \subseteq \omega^2$ such that $\mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive largeness class and $(C \oplus M')' \leq_T M''$.

Proof. By Lemma 2.12 and the relativized low basis theorem [10]. □

Lemma 2.14 For every set $C \subseteq \omega^2$, there is a set $D \subseteq \omega^2$ such that $\langle \mathcal{U}_C^{\mathcal{M}} \rangle = \mathcal{U}_D^{\mathcal{M}}$.

Proof. Let D be the set of all $e, i \in \omega$ such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^{X_i}$ is a largeness class. By Lemma 2.2, $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{U}_e^{X_i}$ is a largeness class if and only if for every finite set $F \subseteq C$, $\mathcal{U}_F^{\mathcal{M}} \cap \mathcal{U}_e^{X_i}$ is a largeness class. By Lemma 2.3, being a largeness class for a $\Sigma^0_1(\mathcal{M})$ class is $\Pi^0_2(M)$, hence $\Pi^0_1(M')$. Thus D is $\Pi^0_1(C \oplus M')$. □

Corollary 2.15 For every set $C \subseteq \omega^2$, and $Z \subseteq \omega$, the relation "$Z \in \langle \mathcal{U}_C^{\mathcal{M}} \rangle$" is $\Pi^0_1((C \oplus Z \oplus M')')$.

Proof. By Lemma 2.14, there is a $\Pi^0_1(C \oplus M')$ set $D \subseteq \omega^2$ such that $\langle \mathcal{U}_C^{\mathcal{M}} \rangle = \mathcal{U}_D^{\mathcal{M}}$. Then $Y \in \langle \mathcal{U}_C^{\mathcal{M}} \rangle$ if and only if for every $e, i \in \omega$, either $\langle e, i \rangle \notin D$ or $Y \in \mathcal{U}_e^{X_i}$. Thus the relation "$Z \in \langle \mathcal{U}_C^{\mathcal{M}} \rangle$" is $\Pi^0_1((C \oplus Z \oplus M')')$. □

2.2. Notion of forcing

Let $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_n$ be countable Scott sets coded by sets $M_0, M_1, \dots, M_n$, respectively. Furthermore, by the relativized low basis theorem [10], assume that M_i is low over $\emptyset^{(i)}$ and $\emptyset^{(i+1)} \in \mathcal{M}_{i+1}$ for every $i < n$. Let $C_0, \dots, C_{n-1}$ be such that $C_i \in \mathcal{M}_{i+1}$ and $\mathcal{U}_{C_i}^{\mathcal{M}_i}$ is an $\mathcal{M}_i$-cohesive largeness class for every $i < n$. The existence of a $C_i \in \mathcal{M}_{i+1}$ is ensured by Lemma 2.12. Furthermore, we require that $\mathcal{U}_{C_{i+1}}^{\mathcal{M}_{i+1}} \subseteq \langle \mathcal{U}_{C_i}^{\mathcal{M}_i} \rangle$. This last property can be satisfied by Lemma 2.14.

Definition 2.16. Fix $n \geq 0$; Let $\mathbb{Q}_n$ be the set of pairs (σ, X) such that
 (a) $X \cap \{0, \dots, |\sigma|\} = \emptyset$; $X \in \mathcal{M}_n$
 (b) X is infinite if $n = 0$ and $X \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ if $n \geq 1$

Note that X is infinite even in the case $n \geq 1$ since $\mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}}$ contains only infinite sets. Mathias forcing builds a single object G by approximations (conditions) which consist in an initial segment σ of G, and an infinite reservoir of integers. The purpose of the reservoir is to restrict the set of elements we are allowed to add to the initial segment. The reservoir therefore enriches the standard Cohen forcing by adding an infinitary negative restrain.

Definition 2.17. The partial order on $\mathbb{Q}_n$ is defined by $(\tau, Y) \leq (\sigma, X)$ if $\sigma \preceq \tau$, $Y \subseteq X$ and $\tau - \sigma \subseteq X$.

Given a collection $\mathcal{F} \subseteq \mathbb{Q}_n$, we let $G_{\mathcal{F}} = \bigcup\{\sigma : (\sigma, X) \in \mathcal{F}\}$.

Definition 2.18. Let $\Phi_e(G, x)$ be a Δ_0 formula with free variable x. Let $p = (\sigma, X) \in \mathbb{Q}_n$.
 (a) $p \Vdash (\exists x)\Phi_e(G, x)$ if $(\exists x)\Phi_e(\sigma, x)$
 (b) $p \Vdash (\forall x)\Phi_e(G, x)$ if $(\forall \tau \subseteq X)(\forall x)\Phi_e(\sigma \cup \tau, x)$

Having defined the forcing relation for Σ^0_1 and Π^0_1 formulas, we extend the forcing relation to arbitrary arithmetical formulas by induction on the level in the hierarchy. A Σ^0_{n+1} formula $(\exists x)\varphi(G, x)$ is forced, where $\varphi(G, x)$ is Π^0_n, if there is some $x \in \omega$ such that the Π^0_n formula $\varphi(G, x)$ is forced. The case of Π^0_{n+1} formulas is more subtle. Intuitively, a Π^0_{n+1} formula $(\forall x)\varphi(G, x)$ is forced, where $\varphi(G, x)$ is Σ^0_n, if for every extension of the condition and every $x \in \omega$, the Π^0_n formula $\neg\varphi(G, x)$ is never forced. The forcing relation is defined by a mutual induction through the following two definitions.

Definition 2.19. Fix $n \geq 0$. Let $\zeta_n : \omega \times 2^{<\omega} \times \omega \to \omega$ be the computable function that takes as a parameter a code for a Δ_0 formula $\Phi_e(G, x_{n+1}, \dots, x_0)$, a string σ and an integer $x_{n+1} \in \omega$, and which gives a code for the $\Sigma^0_{n+1}(\mathcal{M}_n)$ set

$$\{X : (\sigma, X) \nVdash (\forall x_n)(\exists x_{n-1}) \dots (Q x_0)\Phi_e(G, x_{n+1}, \dots, x_0)\}$$

Definition 2.20. Fix $n \geq m \geq 1$. Let $\Phi_e(G, x_m, \dots, x_0)$ be a Δ_0 formula with free variables $x_0, \dots, x_m$. Let $p = (\sigma, X) \in \mathbb{Q}_n$.

(a) $p \Vdash (\exists x_m)(\forall x_{m-1}) \dots (Qx_0)\Phi_e(G, x_m, \dots, x_0)$ if there is some $x_m \in \omega$ such that

$$p \Vdash (\forall x_{m-1}) \dots (Qx_0)\Phi_e(G, x_m, \dots, x_0)$$

(b) $p \Vdash (\forall x_m)(\exists x_{m-1}) \dots (\overline{Q}x_0)\neg\Phi_e(G, x_m, \dots, x_0)$ if for every $\rho \subseteq X$ and every $x_m \in \omega$, $\mathcal{U}^{\mathcal{M}_{m-1}}_{C_{m-1}} \cap \mathcal{U}^{\mathcal{M}_{m-1}}_{\zeta_{m-1}(e,\sigma\cup\rho,x_m)}$ is a largeness class.

Lemma 2.21 Fix $n \geq m \geq 0$. Let $\Phi_e(G, x_m, \dots, x_0)$ be a Δ_0 formula with free variables $x_0, \dots, x_m$. Let $p, q \in \mathbb{Q}_n$ be such that $q \leq p$.

(a) If $p \Vdash (\exists x_m)(\forall x_{m-1}) \dots (Qx_0)\Phi_e(G, x_m, \dots, x_0)$ then so does q.
(b) If $p \Vdash (\forall x_m)(\exists x_{m-1}) \dots (\overline{Q}x_0)\neg\Phi_e(G, x_m, \dots, x_0)$ then so does q.

Proof. We prove (a) and (b) by induction over m. Say $p = (\sigma, X)$ and $q = (\tau, Y)$. The base case $m = 0$ is immediate. Suppose $m \geq 1$.

(a) Since $p \Vdash (\exists x_m)(\forall x_{m-1}) \dots (Qx_0)\Phi_e(G, x_m, \dots, x_0)$, then there is some $x_m \in \omega$ such that $p \Vdash (\forall x_{m-1}) \dots (Qx_0)\Phi_e(G, x_m, \dots, x_0)$. By item (b) of this lemma and induction hypothesis, $q \Vdash (\forall x_{m-1}) \dots (Qx_0)\Phi_e(G, x_m, \dots, x_0)$ hence $q \Vdash (\exists x_m)(\forall x_{m-1}) \dots (Qx_0)$ $\Phi_e(G, x_m, \dots, x_0)$.
(b) Let $\rho = \tau - \sigma$. By definition of $q \leq p$, $\rho \subseteq X$. Let $\rho_1 \subseteq Y$ and $x_m \in \omega$. In particular, $\rho \cup \rho_1 \subseteq X$. By definition of $p \Vdash (\forall x_m)(\exists x_{m-1}) \dots (\overline{Q}x_0)\neg\Phi_e(G, x_m, \dots, x_0)$, $\mathcal{U}^{\mathcal{M}_{m-1}}_{C_{m-1}} \cap \mathcal{U}^{\mathcal{M}_{m-1}}_{\zeta_{m-1}(e,\sigma\cup\rho\cup\rho_1,x_m)}$ is a largeness class. So $\mathcal{U}^{\mathcal{M}_{m-1}}_{C_{m-1}} \cap \mathcal{U}^{\mathcal{M}_{m-1}}_{\zeta_{m-1}(e,\tau\cup\rho_1,x_m)}$ is a largeness class, and $q \Vdash (\forall x_m)(\exists x_{m-1}) \dots (\overline{Q}x_0)\neg\Phi_e(G, x_m, \dots, x_0)$

□

Lemma 2.22 Fix $n \geq 0$. Let $\mathcal{F}$ be a $\mathbb{Q}_n$-filter, let $\Phi_e(G, x)$ be a Δ_0 formula with free variable x, and let $p \in \mathcal{F}$.

(a) If $p \Vdash (\exists x)\Phi_e(G, x)$, then $(\exists x)\Phi_e(G_\mathcal{F}, x)$ holds.
(b) If $p \Vdash (\forall x)\neg\Phi_e(G, x)$, then $(\forall x)\neg\Phi_e(G_\mathcal{F}, x)$ holds.

Proof. Say $p = (\sigma, X)$. (a) By definition of $p \Vdash (\exists x)\Phi_e(G, x)$, there is some $x \in \omega$ such that $\Phi_e(\sigma, x)$ holds. Since $\sigma \subseteq G_\mathcal{F} \subseteq \sigma \cup X$, then by continuity of Φ_e, $\Phi_e(G_\mathcal{F}, x)$ holds. (b) By definition of $p \Vdash (\forall x)\neg\Phi_e(G, x)$, for every $x \in \omega$ and $\rho \subseteq X$, $\Phi_e(\sigma \cup \rho, x)$ does not hold. Since $\sigma \subseteq G_\mathcal{F} \subseteq \sigma \cup X$, by the finite use property, since $\Phi_e(G_\mathcal{F}, x)$ holds for every $x \in \omega$. □

Whenever a Σ^0_1 or a Π^0_1 formula is forced, then it holds over $G_\mathcal{F}$ for every $\mathbb{Q}_n$-filter $\mathcal{F}$. However, the situation is more complex for higher formulas. We need to consider sufficiently generic filters.

2.3. Generic filters

Generic filters are usually defined in terms of intersection of dense sets of conditions. However, given the complexity of the set of conditions, we define n-genericity in terms of deciding every Σ^0_{n+1} property.

Definition 2.23. Fix $n \geq m \geq 0$. A $\mathbb{Q}_n$-filter $\mathcal{F}$ is $(m+1)$-*generic* if for every Σ^0_{m+1} formula $\varphi(G)$, there is some $p \in \mathcal{F}$ such that $p \Vdash \varphi(G)$ or $p \Vdash \neg\varphi(G)$.

Lemma 2.24 Let $\mathcal{F}$ be a 1-generic $\mathbb{Q}_n$-filter. Then $G_\mathcal{F}$ is infinite.

Proof. Suppose for the contradiction that $G_\mathcal{F} \subseteq \{0, \dots, k\}$. Let $\Phi_e(G, x) \equiv x \in G \wedge x > k$. Since $\mathcal{F}$ is 1-generic, there is some $p = (\sigma, X) \in \mathcal{F}$ such that $p \Vdash (\exists x)\Phi_e(G, x)$ or $p \Vdash (\forall x)\neg\Phi_e(G, x)$. If $p \Vdash (\exists x)\Phi_e(G, x)$, then there is some $x \in \omega$ such that $\Phi_e(\sigma, x)$ holds, then $x > k$ and $x \in \sigma \subseteq G_\mathcal{F}$. Contradiction. If $p \Vdash (\forall x)\neg\Phi_e(G, x)$, then for every $x \in \omega$ and every $\rho \subseteq X$, $\Phi_e(\sigma \cup \rho, x)$ does not hold. However, since X is infinite, let $\rho \subseteq X$ be such that $\min \rho > k$. Then letting $x \in \rho$, $\Phi_e(\sigma \cup \rho, x)$ holds. Again, contradiction. □

In general, an $(n+1)$-generic $\mathbb{Q}_n$-filter is not necessarily n-generic. However, in the case $n=1$, we are able to prove this property.

Lemma 2.25 Let $\mathcal{F}$ be a 2-generic $\mathbb{Q}_n$-filter, then $\mathcal{F}$ is 1-generic.

Proof. Let $\Phi_e(G,r)$ be a Δ^0_0 formula with free variable r. We want to show that there is some $(\sigma,X)\in\mathcal{F}$ such that $(\sigma,X)\Vdash(\exists u)\Phi_e(G,r)$ or $(\sigma,X)\Vdash(\forall r)\neg\Phi_e(G,r)$. Let $\Phi_u(G,r,s)=\Phi_e(G,r)$ and $\Phi_v(G,a,b)=\neg\Phi_e(G,b)$ Since $\mathcal{F}$ is 2-generic, there is some $p=(\sigma,X)\in\mathcal{F}$ such that

$$p\Vdash(\exists r)(\forall s)\Phi_u(G,r,s)\text{ or }p\Vdash(\forall r)(\exists s)\neg\Phi_u(G,r,s)$$

and

$$p\Vdash(\exists a)(\forall b)\Phi_v(G,a,b)\text{ or }p\Vdash(\forall a)(\exists b)\neg\Phi_v(G,a,b)$$

We have three cases.

Case 1: $p\Vdash(\exists r)(\forall s)\Phi_u(r,s)$. Unfolding the definition, there is some r such that $p\Vdash(\forall s)\Phi_u(G,r,s)$. In particular, $\Phi_u(\sigma,r,0)$ holds, so $\Phi_e(\sigma,r)$ holds, hence $p\Vdash(\exists r)\Phi_e(G,r)$, and we are done.

Case 2: $p\Vdash(\exists a)(\forall b)\Phi_v(a,b)$. Unfolding the definition, there is some $a\in\omega$ such that $p\Vdash(\forall b)\Phi_v(a,b)$, hence $p\Vdash(\forall s)\neg\Phi_e(G,s)$, and we are done.

Case 3: $p\Vdash(\forall r)(\exists s)\neg\Phi_u(G,r,s)$ and $p\Vdash(\forall a)(\exists b)\neg\Phi_v(a,b)$. Then in particular, letting $a=0$, $\mathcal{U}^{\mathcal{M}_0}_{C_0}\cap\mathcal{U}^{\mathcal{M}_0}_{\zeta_0(v,\sigma,0)}$ is a largeness class. Then $\langle\mathcal{U}^{\mathcal{M}_0}_{C_0}\rangle\subseteq\mathcal{U}_{\zeta_0(v,\sigma,0)}$. Since $X\in\langle\mathcal{U}^{\mathcal{M}_0}_{C_0}\rangle$, then $X\in\mathcal{U}_{\zeta_0(v,\sigma,0)}$. Unfolding the definition of ζ_0, $p\nVdash(\forall n)\Phi_v(G,0,n)$. Thus, there is some $r\in\omega$ and some $\rho\subseteq X$ such that $\Phi_e(\sigma\cup\rho,r)$ holds. Since $p\Vdash(\forall r)(\exists s)\neg\Phi_u(G,r,s)$, $\mathcal{U}^{\mathcal{M}_0}_{C_0}\cap\mathcal{U}_{\zeta_0(u,\sigma\cup\rho,r)}$ is a largeness class. Then $\langle\mathcal{U}^{\mathcal{M}_0}_{C_0}\rangle\subseteq\mathcal{U}_{\zeta_0(u,\sigma\cup\rho,r)}$. Since $X\in\langle\mathcal{U}^{\mathcal{M}_0}_{C_0}\rangle$, $X\in\mathcal{U}_{\zeta_0(u,\sigma\cup\rho,r)}$, $(\sigma\cup\rho,X)\nVdash(\forall s)\Phi_u(G,r,s)$, so $(\sigma\cup\rho,X)\nVdash\Phi_e(G,r)$. Contradiction. □

As explained, the definition of the forcing relation for Π^0_{m+1} formulas $(\forall x)\varphi(G,x)$ asserts that for every extension d of the condition c and every $x\in\omega$, d will not force $\neg\varphi(G,x)$. This is however not sufficient to ensure that $(\forall x)\varphi(G,x)$ will hold, since the filter may not be sufficiently generic to force either $\varphi(G,x)$ or $\neg\varphi(G,x)$. Contrary to Π^0_1 formulas, we therefore need to require that the filter $\mathcal{F}$ is $(s+1)$-generic for every $s<m$ to ensure that whenever a formula is forced, it holds over $G_{\mathcal{F}}$.

Lemma 2.26 Fix $n\geq m\geq 1$. Let $\mathcal{F}$ be an m-generic $\mathbb{Q}_n$-filter and $\Phi_e(G,x_m,\dots,x_0)$ be a Δ_0 formula with free variables $x_m,\dots,x_0$. If $p\Vdash(\forall x_m)(\exists x_{m-1})\dots(\overline{Q}x_0)\neg\Phi_e(G,x_m,\dots,x_0)$ for some $p\in\mathcal{F}$, then for every $x_m\in\omega$, there is some $q\in\mathcal{F}$ such that $q\Vdash(\exists x_{m-1})\dots(\overline{Q}x_0)\neg\Phi_e(G,x_m,\dots,x_0)$.

Proof. Fix $x_m\in\omega$. Since $\mathcal{F}$ is m-generic, there is some $q=(\tau,Y)\in\mathcal{F}$ such that

$$q\Vdash(\exists x_{m-1})\dots(\overline{Q}x_0)\neg\Phi_e(G,m_n,\dots,x_0)\text{ or }q\Vdash(\forall x_{m-1})\dots(Qx_0)\Phi_e(G,x_m,\dots,x_0)$$

Since $\mathcal{F}$ is a filter, we can assume that $q\leq p$. By Lemma 2.21,

$$q\Vdash(\forall x_m)(\exists x_{m-1})\dots(\overline{Q}x_0)\neg\Phi_e(G,x_m,\dots,x_0)$$

so $\mathcal{U}^{\mathcal{M}_{m-1}}_{C_{m-1}}\cap\mathcal{U}^{\mathcal{M}_{m-1}}_{\zeta_{m-1}(e,\tau,x_m)}$ is a largeness class. Thus $\langle\mathcal{U}^{\mathcal{M}_{n-1}}_{C_{n-1}}\rangle\subseteq\langle\mathcal{U}^{\mathcal{M}_{m-1}}_{C_{m-1}}\rangle\subseteq\mathcal{U}^{\mathcal{M}_{m-1}}_{\zeta_{m-1}(e,\tau,x_m)}$. Since $Y\in\langle\mathcal{U}^{\mathcal{M}_{n-1}}_{C_{n-1}}\rangle$, then $Y\in\mathcal{U}^{\mathcal{M}_{m-1}}_{\zeta_{m-1}(e,\tau,x_m)}$. Therefore $q\nVdash(\forall x_{m-1})\dots(Qx_0)\Phi_e(G,x_m,\dots,x_0)$, hence $q\Vdash(\exists x_{m-1})\dots(\overline{Q}x_0)\neg\Phi_e(G,x_m,\dots,x_0)$. □

Lemma 2.27 Fix $n\geq m\geq 1$. Let $\mathcal{F}$ be a $\mathbb{Q}_n$-filter which is is $(s+1)$-generic for every $s<m$. Let $\Phi_e(G,x_m,\dots,x_0)$ be a Δ_0 formula with free variables $x_m,\dots,x_0$, and let $p\in\mathcal{F}$.

(a) If $p\Vdash(\exists x_m)(\forall x_{m-1})\dots(Qx_0)\Phi_e(G,x_m,\dots,x_0)$, then the formula holds for $G_{\mathcal{F}}$.
(b) If $p\Vdash(\forall x_m)(\exists x_{m-1})\dots(\overline{Q}x_0)\neg\Phi_e(G,x_m,\dots,x_0)$, then the formula holds for $G_{\mathcal{F}}$.

Proof. By induction over $m\geq 1$. Say $p=(\sigma,X)$.

(a) By definition, there is some $x_m \in \omega$ such that $p \Vdash (\forall x_{m-1}) \dots (Qx_0)\Phi_e(G, x_m, \dots, x_0)$. By induction hypothesis and by Lemma 2.22(b), $(\forall x_{m-1}) \dots (Qx_0)\Phi_e(G_{\mathcal{F}}, x_m, \dots, x_0)$ holds.

(b) Fix some x_m. By Lemma 2.26, there is some $q \in \mathcal{F}$ such that

$$q \Vdash (\exists x_{m-1}) \dots (\overline{Q}x_0)\neg\Phi_e(G, x_m, \dots, x_0)$$

By induction hypothesis and by Lemma 2.22(a), $(\exists x_{m-1}) \dots (\overline{Q}x_0)\neg\Phi_e(G_{\mathcal{F}}, x_m, \dots, x_0)$ holds.

□

3. Generalized Pigeonhole forcing

In this section, we adapt the generalized notion of Mathias of forcing to design a notion of forcing producing solutions to the infinite pigeonhole principle while controlling iterated jumps of the solutions. In what follows, we fix 2-partition $A_0 \sqcup A_1 = \omega$ representing an instance of the infinite pigeonhole principle.

3.1. Notion of forcing

Here again, we assume fix a countable Scott sets $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_n$ coded by sets $M_0, M_1, \dots, M_n$, respectively, such that M_i is low over $\emptyset^{(i)}$ and $\emptyset^{(i+1)} \in \mathcal{M}_{i+1}$ for every $i < n$. We also have fixed sets $C_0, \dots, C_{n-1}$ such that $C_i \in \mathcal{M}_{i+1}$ and $\mathcal{U}_{C_i}^{\mathcal{M}_i}$ is an $\mathcal{M}_i$-cohesive largeness class for every $i < n$. We also require that $\mathcal{U}_{C_{i+1}}^{\mathcal{M}_{i+1}} \subseteq \langle \mathcal{U}_{C_i}^{\mathcal{M}_i} \rangle$.

In order to obtain low$_n$ solutions to Δ_n^0 instances of the pigeonhole principle, we need to provide a careful analysis of the effectiveness of the dense sets considered. For this, we need to fix a set P of PA degree relative to M_n'. This set will basically enable us to pick, given a cover $Y_0 \cup \dots \cup Y_{k-1}$ of a set $X \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$, some $j < k$ such that $Y_j \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$.

Definition 3.1. Fix $n \geq 0$. Let $\mathbb{P}_n$ denote the set of conditions (σ^0, σ^1, X) such that

(a) $\sigma^i \subseteq A^i$ for every $i < 2$

(b) $X \cap \{0, \dots, \max_i |\sigma^i|\} = \emptyset$; $X \in \mathcal{M}_n$

(c) X is infinite if $n = 0$ and $X \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ if $n \geq 1$.

By definition of a Turing ideal $\mathcal{M}$ countable coded by a set M, then $\mathcal{M}$ can be written as $\{Z_0, Z_1, \dots\}$ with $M = \bigoplus_i Z_i$. We then say that i is an M-*index* of Z_i. Thanks to the notion of index, any $\mathbb{P}_n$-condition can be finitely presented as follows. An *index* of a $\mathbb{P}_n$-condition $c = (\sigma^0, \sigma^1, X)$ is a tuple (σ^0, σ^1, a) where a is an M_n-index for X.

Definition 3.2. The partial order on $\mathbb{P}_n$ is defined by

$$(\tau^0, \tau^1, Y) \leq (\sigma^0, \sigma^1, X)$$

if for every $i < 2$, $(\tau^i, Y) \leq (\sigma^i, X)$.

Given a condition $c = (\sigma^0, \sigma^1, X)$ and $i < 2$, we write $c^{[i]} = (\sigma^i, X)$. Each $\mathbb{P}_n$-condition c represents two $\mathbb{Q}_n$-conditions $c^{[0]}$ and $c^{[1]}$.

Definition 3.3. Let $\mathcal{F} \subseteq \mathbb{P}_n$ be a collection. We write $\mathcal{F}^{[i]} = \{c^{[i]} : c \in \mathcal{F}\}$.

3.2. Forcing question for $\mathbb{P}_0$

We now design a disjunctive forcing question which is an abstraction of the first jump control of Cholak, Jocksuch and Slaman [2].

Definition 3.4. Let $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_0$ and let $\Phi_{e_0}(G, x)$ and $\Phi_{e_1}(G, x)$ be two Δ_0 formulas. Define the relation

$$c \mathrel{?\vdash} (\exists x)\Phi_{e_0}(G^0, x) \vee (\exists x)\Phi_{e_1}(G^1, x)$$

to hold if for every 2-cover $Z^0 \cup Z^1 = X$, there is some side $i < 2$, some finite set $\rho \subseteq Z^i$ and some $x \in \omega$ such that $\Phi_{e_i}(\sigma^i \cup \rho, x)$ holds.

This forcing relation satisfies the following disjunctive property.

Lemma 3.5 (Cholak, Jockusch and Slaman [2]) Let $c \in \mathbb{P}_0$ and let $\Phi_{e_0}(G, x)$ and $\Phi_{e_1}(G, x)$ be two Δ_0 formulas.

(a) If $c \mathrel{?{\vdash}} (\exists x)\Phi_{e_0}(G^0, x) \vee (\exists x)\Phi_{e_1}(G^1, x)$, then there is some $d \leq c$ and some $i < 2$ such that $d^{[i]} \Vdash (\exists x)\Phi_{e_i}(G, x)$.

(b) If $c \mathrel{?{\nvdash}} (\exists x)\Phi_{e_0}(G^0, x) \vee (\exists x)\Phi_{e_1}(G^1, x)$, then there is some $d \leq c$ and some $i < 2$ such that $d^{[i]} \Vdash (\forall x)\neg\Phi_{e_i}(G, x)$.

Proof. Suppose $c \mathrel{?{\vdash}} (\exists x)\Phi_{e_0}(G^0, x) \vee (\exists x)\Phi_{e_1}(G^1, x)$ holds. Then letting $Z^0 = X \cap A^0$ and $Z^1 = X \cap A^1$, there is some side $i < 2$, some finite set $\rho \subseteq X \cap A^i$ and some $x \in \omega$ such that $\Phi_{e_i}(\sigma^i \cup \rho, x)$ holds. The condition $d = (\sigma^i \cup \rho, \sigma^{1-i}, X \cap (\max \rho, \infty))$ is an extension of c such that $d^{[i]} \Vdash (\exists x)\Phi_{e_i}(G, x)$.

Suppose now that $c \mathrel{?{\nvdash}} (\exists x)\Phi_{e_0}(G^0, x) \vee (\exists x)\Phi_{e_1}(G^1, x)$. Let $\mathcal{P}$ be the collection of all the sets $Z^0 \oplus Z^1$ such that $Z^0 \cup Z^1 = X$ and such that for every $i < 2$, every finite set $\rho \subseteq Z^i$ and every $x \in \omega$, $\Phi_{e_i}(\sigma^i \cup \rho, x)$ does not hold. $\mathcal{P}$ is a non-empty $\Pi^{0,X}_1$ class, so since $X \in \mathcal{M}_0 \models \mathsf{WKL}$, there is some 2-cover $Z^0 \oplus Z^1 \in \mathcal{P} \cap \mathcal{M}_0$. Let $i < 2$ be such that Z^i is infinite. Then the condition $d = (\sigma^0, \sigma^1, Z^i)$ is an extension of c such that $d^{[i]} \Vdash (\forall x)\neg\Phi_{e_i}(G, x)$. ☐

By a pairing argument (if for every pair $m, n \in \omega$, $m \in A$ or $n \in B$, then $A = \omega$ or $B = \omega$), if a filter $\mathcal{F}$ is sufficiently generic, there is some side i such that for every Σ^0_1 formula $\varphi(G)$, there is some $c \in \mathcal{F}$ such that $c^{[i]} \Vdash \varphi(G)$ or $c^{[i]} \Vdash \neg\varphi(G)$. We therefore get the following lemma.

Lemma 3.6 For every sufficiently generic $\mathbb{P}_1$-filter $\mathcal{F}$, there is a side $i < 2$ such that $\mathcal{F}^{[i]}$ is a 1-generic $\mathbb{Q}_0$-filter.

3.3. Forcing question for $\mathbb{P}_n$

We now generalize the first jump control of Cholak, Jocksuch and Slaman [2] to iterated jumps with a disjunctive forcing question for Σ^0_{n+1} formulas.

Definition 3.7. Let $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_n$ and let $\Phi_{e_0}(G, x_n, \dots, x_0)$ and $\Phi_{e_1}(G, x_n, \dots, x_0)$ be two Δ_0 formulas. Define the relation

$$c \mathrel{?{\vdash}} (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_0}(G^0, x_n, \dots, x_0) \vee (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_1}(G^1, x_n, \dots, x_0)$$

to hold if for every $Z^0 \cup Z^1 = X$, there is some $i < 2$, some $\rho \subseteq Z^i$ and $x_n \in \omega$ such that

$$\mathcal{U}^{\mathcal{M}_{n-1}}_{C_{n-1}} \cap \mathcal{U}^{\mathcal{M}_{n-1}}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}$$

is not a largeness class.

Lemma 3.8 Let $c \in \mathbb{P}_n$ and let $\Phi_{e_0}(G, x_n, \dots, x_0)$ and $\Phi_{e_1}(G, x_n, \dots, x_0)$ be two Δ_0 formulas. The relation

$$c \mathrel{?{\vdash}} (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_0}(G^0, x_n, \dots, x_0) \vee (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_1}(G^1, x_n, \dots, x_0)$$

is $\Sigma^0_1(\mathcal{M}_n)$.

Proof. By compactness, the relation holds if there is a finite set $E \subseteq X$ such that for every $E_0 \cup E_1 = E$, there is some $i < 2$, some finite set $F \subseteq C_{n-1}$, some $\rho \subseteq E_i$ and $x_n \in \omega$ such that $\mathcal{U}^{\mathcal{M}_{n-1}}_F \cap \mathcal{U}^{\mathcal{M}_{n-1}}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}$ is not a largeness class. By Lemma 2.3, given a finite set F, the statement "$\mathcal{U}^X_F$ is not a largeness class" is $\Sigma^{0,X}_2$, thus $\Sigma^{0,X'}_1$. As $M'_{n-1} \in \mathcal{M}_n$ where M_{n-1} codes for $\mathcal{M}_{n-1}$ thus the overall relation is $\Sigma^0_1(\mathcal{M}_n)$. ☐

Lemma 3.9 Let $c \in \mathbb{P}_n$ and let $\Phi_{e_0}(G, x_n, \dots, x_0)$ and $\Phi_{e_1}(G, x_n, \dots, x_0)$ be two Δ_0 formulas.

(a) If $c \mathrel{?{\vdash}} (\exists x_n) \dots (Qx_0)\Phi_{e_0}(G^0, x_n, \dots, x_0) \vee (\exists x_n) \dots (Qx_0)\Phi_{e_1}(G^1, x_n, \dots, x_0)$, then there is some $d \leq c$ and some $i < 2$ such that

$$d^{[i]} \Vdash (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_i}(G^i, x_n, \dots, x_0)$$

(b) If $c \mathrel{?}\!\nVdash (\exists x_n)\dots(Qx_0)\Phi_{e_0}(G^0, x_n, \dots, x_0) \vee (\exists x_n)\dots(Qx_0)\Phi_{e_1}(G^1, x_n, \dots, x_0)$, then there is some $d \leq c$ and some $i < 2$ such that

$$d^{[i]} \Vdash (\forall x_n)(\exists x_{n-1})\dots(\overline{Q}x_0)\neg\Phi_{e_i}(G^i, x_n, \dots, x_0)$$

Moreover, an index of d can be found $A \oplus P$-uniformly in an index of c, e_0 and e_1.

Proof. Say $c = (\sigma^0, \sigma^1, X)$.

(a) Let $Z^0 = X \cap A^0$ and $Z^1 = X \cap A^1$. Unfolding the definition of the forcing question, there is some $i < 2$, some finite set $F \subseteq C_{n-1}$, some $\rho \subseteq Z^i$ and $x_n \in \omega$ such that

$$\mathcal{U}_F^{\mathcal{M}_{n-1}} \cap \mathcal{U}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}^{\mathcal{M}_{n-1}}$$

is not a largeness class. Since $\mathcal{M}_{n-1} \models \mathsf{WKL}$, there is a cover $R_0 \cup \dots \cup R_{\ell-1} \supseteq \omega$ in $\mathcal{M}_{n-1}$ such that for every $t < \ell$, $R_t \notin \mathcal{U}_F^{\mathcal{M}_{n-1}} \cap \mathcal{U}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}^{\mathcal{M}_{n-1}}$.

Since $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ is a partition regular class containing X, there is some $t < \ell$ such that $X \cap R_t \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. Moreover, since $\{0, \dots, \max\rho\}$ is finite, the set $Y = (X \cap R_t) - \{0, \dots, \max\rho\}$ belongs to $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. Define the $\mathbb{P}_n$-condition $d = (\sigma^i \cup \rho, \sigma^{1-i}, Y)$. In particular,

$$d^{[i]} \Vdash (\exists x_n)(\forall x_{n-1})\dots(Qx_0)\Phi_{e_i}(G, x_n, \dots, x_0)$$

(b) Let $\mathcal{D}$ be the $\Pi^0_1(\mathcal{M}_n)$ class of all $Z^0 \oplus Z^1$ with $Z^0 \cup Z^1 = X$, such that for every $i < 2$, every $\rho \subseteq Z^i$, and every $x_n \in \omega$,

$$\mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \cap \mathcal{U}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}^{\mathcal{M}_{n-1}}$$

is a largeness class. Since $\mathcal{M}_n \models \mathsf{WKL}$, there is some $Z^0 \oplus Z^1 \in \mathcal{D} \cap \mathcal{M}_n$.

Since $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ is a partition regular class containing X, there is some $i < 2$ such that $Z^i \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$.

Define the $\mathbb{P}_n$-condition $d = (\sigma^0, \sigma^1, Z^i)$. Then

$$d^{[i]} \Vdash (\forall x_n)(\exists x_{n-1})\dots(\overline{Q}x_0)\neg\Phi_{e_i}(G, x_n, \dots, x_0)$$

This completes the proof of the lemma. ☐

3.4. **Validity and genericity**

By Lemma 3.9, the disjunctive forcing question ensures that for every sufficiently generic $\mathbb{P}_n$-filter $\mathcal{F}$, there is some $i < 2$ such that $\mathcal{F}^{[i]}$ is $(n+1)$-generic. This is however not sufficient to ensure that the forced formulas will hold. Lemma 2.27 uses the fact that $\mathcal{F}^{[i]}$ is $(s+1)$-generic for every $s < n$. This genericity constraint holds for the side i whenever for every condition $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_n$, $X \cap A^i \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. This motivates the following definition.

Definition 3.10. The side $i < 2$ of a condition $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_n$ is *valid* if $X \cap A^i \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. Otherwise, there is some $\mathcal{U}_e^{\mathcal{M}_{n-1}} \supseteq \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ such that $X \cap A^i \notin \mathcal{U}_e^{\mathcal{M}_{n-1}}$, in which case we say that the side i of c is *e-invalid*.

Given a condition $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_n$, since $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ is a partition regular class containing X, then either $X \cap A^0$ or $X \cap A^1$ belongs to $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. Therefore every condition must have a valid side. However, it is not immediate to see that the $(n+1)$-generic side ensured by the disjunctive forcing question and the valid side of a condition will coincide. It is not very hard to show that if A is Kurtz random relative to M_{n-1}, both sides are valid, and therefore it suffices to choose the generic side. It is however not necessarily the case in general, and the following asymmetric forcing question handles the "degenerate" case where one side is not valid.

In the following definition, the class $\mathcal{U}_{e_{1-i}}^{\mathcal{M}_{n-1}}$ corresponds to a witness that the side $1-i$ is not valid in the condition c. Necessarily, the side i must be valid in c. Thanks to this asymmetric forcing question, there will be able to do some progress in $(n+1)$-genericity on the side i of c.

Definition 3.11. Let $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_n$, $i < 2$, $\mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$ be an upward-closed class and $\Phi_{e_i}(G, x_n, \dots, x_0)$ be a Δ_0 formula. Define the relation

$$c \,?{\vdash} (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_i}(G^i, x_n, \dots, x_0) \vee G^{1-i} \in \mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$$

to hold if for every $Z^0 \cup Z^1 = X$, such that $Z^{1-i} \notin \mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$, there is some $\rho \subseteq Z^i$ and $x_n \in \omega$ such that

$$\mathcal{U}^{\mathcal{M}_{n-1}}_{C_{n-1}} \cap \mathcal{U}^{\mathcal{M}_{n-1}}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}$$

is not a largeness class.

Lemma 3.12 Let $c \in \mathbb{P}_n$ and let $\Phi_{e_0}(G, x_n, \dots, x_0)$ and $\Phi_{e_1}(G, x_n, \dots, x_0)$ be two Δ_0 formulas. The relation

$$c \,?{\vdash} (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_i}(G^i, x_n, \dots, x_0) \vee G^{1-i} \in \mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$$

is $\Sigma^0_1(\mathcal{M}_n)$.

Proof. By compactness, the relation holds if there is a finite set $E \subseteq X$ such that for every $E_0 \cup E_1 = E$, either $E_{1-i} \in \mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$ or there is some finite set $F \subseteq C_{n-1}$, some $\rho \subseteq Z^i$ and $x_n \in \omega$ such that $\mathcal{U}^{\mathcal{M}_{n-1}}_F \cap \mathcal{U}^{\mathcal{M}_{n-1}}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}$ is not a largeness class. By Lemma 2.3, given a finite set F, the statement "$\mathcal{U}^X_F$ is not a largeness class" is $\Sigma^{0,X}_2$, thus the overall relation is $\Sigma^0_1(\mathcal{M}_n)$. ☐

Lemma 3.13 Let $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_n$ and $i < 2$ be such that the side $1 - i$ of c is e_{1-i}-invalid. Let $\Phi_{e_i}(G, x_n, \dots, x_0)$ be a Δ_0 formula.

(a) If $c \,?{\vdash} (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_i}(G^i, x_n, \dots, x_0) \vee G^{1-i} \in \mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$, then there some $d \leq c$ such that

$$d^{[i]} \Vdash (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_i}(G^i, x_n, \dots, x_0)$$

(b) If $c \,?{\nvdash} (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_i}(G^i, x_n, \dots, x_0) \vee G^{1-i} \in \mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$, then there is some $d \leq c$ such that

$$d^{[i]} \Vdash (\forall x_n)(\exists x_{n-1}) \dots (\overline{Q}x_0)\neg\Phi_{e_i}(G^i, x_n, \dots, x_0)$$

Moreover, an index of d can be found $A \oplus P$-uniformly in an index of c, e_0 and e_1.

Proof. Say $c = (\sigma^0, \sigma^1, X)$.

(a) Let $Z^0 = X \cap A^0$ and $Z^1 = X \cap A^1$. Since the side $1 - i$ of c is e_{1-i}-invalid, then $Z^{1-i} \notin \mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$, so unfolding the definition of the forcing question, there is some finite set $F \subseteq C_{n-1}$, some $\rho \subseteq Z^i$ and $x_n \in \omega$ such that

$$\mathcal{U}^{\mathcal{M}_{n-1}}_F \cap \mathcal{U}^{\mathcal{M}_{n-1}}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}$$

is not a largeness class. Since $\mathcal{M}_{n-1} \models \mathsf{WKL}$, there is a cover $R_0 \cup \dots \cup R_{\ell-1} \supseteq \omega$ in $\mathcal{M}_{n-1}$ such that for every $t < \ell$, $R_t \notin \mathcal{U}^{\mathcal{M}_{n-1}}_F \cap \mathcal{U}^{\mathcal{M}_{n-1}}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}$.

Since $\langle \mathcal{U}^{\mathcal{M}_{n-1}}_{C_{n-1}} \rangle$ is a partition regular class containing X, there is some $t < \ell$ such that $X \cap R_t \in \langle \mathcal{U}^{\mathcal{M}_{n-1}}_{C_{n-1}} \rangle$. Moreover, since $\{0, \dots, \max \rho\}$ is finite, the set $Y = (X \cap R_t) - \{0, \dots, \max \rho\}$ belongs to $\langle \mathcal{U}^{\mathcal{M}_{n-1}}_{C_{n-1}} \rangle$. Define the $\mathbb{P}_n$-condition $d = (\sigma^i \cup \rho, \sigma^{1-i}, Y)$. In particular,

$$d^{[i]} \Vdash (\exists x_n)(\forall x_{n-1}) \dots (Qx_0)\Phi_{e_i}(G, x_n, \dots, x_0)$$

(b) Let $\mathcal{D}$ be the $\Pi^0_1(\mathcal{M}_n)$ class of all $Z^0 \oplus Z^1$ with $Z^0 \cup Z^1 = X$, such that $Z^{1-i} \notin \mathcal{U}^{\mathcal{M}_{n-1}}_{e_{1-i}}$ and for every $\rho \subseteq Z^i$, and every $x_n \in \omega$,

$$\mathcal{U}^{\mathcal{M}_{n-1}}_{C_{n-1}} \cap \mathcal{U}^{\mathcal{M}_{n-1}}_{\zeta_{n-1}(e_i, \sigma^i \cup \rho, x_n)}$$

is a largeness class. Since $\mathcal{M}_n \models \mathsf{WKL}$, there is some $Z^0 \oplus Z^1 \in \mathcal{D} \cap \mathcal{M}_n$.

Since $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ is a partition regular class containing X and $Z^{1-i} \not\in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$, then $Z^i \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$.

Define the $\mathbb{P}_n$-condition $d = (\sigma^0, \sigma^1, Z^i)$. Then

$$d^{[i]} \Vdash (\forall x_n)(\exists x_{n-1}) \dots (\overline{Q} x_0) \neg \Phi_{e_i}(G, x_n, \dots, x_0)$$

This completes the proof of the lemma. ☐

Contrary to the fact that every 2-generic $\mathbb{Q}_n$-filter is 1-generic, it is not clear that every 3-generic $\mathbb{Q}_n$-filter is 2-generic. The following lemma states that whenever $\mathcal{F}$ is a sufficiently generic $\mathbb{P}_n$-filter, then if a side $i < 2$ is valid, the $\mathbb{Q}_n$-filter $\mathcal{F}^{[i]}$ is $(s+1)$-generic for every $s < n$. By Lemma 2.25, it would be sufficient to prove the following lemma for $s \in \{1, \dots, n-1\}$, but the proof also holds for the case $s = 0$.

Lemma 3.14 Let $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_n$ and $i < 2$ be such that the side i of c is valid. Fix $s \in \{0, \dots, n-1\}$ and let $\Phi_e(G, x_s, \dots, x_0)$ be a Δ_0 formula with free variables $x_s, \dots, x_0$. Then there is an extension $d \leq c$ such that either

$$d^{[i]} \Vdash (\exists x_s)(\forall x_{s-1}) \dots (Q x_0) \Phi_e(G, x_s, \dots, x_0)$$

or

$$d^{[i]} \Vdash (\forall x_s)(\exists x_{s-1}) \dots (\overline{Q} x_0) \neg \Phi_e(G, x_s, \dots, x_0)$$

Moreover, an index for d can be found $A \oplus P$-uniformly from an index for c and e.

Proof. Let $\mathcal{U}_a^{\mathcal{M}_s}$ be the upward closed $\Sigma_1^0(\mathcal{M}_s)$ class of all X such that

$$(\sigma^i, X) \nVdash (\forall x_s)(\exists x_{s-1}) \dots (\overline{Q} x_0) \neg \Phi_e(G, x_s, \dots, x_0)$$

We have two cases.

Case 1: $\mathcal{U}_{C_s}^{\mathcal{M}_s} \cap \mathcal{U}_a^{\mathcal{M}_s}$ is a largeness class. Then $\langle \mathcal{U}_{C_s}^{\mathcal{M}_s} \rangle \subseteq \mathcal{U}_a^{\mathcal{M}_s}$. Since $X \cap A^i \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle \subseteq \langle \mathcal{U}_{C_s}^{\mathcal{M}_s} \rangle \subseteq \mathcal{U}_a^{\mathcal{M}_s}$, then in particular $X \cap A^i \in \mathcal{U}_a^{\mathcal{M}_s}$. Unfolding the definition, in case $s = 0$, there is some $\rho \subseteq X \cap A^i$ and some $x_0 \in \omega$ such that $\Phi_e(\sigma \cup \rho, x_0)$ holds. Since ρ is finite, the set $Y = X - \{0, \dots, |\rho|\}$ belongs to $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. Thus $d = (\sigma^i \cup \rho, \sigma^{1-i}, Y)$ is an extension of c such that $d^{[i]} \Vdash (\exists x_0) \Phi_e(G, x_0)$. In case $s > 0$, there is some $\rho \subseteq X \cap A^i$ and some $x_s \in \omega$ such that $\mathcal{U}_{C_{s-1}}^{\mathcal{M}_{s-1}} \cap \mathcal{U}_{\zeta_{s-1}(e, \sigma^i \cup \rho, x_s)}$ is not a largeness class. Let $R_0, \dots, R_{\ell-1}$ be a cover of ω in $\mathcal{M}_{s-1}$ such that for every $t < \ell$, $R_t \not\in \mathcal{U}_{C_{s-1}}^{\mathcal{M}_{s-1}} \cap \mathcal{U}_{\zeta_{s-1}(e, \sigma^i \cup \rho, x_s)}$. Since $X \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$, which is a partition regular class, there is some $t < \ell$ such that $X \cap R_t \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. Let $Y = X \cap R_t - \{0, \dots, |\rho|\}$. Since ρ is finite, $Y \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. Then $d = (\sigma^i \cup \rho, \sigma^{1-i}, Y)$ is an extension of c such that $d^{[i]} \Vdash (\exists x_s)(\forall x_{s-1}) \dots (Q x_0) \Phi_e(G, x_s, \dots, x_0)$.

Case 2: $\mathcal{U}_{C_s}^{\mathcal{M}_s} \cap \mathcal{U}_a^{\mathcal{M}_s}$ is not a largeness class. Let $R_0, \dots, R_{\ell-1}$ be a cover of ω in $\mathcal{M}_s$ such that for every $t < \ell$, $R_t \not\in \mathcal{U}_{C_s}^{\mathcal{M}_s} \cap \mathcal{U}_a^{\mathcal{M}_s}$. Since $X \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$, which is a partition regular class, there is some $t < \ell$ such that $X \cap R_t \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. Then $d = (\sigma^0, \sigma^1, X \cap R_t)$ is an extension of c such that $d^{[i]} \Vdash (\forall x_s)(\exists x_{s-1}) \dots (\overline{Q} x_0) \neg \Phi_e(G, x_s, \dots, x_0)$. ☐

The following lemma states that whenever $\mathcal{F}$ is a sufficiently generic $\mathbb{P}_n$-filter, then if a side $i < 2$ is valid, letting $\mathcal{G} = \mathcal{F}^{[i]}$, the set $G_{\mathcal{G}}$ belongs to $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$.

Lemma 3.15 Let $c = (\sigma^0, \sigma^1, X) \in \mathbb{P}_n$ and $i < 2$ be such that the side i of c is valid. Let D be such that $\mathcal{U}_D^{\mathcal{M}_{n-1}} = \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ and fix $\langle e, i \rangle \in D$. Then there is an extension $d = (\tau^0, \tau^1, Y) \leq c$ such that $\tau^i \in \mathcal{U}_e^{X_i}$. Moreover, an index for d can be found $A \oplus P$-uniformly from an index for c, e and i.

Proof. Since the side i of c is valid, then $X \cap A^i \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle \subseteq \mathcal{U}_e^{X_i}$. Thus there is a finite set $\rho \subseteq X \cap A^i$ such that $\rho \in \mathcal{U}_e^{X_i}$. By upward closure of $\mathcal{U}_e^{X_i}$, $\sigma^i \cup \rho \in \mathcal{U}_e^{X_i}$. By Corollary 2.11,

$\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ is a partition regular class, so $X - \{0, \dots, \max\rho\} \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. The condition $d = (\sigma^i \cup \rho, \sigma^{1-i}, X - \{0, \dots, \max\rho\})$ is the desired extension. □

4. Applications

In this section, we apply the framework developed in section 3 to derive our main theorems.

4.1. Preservation of non-Σ^0_n definitions

Our first application shows the existence, for every instance of the pigeonhole principle, of a solution which does not collapse the definition of a non-Σ^0_n set into a Σ^0_n one. This corresponds to preservation of one non-Σ^0_n definition, following the terminology of Wang [31].

Theorem 4.1 Fix $n \geq 1$ and let B be a non-Σ^0_n set. For every set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that B is not $\Sigma^{0,H}_n$.

Fix B and A, and let $A^0 = \overline{A}$ and $A^1 = A$. As in Section 3, let $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_n$ be countable Scott sets coded by sets $M_0, M_1, \dots, M_n$, respectively. Again, assume that M_i is low over $\emptyset^{(i)}$ and $\emptyset^{(i+1)} \in \mathcal{M}_{i+1}$ for every $i < n$. Let $C_0, \dots, C_{n-1}$ be such that $C_i \in \mathcal{M}_{i+1}$ and $\mathcal{U}_{C_i}^{\mathcal{M}_i}$ is an $\mathcal{M}_i$-cohesive largeness class for every $i < n$. Furthermore, we require that $\mathcal{U}_{C_{i+1}}^{\mathcal{M}_{i+1}} \subseteq \langle \mathcal{U}_{C_i}^{\mathcal{M}_i} \rangle$. By Wang [31, Theorem 3.6.], we can also assume that B is not $\Sigma^0_1(\mathcal{M}_n)$. We build our infinite set by the notion of forcing $\mathbb{P}_n$.

Fix an enumeration $\varphi_0(G, u), \varphi_1(G, u)$ of all Σ^0_2 formulas with one set parameter G and one integer parameter u.

Lemma 4.2 Let $c \in \mathbb{P}_n$. For every pair of Σ^0_n formulas $\varphi_0(G, u)$ and $\varphi_1(G, u)$, there is some $i < 2$ and some $d \leq c$ such that

$$(\exists u \notin B) d^{[i]} \Vdash \varphi_i(G, u) \quad \text{or} \quad (\exists u \in B) d^{[i]} \Vdash \neg\varphi_i(G, u)$$

Proof. Let $W = \{u : c \mathrel{?\vdash} \varphi_0(G^0, u) \vee \varphi_1(G^1, u)\}$. By Lemma 3.8, the set W is $\Sigma^0_1(\mathcal{M}_n)$ but B is not, therefore $W \neq B$. Let $u \in W \Delta B = (W - B) \cup (B - W)$. We have two cases.

Case 1: $u \in W - B$, then $c \mathrel{?\vdash} \varphi_0(G, u) \vee \varphi_1(G, u)$. By Lemma 3.9(a), there is an extension d of c such that $d^{[i]} \Vdash \varphi_i(G, u)$ for some $i < 2$.

Case 2: $u \in B - W$, then $c \mathrel{?\nvdash_s} \varphi_0(G, u) \vee \varphi_1(G, u)$. By Lemma 3.9(b), there is an extension d of c such that $d^{[i]} \Vdash \neg\varphi_i(G, u)$ for some $i < 2$. □

Lemma 4.3 Let $c \in \mathbb{P}_n$ and $i < 2$ be such that the side $1 - i$ of c is e-invalid. For every Σ^0_n formula $\varphi(G, u)$, there is some $d \leq c$ such that

$$(\exists u \notin B) d^{[i]} \Vdash \varphi(G, u) \quad \text{or} \quad (\exists u \in B) d^{[i]} \Vdash \neg\varphi(G, u)$$

Proof. Let $W = \{u : c \mathrel{?\vdash} \varphi(G^i, u) \vee G^{1-i} \notin \mathcal{U}_e^{\mathcal{M}_{n-1}}\}$. By Lemma 3.12, the set W is $\Sigma^0_1(\mathcal{M}_n)$ but B is not, therefore $W \neq B$. Let $u \in W \Delta B = (W - B) \cup (B - W)$. We have two cases.

Case 1: $u \in W - B$, then $c \mathrel{?\vdash} \varphi(G^i, u) \vee G^{1-i} \notin \mathcal{U}_e^{\mathcal{M}_{n-1}}$. By Lemma 3.13(a), there is an extension d of c such that $d^{[i]} \Vdash \varphi(G, u)$.

Case 2: $u \in B - W$, then $c \mathrel{?\nvdash_s} \varphi(G^i, u) \vee G^{1-i} \notin \mathcal{U}_e^{\mathcal{M}_{n-1}}$. By Lemma 3.13(b), there is an extension d of c such that $d^{[i]} \Vdash \neg\varphi(G, u)$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\mathcal{F}$ be a sufficiently generic $\mathbb{P}_n$-filter. By Lemma 4.2 and Lemma 4.3, there is some $i < 2$ such that

(a) The side i of c is valid for every $c \in \mathcal{F}$;
(b) For every Σ^0_n formula $\varphi(G, u)$, there is some $d \in \mathcal{F}$ such that

$$(\exists u \notin B) d^{[i]} \Vdash \varphi(G, u) \quad \text{or} \quad (\exists u \in B) d^{[i]} \Vdash \neg\varphi(G, u)$$

Let $\mathcal{G} = \mathcal{F}^{[i]}$; In particular, $\mathcal{G}$ is an $(n+1)$-generic $\mathbb{Q}_n$-filter. By Lemma 3.14, $\mathcal{G}$ is $(s+1)$-generic for every $s \in \{0, \dots, n-1\}$ and by Lemma 2.25. By Lemma 2.24, $G_\mathcal{G}$ is infinite, and by Lemma 2.27, B is not $\Sigma^{0,H}_n$. By definition of $\mathbb{P}_n$, $G_\mathcal{G} \subseteq A^i$. This completes the proof of Theorem 4.1. □

The following corollary would correspond to strong iterated jump cone avoidance of RT^1_2, following the terminology of Wang [32].

Corollary 4.4 Fix a non-Δ^0_n set B. For every set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that B is not $\Delta^{0,H}_n$.

Proof. Given a non-Δ^0_n set B, either B or $\overline{B}$ is not Σ^0_n. By Theorem 4.1, for every set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that either B or $\overline{B}$ is not $\Sigma^{0,H}_n$, hence such that B is not $\Delta^{0,H}_n$. □

4.2. Preservation of Δ^0_n hyperimmunities

Our second application concerns the ability to prevent solutions from computing fast-growing functions. Recall the definition of hyperimmunity.

Definition 4.5. A function f *dominates* a function g if $f(x) \geq g(x)$ for every x. A function f is *X-hyperimmune* if it is not dominated by any X-computable function.

The following lemma is proven by Downey et al. [5, Lemma 3.3].

Lemma 4.6 ([5]) For every $k \leq \omega$ and every Z, for any nondecreasing functions $(f_i)_{i<k}$ which are Z-hyperimmune, there is a G and sets $(A_i)_{i<k}$ such that none of the A_i is $\Sigma^0_1(Z \oplus G)$, but for any i and any function h dominating f_i, A_i is $\Sigma^0_1(Z \oplus G \oplus h)$.

Theorem 4.7 Fix a $\emptyset^{(n)}$-hyperimmune function f. For every set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that f is $H^{(n)}$-hyperimmune.

Proof. By Lemma 4.6, letting $Z = \emptyset^{(n)}$, there is a set G and a set B such that B is not $\Sigma^0_1(\emptyset^{(n)} \oplus G)$ but for any function h dominating f, B is $\Sigma^0_1(\emptyset^{(n)} \oplus G \oplus h)$. By the jump inversion theorem, there is a set Q such that $Q^{(n)} \equiv_T \emptyset^{(n)} \oplus G$. In particular, B is not $\Sigma^0_1(Q^{(n)})$, so it is not $\Sigma^0_{n+1}(Q)$. By Theorem 4.1, there is an infinite set $H \subseteq H$ or $H \subseteq \overline{A}$ such that B is not $\Sigma^0_{n+1}(H \oplus Q)$. In particular B is not $\Sigma^0_1((H \oplus Q)^{(n)}$ and therefore not $\Sigma^0_1(H^{(n)} \oplus G)$. Suppose for the contradiction that f is dominated by an $H^{(n)}$-computable function h. Then B is $\Sigma^0_1(\emptyset^{(n)} \oplus G \oplus h)$, hence B is $\Sigma^0_1(H^{(n)} \oplus \oplus G)$. Contradiction. □

4.3. Low$_n$ solutions

An effectivization of the forcing construction enables us to obtain lowness results for the infinite pigeonhole principle. The existence of low$_2$ solutions for Δ^0_2 sets, and of low$_2$ cohesive sets for computable sequences of sets, was proven by Cholak, Jockusch and Slaman [2, sections 4.1 and 4.2]. The existence of low$_3$ cohesive sets for Δ^0_2 sequences of sets was proven by Wang [30, Theorem 3.4]. Wang [30, Questions 6.1 and 6.2] and the second author [18, Question 5.4] asked whether such results can be generalized for every Δ^0_{n+1} instances of the pigeonhole and every Δ^0_n instances of cohesiveness. We answer positively both questions.

Theorem 4.8 For every Δ^0_{n+2} set A and every $P \gg \emptyset^{(n+1)}$, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that $H^{(n+1)} \leq_T P$.

Proof. The case $n = 0$ is proven by Cholak, Jockusch and Slaman [2, sections 4.1 and 4.2]. Suppose $n > 0$. Fix P and A, and let $A^0 = \overline{A}$ and $A^1 = A$. As in Section 3, let $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_n$ be countable Scott sets coded by sets $M_0, M_1, \dots, M_n$, respectively. Again, assume that M_i is low over $\emptyset^{(i)}$ and $\emptyset^{(i+1)} \in \mathcal{M}_{i+1}$ for every $i < n$. We also require M_n to be low over $\emptyset^{(n)}$. Let

$C_0, \dots, C_{n-1}$ be such that $C_i \in \mathcal{M}_{i+1}$ and $\mathcal{U}_{C_i}^{\mathcal{M}_i}$ is an $\mathcal{M}_i$-cohesive largeness class for every $i < n$. Furthermore, we require that $\mathcal{U}_{C_{i+1}}^{\mathcal{M}_{i+1}} \subseteq \langle \mathcal{U}_{C_i}^{\mathcal{M}_i} \rangle$.

Note that by Corollary 2.15, the class $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ is $\Pi^0_2(C_{n-1} \oplus M'_{n-1})$. Since $C_{n-1} \oplus M'_{n-1} \leq_T M_n$, $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ is $\Pi^0_2(M_n)$. If $X \in M_n$, the relation "$X \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$" is $\Pi^0_1((M_n)')$, hence $\Pi^0_1(\emptyset^{(n+1)})$. Therefore, P computes a total function $h : \omega^2 \to 2$ such that if e_0 and e_1 are M_n-indices of X_{e_0} and X_{e_1} such that $X_{e_0} \cup X_{e_1} \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$, then $h(e_0, e_1)$ is some $i < 2$ such that $X_{e_i} \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. We have two constructions, based on whether every condition have both valid sides or not.

Symmetric case. Suppose that for every $\mathbb{P}_n$-condition $c = (\sigma^0, \sigma^1, X)$ both sides are valid, that is, for every $i < 2$, $X \cap A^i \in \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$.

Define an infinite decreasing sequence of $\mathbb{P}_n$-conditions $c_0 \geq c_1 \geq \dots$ such that for every s, there is some $i < 2$ such that

$$c_s^{[i]} \Vdash (\exists x_t)(\forall x_{t-1}) \dots (Q x_0) \Phi_s(G, x_n, \dots, x_0) \text{ or } c_s^{[i]} \Vdash (\forall x_n)(\exists x_{n-1}) \dots (\overline{Q} x_0) \neg \Phi_s(G, x_n, \dots, x_0)$$

By Lemma 3.9, this sequence can be $A \oplus P$-computable, hence P-computable. Moreover, we require that for every $t \in \{1, \dots, n\}$ and every $i < 2$,

$$c_s^{[i]} \Vdash (\exists x_t)(\forall x_{t-1}) \dots (Q x_0) \Phi_e(G, x_t, \dots, x_0) \text{ or } c_s^{[i]} \Vdash (\forall x_t)(\exists x_{t-1}) \dots (\overline{Q} x_0) \neg \Phi_e(G, x_t, \dots, x_0)$$

This can be ensured by Lemma 3.14 and the assumption that both sides of every $\mathbb{P}_n$-condition are valid. By a pairing argument, there is some $i < 2$ such that the upward-closure $\mathcal{G}$ of the collection $\{c_s^{[i]} : s \in \omega\}$ is an $(n+1)$-generic $\mathbb{Q}_n$-filter. Moreover, $\mathcal{G}{\upharpoonright}_{\mathbb{Q}_t}$ is $(t+1)$-generic for every $t < n$. By Lemma 2.24, $G_\mathcal{G}$ is infinite, and by definition of a $\mathbb{P}_n$-condition, $G_\mathcal{G} \subseteq A^i$. Moreover, by Lemma 2.27, $\varphi(G_\mathcal{G})$ holds for a Σ^0_{n+1} (Π^0_{n+1}) formula φ if and only if there is some stage s such that $c_s^{[i]} \Vdash \varphi(G)$. Therefore, to decide $\varphi(G_\mathcal{G})$, one can search P-effectively for some s such that $c_s^{[i]} \Vdash \varphi(G)$ or $c_s^{[i]} \Vdash \neg\varphi(G)$ and decide in which case we are. It follows that $G_\mathcal{G}^{(n+1)} \leq P$. This completes the symmetric construction.

Asymmetric case. Suppose that there is a $\mathbb{P}_n$-condition $c_0 = (\sigma^0, \sigma^1, X)$ and a side $i < 2$ such that the side $1-i$ is e-invalid for some $e \in \omega$, that is, $X \cap A^{1-i} \notin \mathcal{U}_e^{\mathcal{M}_{n-1}} \supseteq \langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$. The construction is very similar to the symmetric case. However, we can already fix the side i by using the asymmetric forcing question. Thanks to Lemma 3.13, we can define an infinite decreasing sequence of $\mathbb{P}_n$-conditions $c_0 \geq c_1 \geq \dots$ such that letting $\mathcal{G}$ be the upward-closure of the collection $\{c_s^{[i]} : s \in \omega\}$, $\mathcal{G}$ is an $(n+1)$-generic $\mathbb{P}_n$-filter which is $(t+1)$-generic for every $t < n$. The verification is the same as in the symmetric case. This completes the proof of Theorem 4.8. ☐

Corollary 4.9 Fix $n \geq 1$. For every Δ^0_n set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ of low$_n$ degree.

Proof. This is trivially true for $n = 1$. We prove it in the case $n \geq 2$. By the relativized low basis theorem [10], there is some $P \gg \emptyset^{(n-1)}$ such that $P' \leq_T \emptyset^{(n)}$. By Theorem 4.8, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that $H^{(n-1)} \leq_T P$. In particular, $H^{(n)} \leq_T P' \leq_T \emptyset^{(n)}$. ☐

4.4. **Arithmetical reductions**

We now prove that the infinite pigeonhole principle admits strong cone avoidance for arithmetical reductions.

Theorem 4.10 Let B be a non-arithmetical set. For every set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that B is not arithmetical in H.

Fix B and A, and let $A^0 = \overline{A}$ and $A^1 = A$. Let $\mathcal{M}_0, \mathcal{M}_1, \dots$ be a countable sequence of countable Scott sets coded by sets $M_0, M_1, \dots$, respectively. Assume that M_i is low over $\emptyset^{(i)}$

and $\emptyset^{(i+1)} \in \mathcal{M}_{i+1}$ for every $i \in \omega$. Let $C_0, C_1, \dots$ be such that $C_i \in \mathcal{M}_{i+1}$ and $\mathcal{U}_{C_i}^{\mathcal{M}_i}$ is an $\mathcal{M}_i$-cohesive largeness class for every $i \in \omega$. Furthermore, we require that $\mathcal{U}_{C_{i+1}}^{\mathcal{M}_{i+1}} \subseteq \langle \mathcal{U}_{C_i}^{\mathcal{M}_i} \rangle$.

Let $\mathcal{A} = \bigcap_n \mathcal{U}_{C_n}^{\mathcal{M}_n}$. Note that $\mathcal{A}$ is a largeness class by Lemma 2.2 and that $\mathcal{L}(\mathcal{A})$ is the largest partition regular subclass of $\mathcal{A}$ by Lemma 2.10. Consider the following notion of forcing:

Definition 4.11. Let $\mathbb{P}_\omega$ denote the set of conditions (σ^0, σ^1, X) such that

(a) $\sigma^i \subseteq A^i$ for every $i < 2$
(b) $X \cap \{0, \dots, \max_i |\sigma^i|\} = \emptyset$
(c) $X \in \mathcal{L}(\mathcal{A})$.
(d) $X \in \bigcup_n \mathcal{M}_n$

Note that $\mathbb{P}_\omega \subseteq \bigcup_n \mathbb{P}_n$. The partial order on $\mathbb{P}_\omega$ is the standard Mathias extension. All the proofs remain the same, except the replacement of $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$ by $\mathcal{L}(\mathcal{A})$ whenever one has pick a part of a cover of a set belonging to $\langle \mathcal{U}_{C_{n-1}}^{\mathcal{M}_{n-1}} \rangle$.

We define $\mathbb{Q}_\omega$ similarly, and let $c^{[i]} = (\sigma^i, X) \in \mathbb{Q}_\omega$.

Proof of Theorem 4.10. Let $\mathcal{F}$ be a sufficiently generic $\mathbb{P}_\omega$-filter. Note that for every $c \in \mathbb{P}_\omega$, there is some $n \in \omega$ such that $c \in \mathbb{P}_n$. In particular, B is not $\Sigma_1^0(\mathcal{M}_n)$, so we can apply Lemma 4.2 and Lemma 4.3. Moreover, every Σ_n^0 formula can be seen as a Σ_m^0 formula with $m \geq n$ by adding dummy quantifiers. Therefore there is some $i < 2$ such that

(a) The side i of c is valid for every $c \in \mathcal{F}$;
(b) For every $n \in \omega$ and every Σ_n^0 formula $\psi(G, u)$, there is some $m \geq n$, some Σ_{m+1}^0 formula $\varphi(G, u)$ logically equivalent to $\psi(G, u)$ and some $d \in \mathcal{F} \cap \mathbb{P}_m$ such that

$$(\exists u \notin B) d^{[i]} \Vdash \varphi(G, u) \vee (\exists u \in B) d^{[i]} \Vdash \neg\varphi(G, u)$$

Let $\mathcal{G} = \mathcal{F}^{[i]}$. In particular, $\mathcal{G}$ is an $(n+1)$-generic $\mathbb{Q}_\omega$-filter. By Lemma 3.14, $\mathcal{G}$ is $(s+1)$-generic for every $s \in \{1, \dots, n-1\}$ and by Lemma 2.25, $\mathcal{G}$ is 1-generic. By Lemma 2.24, $G_\mathcal{G}$ is infinite, and by Lemma 2.27, B is not $\Sigma_n^{0,H}$ for any $n \in \omega$. By definition of $\mathbb{P}_\omega$, $G_\mathcal{G} \subseteq A^i$. This completes the proof of Theorem 4.10. ☐

5. Hyperarithmetical reductions

In this section, we extend the jump control of solutions to the pigeonhole principle to ordinal iterations of the jump. We then derive a proof of strong cone avoidance for hyperarithmetical reductions.

5.1. Background

5.1.1. *Computable ordinals.* We let ω_1^{ck} denote the first non-computable ordinal. There is a Π_1^1 set $\mathcal{O}_1 \subseteq \omega$ such that each $o \in \mathcal{O}_1$ codes for an ordinal $\alpha < \omega_1^{ck}$ and each ordinal $\alpha < \omega_1^{ck}$ has a unique code in $\mathcal{O}_1$. Furthermore given that $o \in \mathcal{O}_1$, one can computably recognize if o codes for 0, if o codes for a successor ordinal $\alpha + 1$, in which case we can uniformly and computably produce a code in $\mathcal{O}_1$ for α, and if o codes for a limit ordinal $\sup_n \beta_n$, in which case we can uniformly and computably produce for each n codes in $\mathcal{O}_1$ for β_n. See [23] for more details about $\mathcal{O}_1$. In this section, we manipulate each ordinal $\alpha < \omega_1^{ck}$ via its respective code in $\mathcal{O}_1$. To simplify the reading, we use the notation α instead of the code for α.

5.1.2. *The effective Borel sets.* We also use codes for effective Borel subsets of ω or of 2^ω : For $\alpha < \omega_1^{ck}$ a code for a $\Sigma_{\alpha+1}^0$ set $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ is the code of a function that effectively enumerate codes for each Π_α^0 set $\mathcal{B}_n$. A code for a $\Pi_{\alpha+1}^0$ set $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_n$ is the code of a function that effectively enumerate codes for each Σ_α^0 set $\mathcal{B}_n$. For $\alpha = \sup_n \beta_n$ limit a code of a Σ_α^0 set $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ is the code of a function that effectively enumerate codes for each $\Pi_{\beta_n}^0$ set $\mathcal{B}_{\beta_n}$ with $\sup_n \beta_n = \alpha$. The code of a Π_α^0 set $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$ is the code of a function that effectively enumerate codes for each $\Sigma_{\beta_n}^0$ set $\mathcal{B}_{\beta_n}$ with $\sup_n \beta_n = \alpha$. We also assume the codes for effective Borel sets include some information so that we can computably distinguish Π_α^0 from Σ_α^0 codes as well as distinguish if $\alpha = 1$, if α is successor or if it is limit.

5.1.3. *The iterated jumps.* We use such codes to iterate the jump through the ordinals:

(1) $\emptyset^{(0)} = \emptyset$
(2) $\emptyset^{(\alpha+1)} = (\emptyset^{(\alpha)})'$
(3) $\emptyset^{(\sup_n \alpha_n)} = \oplus_{n\in\omega}\emptyset^{(\alpha_n)}$

Note that for $n < \omega$ the set $\emptyset^{(n)}$ is Σ^0_n and complete for Σ^0_n questions. Above the first limit ordinal the situation is slightly different : $\emptyset^{(\omega)}$ is Δ^0_ω and not Σ^0_ω. Also given $\alpha \geq \omega$ we have that $\emptyset^{(\alpha+1)}$ is Σ^0_α and complete for Σ^0_α questions.

Proposition 5.1 Let $n \in \omega$.

(1) Let $m > 0$. The set $\{X \ : \ n \in X^{(m)}\}$ is a Σ^0_m class.
(2) Let α be limit. The set $\{X \ : \ n \in X^{(\alpha)}\}$ is a Δ^0_β class for some $\beta < \alpha$.
(3) Let $\alpha = \beta + 1$ with $\beta \geq \omega$. The set $\{X \ : \ n \in X^{(\alpha)}\}$ is a Σ^0_β class.

Proof. The set $\{X \ : \ n \in X'\}$ is clearly Σ^0_1. Let $m > 1$. the set $\{X \ : \ n \in X^{(m)}\}$ equals

$$\bigcup_{\{\sigma \ : \ \Phi_n(\sigma,n)\downarrow\}} \bigcap_{\{i \ : \ \sigma(i)=0\}} \{X \ : \ i \notin X^{(m-1)}\} \cap \bigcap_{\{i \ : \ \sigma(i)=1\}} \{X \ : \ i \in X^{(m-1)}\}$$

This is by induction a Σ^0_m set.

Let α be limit. Let p_1, p_2 be projections of the pairing function, that is, $x = \langle p_1(x), p_2(x)\rangle$. Then $\{X \ : \ n \in X^{(\alpha)}\}$ equals $\{X \ : \ p_1(n) \in X^{(p_2(n))}\}$, which is a Δ^0_β set for $\beta < \alpha$.

Let $\alpha = \beta + 1$. The set $\{X \ : \ n \in X^{(\beta+1)}\}$ equals

$$\bigcup_{\{\sigma \ : \ \Phi_n(\sigma,n)\downarrow\}} \bigcap_{\{i \ : \ \sigma(i)=0\}} \{X \ : \ i \notin X^{(\beta)}\} \cap \bigcap_{\{i \ : \ \sigma(i)=1\}} \{X \ : \ i \in X^{(\beta)}\}$$

This is by induction a Σ^0_β class. □

Proposition 5.2 Let Φ be a functional. Let $n, i \in \omega$.

(1) Let $m > 0$. The set $\{X \ : \ \exists t \ \Phi(X^{(m)}, n)[t] \downarrow= i\}$ is a Σ^0_{m+1} class.
(2) Let $\alpha \geq \omega$. The set $\{X \ : \ \exists t \ \Phi(X^{(\alpha)}, n)[t] \downarrow= i\}$ is a Σ^0_α class.

Proof. Trivial using Proposition 5.1 □

5.1.4. Π^1_1 *and* Σ^1_1 *sets of integers.* We previously mentioned a Π^1_1 set $\mathcal{O}_1$ of unique notations for ordinals. This set is included in Kleene's $\mathcal{O}$, the set of all the constructible codes for the computable ordinals. Given an ordinal $\alpha < \omega_1^{ck}$, let $\mathcal{O}_{<\alpha}$ denote the elements of $\mathcal{O}$ which code for an ordinal strictly smaller than α. Each $\mathcal{O}_{<\alpha}$ is Δ^1_1 uniformly in α (it actually is always a $\Sigma^0_{\alpha+1}$ set [14]). It is well-known that $\mathcal{O}$ is a Π^1_1-complete set [23], that is, for any Π^1_1 set $B \subseteq \omega$ there is a computable function $f : \omega \to \omega$ such that $n \in B \leftrightarrow f(n) \in \mathcal{O}$. Let us define $B_\alpha = \{n \ : \ f(n) \in \mathcal{O}_{<\alpha}\}$. In particular, each B_α is Δ^1_1 uniformly in α and $B = \bigcup_{\alpha<\omega_1^{ck}} B_\alpha$. In particular B is a $\Sigma^0_{\omega_1^{ck}}$ set. Note that contrary to Σ^0_α sets for $\alpha < \omega_1^{ck}$, the $\Sigma^0_{\omega_1^{ck}}$ are not described with a computable code, but rather with a Π^1_1 set of codes for all the Π^0_α that constitutes the $\Sigma^0_{\omega_1^{ck}}$ set B. With a little hack, we can even make sure that at most one new element appears in each B_α. For this reason, we often see Π^1_1 sets as enumerable along the computable ordinals.

By complementation a Σ^1_1 set $B \subseteq \omega$ can be seen as co-enumerable along the computable ordinals and we have $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ where each B_α is Δ^1_1 uniformly in α. We also say in this case that B is $\Pi^0_{\omega_1^{ck}}$.

5.1.5. Σ^1_1-*boundedness.* A central theorem when working with Σ^1_1 and Π^1_1 sets is Σ^1_1-boundedness:

Theorem 5.3 (Σ^1_1-boundedness [26]) Let B be a Σ^1_1 set of codes for ordinals, then the supremum of the ordinals coded by elements of B is strictly smaller than ω_1^{ck}.

We mostly here use the following corollary:

Corollary 5.4 Let $f : \omega \to \omega_1^{ck}$ be a total Π^1_1 function. Then $\sup_n f(n) = \alpha < \omega_1^{ck}$.

Note that $f : \omega \to \omega_1^{ck}$ means the range of f is a subset of $\mathcal{O}_1$. The corollary comes from the fact that if f is total, then it becomes Δ^1_1 and its range is then a Σ^1_1 set of codes for ordinals. As an example we apply here Σ^1_1-boundedness to show a simple fact that will be needed later : adding an ω-bounded quantifier to a $\Sigma^0_{\omega_1^{ck}}$ or a $\Pi^0_{\omega_1^{ck}}$ set does not change its complexity.

Lemma 5.5 Every $\Sigma^0_{\omega_1^{ck}+1}$ set of integers is $\Pi^0_{\omega_1^{ck}}$.

Proof. Let B be $\Sigma^0_{\omega_1^{ck}+1}$, that is, $B = \bigcup_{n\in\omega} \bigcap_{\alpha\in\omega_1^{ck}} B_{n,\alpha}$ where each $B_{n,\alpha}$ is Σ^0_α uniformly in α. Then B is $\Pi^0_{\omega_1^{ck}}$ via the following equality : $\bigcup_{n\in\omega} \bigcap_{\alpha\in\omega_1^{ck}} B_{n,\alpha} = \bigcap_{\alpha\in\omega_1^{ck}} \bigcup_{n\in\omega} \bigcap_{\beta\in\alpha} B_{n,\beta}$. $\square$

It is clear that if m is in the leftmost set it is also in the rightmost set. The reader should have no trouble to apply Σ^1_1-boundedness to show that if m is not in the leftmost set, then it is not in the rightmost one.

5.1.6. Π^1_1 *and* Σ^1_1 *sets of reals.* Given $X \in 2^\omega$ we let $\mathcal{O}^X$ be the set of X-constructible codes for X-computable ordinals. We let $\omega_1^X \geq \omega_1^{ck}$ be the smallest non X-computable ordinal. For $\alpha < \omega_1^X$, we let $\mathcal{O}^X_{<\alpha}$ be the elements of $\mathcal{O}^X$ coding for an ordinal strictly smaller than α.

One can show that a set $\mathcal{B} \subseteq 2^\omega$ is Π^1_1 iff there exists some $e \in \omega$ such that $\mathcal{B} = \{X \ : \ e \in \mathcal{O}^X\}$, that is, $\mathcal{B}$ is the set of elements relative to which e codes for an X-computable ordinal. In particular, $\mathcal{B} = \bigcup_{\alpha<\omega_1} \{X \ : \ e \in \mathcal{O}^X_{<\alpha}\}$. Note that the union may go up to ω_1, indeed, Π^1_1 sets of reals are not necessarily Borel.

A Π^1_1 set of particular interest is the set of element X such that $\omega_1^X > \omega_1^{ck}$. The set is Borel, but not effectively. One can even prove that it contains no non-empty Σ^1_1 subset : this is known as the Gandy Basis theorem (see Sacks [23, III.1.5]):

Theorem 5.6 (Gandy Basis theorem) Let $\mathcal{B} \subseteq 2^\omega$ be a non-empty Σ^1_1 set. Then there exists $X \in \mathcal{B}$ such that $\omega_1^X = \omega_1^{ck}$.

5.1.7. *The general strategy to show hyperarithmetic cone avoidance.* Let Z be non Δ^1_1. Our goal is to build a generic $G \subseteq A$ or $G \subseteq \omega - A$ such that Z is not $\Delta^1_1(G)$. This is done in two steps: first show that Z is not $G^{(\alpha)}$-computable for any $\alpha < \omega_1^{ck}$ and second show that $\omega_1^G = \omega_1^{ck}$, so in particular we cannot have that Z is $G^{(\alpha)}$-computable for $\omega_1^{ck} \leq \alpha < \omega_1^G$.

The first part is simply an iteration of the forcing through the computable ordinals, and raises no particular issue. This is done in Section 5.2.

The second part is a little bit trickier but still follows a canonical technic, which has often been used, up to some cosmetic changes in its presentation, to show this kind of preservation theorem (see for instance [8], [22] or [27]) : Suppose $\omega_1^G > \omega_1^{ck}$, in particular there is an element $e \in \mathcal{O}^G$ which codes for ω_1^{ck}, that is e is the code of a functional with $\forall n\ \Phi_e(G, n) \downarrow \in \mathcal{O}^G_{<\omega_1^{ck}}$ with $\sup_n |\Phi_e(G, n)| = \omega_1^{ck}$ where $|\Phi_e(G, n)|$ is the ordinal coded by $\Phi_e(G, n)$. All we have to do is to show that such a code e does not exist. Given e we show that one of the following holds:

(1) $\exists n\ \forall \alpha < \omega_1^{ck}\ \Phi_e(G, n) \notin \mathcal{O}^G_{<\alpha}$
(2) $\exists \alpha < \omega_1^{ck}\ \forall n\ \Phi_e(G, n) \in \mathcal{O}^G_{<\alpha}$

Each set $\{X \ : \ \Phi_e(X, n) \notin \mathcal{O}^X_{<\alpha}\}$ is Δ^1_1 uniformly in α. It follows that the set $\{X \ : \ \exists n\ \forall \alpha < \omega_1^{ck}\ \Phi_e(X, n) \notin \mathcal{O}^X_{<\alpha}\}$ is a $\Sigma^0_{\omega_1^{ck}+1}$ set of reals. Contrary to $\Sigma^0_{\omega_1^{ck}+1}$ sets of integers, such sets cannot be simplified. We are then required to extend our forcing questions in order to control the truth of $\Sigma^0_{\omega_1^{ck}+1}$-statements. This is what will be done in Section 5.3.

5.2. The forcing

We now design a notion of forcing for controlling the α-jump of solutions to the pigeonhole principle. Unlike the notion of forcing for controlling finite iterations of the jump, this notion is non-disjunctive and initially fixes the side of the instance A from which we will construct a solution. This is at the cost of a forcing question whose definitional complexity is higher than the question it asks.

Proposition 5.7 There is a sequence of sets $\{M_\alpha\}_{\alpha<\omega_1^{ck}}$ such that:

(1) M_α codes for a countable Scott set $\mathcal{M}_\alpha$
(2) $\emptyset^{(\alpha)}$ is uniformly coded by an element of $\mathcal{M}_\alpha$
(3) Each M'_α is uniformly computable in $\emptyset^{(\alpha+1)}$

Proof. Let us show the following: there is a functional $\Phi : 2^\omega \to 2^\omega$ such that for any oracle X, we have that $M' = \Phi(X')$ is such that $M = \oplus_{n\in\omega} X_n$ codes for a Scott set $\mathcal{M}$ with $X_0 = X$.

Fix a uniformly computable enumeration $\mathcal{C}_0^Y, \mathcal{C}_1^Y, \dots$ of all non-empty $\Pi^0_1(Y)$ classes. Let $\mathcal{D}_X$ be the $\Pi^0_1(X)$ class of all $\bigoplus_n Y_n$ such that $Y_0 = X$ and for every $n = \langle a, b\rangle \in \omega$, $Y_{n+1} \in \mathcal{C}_a^{\bigoplus_{j\leq b} Y_j}$. Note that this $\Pi^0_1(X)$ class is uniform in X and any member of $\mathcal{D}_X$ is a code of a Scott set whose first element is X. Using the Low basis theorem [10], there is a Turing functional Φ such that for any X, $\Phi(X')$ is the jump of a member of $\mathcal{D}_X$.

Using this function Φ, it is clear that uniformly in $\emptyset^{(\alpha+1)}$ one can compute the jump of a set M_α coding for a Scott set $\mathcal{M}_\alpha$ and containing $\emptyset^{(\alpha)}$ as its first element. □

Note $\emptyset^{(\beta)}$ is computable in $\emptyset^{(\alpha)}$ for $\beta < \alpha$ in a uniform way : there is a unique computable function $f(\emptyset^{(\alpha)}, \alpha, \beta)$ which outputs $\emptyset^{(\beta)}$ for every $\beta < \alpha$. Also Proposition 5.7 implies that M_β is computable in $\emptyset^{(\alpha)}$ for $\beta < \alpha$ and similarly, the computation is uniform in β, α.

Proposition 5.8 There is a sequence of sets $\{C_\alpha\}_{\alpha<\omega_1^{ck}}$ such that:

(1) $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is an $\mathcal{M}_\alpha$-cohesive largeness class
(2) $\beta < \alpha$ implies $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta} \rangle$
(3) Each C_α is coded by an element of $\mathcal{M}_{\alpha+1}$ uniformly in α and $M_{\alpha+1}$.

In order to prove Proposition 5.8 we use the two following uniformity lemmas:

Lemma 5.9 There is a functional $\Phi : 2^\omega \times \omega \to 2^\omega$ such that for any set M coding for a Scott set $\mathcal{M}$, for any e such that $C = \Phi_e(M'')$ is such that $\mathcal{U}_C^{\mathcal{M}}$ is a largeness class, $D = \Phi(M'', e)$ is such that $C \subseteq D$ and $\mathcal{U}_D^{\mathcal{M}} = \langle \mathcal{U}_C^{\mathcal{M}} \rangle$.

Proof. Say $\mathcal{M} = \{X_0, X_1, \dots\}$ with $M = \bigoplus_i X_i$. Let $\langle e_t, i_t\rangle$ be an enumeration of $\omega \times \omega$. Suppose that at stage t a finite set $D_t \subseteq \{\langle e_0, i_0\rangle, \dots, \langle e_t, i_t\rangle\}$ has been defined such that $\mathcal{U}_{D_t}^{\mathcal{M}} \cap \mathcal{U}_C^{\mathcal{M}}$ is a largeness class and such that for any $s \leq t$, $\langle e_s, i_s\rangle \notin D_t$ implies that $\mathcal{U}_{e_s}^{X_{i_s}} \cap \mathcal{U}_{D_t}^{\mathcal{M}} \cap \mathcal{U}_C^{\mathcal{M}}$ is not a largeness class.

Then at stage $t+1$, we ask M'' if $\mathcal{U}_{e_{t+1}}^{X_{i_{t+1}}} \cap \mathcal{U}_{D^t}^{\mathcal{M}} \cap \mathcal{U}_C^{\mathcal{M}}$ is a largeness class. If so we define $D^{t+1} = D^t \cup \{\langle e_{t+1}, i_{t+1}\rangle\}$. Otherwise we define $D^{t+1} = D^t$. Then $D = C \cup \bigcup_t D^t$ is uniformly M''-computable and $\mathcal{U}_D^{\mathcal{M}}$ is a largeness class. □

Lemma 5.10 There is a functional $\Phi : 2^\omega \times \omega \times \omega \to \omega$ such that for any set M coding for a Scott set $\mathcal{M}$, for any set N coding for a Scott set $\mathcal{N}$ such that $M' \in \mathcal{N}$ with N-index i_M, for any $C \in \mathcal{N}$ with N-index i_C, such that $\mathcal{U}_C^{\mathcal{M}}$ is a largeness and partition regular class, $\Phi(N, i_M, i_C)$ is an M-index for $D \supseteq C$ such that $\mathcal{U}_D^{\mathcal{M}}$ is a $\mathcal{M}$-cohesive largeness class.

Proof. The functional Φ does the following : It looks for M' at index i_M inside $\mathcal{N}$. From M' it computes $M = \oplus_n X_n$. It then computes with M' the tree T containing all the elements σ such that

$$\left(\bigcap_{\sigma(i)=0} 2^\omega - X_i\right) \cap \left(\bigcap_{\sigma(i)=1} X_i\right) \in \bigcap_{\langle e,j\rangle \in C\upharpoonright|\sigma|} \mathcal{U}_e^{X_j}$$

Clearly $[T]$ is not empty. The functional Φ then finds an N-index for an element $Y \in [T]$. For $\sigma \prec Y$ let $X_\sigma = (\bigcap_{\sigma(i)=0}(2^\omega - X_i) \cap (\bigcap_{\sigma(i)=0} X_i)$. We must have for every $\sigma \prec Y$ that $X_\sigma \in \mathcal{U}_C^{\mathcal{M}}$. It follows as $\mathcal{U}_C^{\mathcal{M}}$ is partition regular, that for every $\sigma \prec Y$, $\mathcal{L}_{X_\sigma} \cap \mathcal{U}_C^{\mathcal{M}}$ is a largeness class. Thus $\bigcap_{\sigma \prec Y} \mathcal{L}_{X_\sigma} \cap \mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive largeness class. Also $M \oplus Y \oplus C$ uniformly compute a set D such that $\mathcal{U}_D^{\mathcal{M}} = \bigcap_{\sigma \prec Y} \mathcal{L}_{X_\sigma} \cap \mathcal{U}_C^{\mathcal{M}}$. The function Φ then returns an N-index for D. ☐

Proof of Proposition 5.8. Let X_i^α be the element of $\mathcal{M}_\alpha$ of code i, so that each $M_\alpha = \oplus_i X_i^\alpha$. Let us argue that there is a computable function $f : \omega_1^{ck} \times \omega_1^{ck} \times \omega$ such that whenever $\beta < \alpha$, then $X_i^\beta = X^\alpha_{f(\alpha,\beta,i)}$: Given an ordinal α the function f considers the M_α-code of $\emptyset^{(\alpha)}$ (which is uniformly coded in $\mathcal{M}_\alpha$) and uses it produce an M_α-code of $M_\beta = \oplus_i X_i^\beta$ (as M_β is computable in $\emptyset^{(\alpha)}$, uniformly in β, α) and then returns an M_α-code of X_i^β. Given $\alpha < \beta$ and $C \subseteq \omega^2$, we then let $g(\alpha, \beta, C) = \{\langle e, f(\alpha,\beta,i)\rangle : \langle e,i\rangle \in C\}$. In particular, $\mathcal{U}^{\mathcal{M}_\alpha}_{g(\alpha,\beta,C)} = \mathcal{U}_C^{\mathcal{M}_\beta}$.

Suppose that stage α we have defined by induction sets C_β for each $\beta < \alpha$, verifying (1)(2) and (3). Let us proceed and define C_α.

Suppose first that $\alpha = \beta + 1$ is successor. Note that the set C_β is coded by an element of $\mathcal{M}_{\beta+1}$ uniformly in β, and thus that C_β is uniformly computable in $\emptyset^{(\beta+2)}$ and then uniformly computable in M''_β. Using Lemma 5.9 we define $D_\beta \supseteq C_\beta$ to be such that $\mathcal{U}_{D_\beta}^{\mathcal{M}_\beta} = \langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}\rangle$ and such that D_β is uniformly M''_β-computable. We define E_α to be $g(\alpha, \beta, D_\beta)$, so that $\mathcal{U}_{E_\alpha}^{\mathcal{M}_\alpha} = \mathcal{U}_{D_\beta}^{\mathcal{M}_\beta}$. Note that as E_α is uniformly computable in M''_β and thus in $\emptyset^{(\alpha+1)}$, it is uniformly coded by an element of $\mathcal{M}_{\alpha+1}$. Note also that $\mathcal{U}_{E_\alpha}^{\mathcal{M}_\alpha}$ is partition regular as it equals $\langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}\rangle$. Using Lemma 5.10 we uniformly find an $\mathcal{M}_{\alpha+1}$-index of $C_\alpha \supseteq E_\alpha$ to be such that $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is an $\mathcal{M}_\alpha$-cohesive largeness class.

At limit stage $\alpha = \sup_n \beta_n$, each set C_{β_n} is coded by an element of $\mathcal{M}_{\beta_n+1}$ uniformly in β_n and that $\mathcal{M}_{\beta_n+1}$ is uniformly computable in $\emptyset^{(\alpha)}$. It follows that $\bigcup_n C_{\beta_n}$ is uniformly computable in $\emptyset^{(\alpha)}$. We define D_α to be $\bigcup_n g(\alpha, \beta_n, C_{\beta_n})$. Note that D_α is uniformly computable in $\emptyset^{(\alpha)}$ and thus coded by an element of $\mathcal{M}_\alpha$ uniformly in α. Note also that $\mathcal{U}_{D_\alpha}^{M_\alpha} = \bigcap_{n\in\omega} \mathcal{U}_{C_{\beta_n}}^{M_{\beta_n}} = \bigcap_{n\in\omega} \langle \mathcal{U}_{C_{\beta_n}}^{M_{\beta_n}}\rangle$. As an intersection of partition regular class, $\mathcal{U}_{D_\alpha}^{M_\alpha}$ is partition regular. Using Lemma 5.10 there is a set $C_\alpha \supseteq D_\alpha$ such that $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is $\mathcal{M}_\alpha$-cohesive and such that C_α is uniformly coded by an element of $\mathcal{M}_{\alpha+1}$. ☐

From now on, fix sequences $\{\mathcal{M}_\alpha\}_{\alpha<\omega_1^{ck}}$ and $\{C_\alpha\}_{\alpha<\omega_1^{ck}}$ which verify Proposition 5.7 and Proposition 5.8, respectively. Assume also that we have a class $\mathcal{S} \subseteq \bigcap_{\beta<\omega_1^{ck}} \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}$ which is partition regular and that will be detailed later.

Let $A^0 \cup A^1 = \omega$. Note that there must be $i < 2$ such that $A^i \in \mathcal{S}$. Let then $A = A^i$ for some i such that $A^i \in \mathcal{S}$.

Definition 5.11. Let $\mathbb{P}_{\omega_1^{ck}}$ be the set of conditions (σ, X) such that:

(1) $\sigma \subseteq A$
(2) $X \subseteq A$
(3) $X \cap \{0, \ldots, |\sigma|\} = \emptyset$.
(4) $X \in \mathcal{S}$

Given two conditions $(\sigma, X), (\tau, Y) \in \mathbb{P}_{\omega_1^{ck}}$ we let $(\sigma, X) \leq (\tau, Y)$ be the usual Mathias extension, that is, $\sigma \succeq \tau$, $X \subseteq Y$ and $\sigma - \tau \subseteq Y$.

We now define an abstract forcing question for Σ^0_α sets, and deciding whether there is an extension forcing the generic set G to belong or not to belong to the set. Contrary to the forcing question for arithmetical sets where the question was disjunctive, asking whether for every 2-cover of ω, there is a side $i < 2$ and an extension of the stem forcing the generic set G^i to belong to the Σ^0_α set $\mathcal{B}_i$, we ask whether the collection of sets such that there is an extension forcing G to belong to $\mathcal{B}$ is a large class. The cost is a forcing question of higher definitional complexity.

Definition 5.12. Let $\sigma \in 2^{<\omega}$. Given a Σ^0_1 class $\mathcal{U}$, let $\sigma \mathrel{?{\vdash}} \mathcal{U}$ hold if

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ [\sigma \cup \tau] \subseteq \mathcal{U}\} \cap \mathcal{U}^{\mathcal{M}_0}_{C_0}$$

is a largeness class. Then inductively, given a Σ^0_m class $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ with $1 < m < \omega$, we let $\sigma \mathrel{?{\vdash}} \mathcal{B}$ hold if

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_n\} \cap \mathcal{U}^{\mathcal{M}_{m-1}}_{C_{m-1}}$$

is a largeness class. Then inductively, given a Σ^0_α class $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ with $\omega \leq \alpha < \omega_1^{ck}$, we define $\sigma \mathrel{?{\vdash}} \mathcal{B}$ if

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_n}\} \cap \mathcal{U}^{\mathcal{M}_\alpha}_{C_\alpha}$$

is a largeness class.

For a condition $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$ and an effectively Borel set $\mathcal{B}$, we write $p \mathrel{?{\vdash}} \mathcal{B}$ if $\sigma \mathrel{?{\vdash}} \mathcal{B}$.

We shall now study the effectivity of the relation $?{\vdash}$. To do so we introduce the following notation.

Definition 5.13. Let $\sigma \in 2^{<\omega}$. Given a Σ^0_1 class $\mathcal{B}$, we write $\mathcal{U}(\mathcal{B}, \sigma)$ for the open set:

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ [\sigma \cup \tau] \subseteq \mathcal{B}\}$$

Given a Σ^0_α class $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ for $1 < \alpha < \omega_1^{ck}$ we write $\mathcal{U}(\mathcal{B}, \sigma)$ for the open set:

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_n}\}$$

Let us now study the complexity of the relation $?{\vdash}$ together with the complexity of the sets $\mathcal{U}(\mathcal{B}, \sigma)$. Note that the difference between (1b) and (2b) in the following proposition may give the wrong impression that the complexity of the relation grows by one additional jump beyond Σ^0_ω classes. This is due to the fact that for $\alpha \geq \omega$, the complete set for Σ^0_α questions is $\emptyset^{(\alpha+1)}$ and not $\emptyset^{(\alpha)}$.

Proposition 5.14 Let $\sigma \in 2^{<\omega}$.

(1) Let $\mathcal{B}$ be a Σ^0_m class for $0 < m < \omega$
   (a) The set $\mathcal{U}(\mathcal{B}, \sigma)$ is an upward-closed $\Sigma^0_1(C_{m-2} \oplus \emptyset^{(m-1)})$ open set if $m > 1$ and an upward-closed Σ^0_1 open set if $m = 1$.
   (b) The relation $\sigma \mathrel{?{\vdash}} \mathcal{B}$ is $\Pi^0_1(C_{m-1} \oplus \emptyset^{(m)})$.
(2) Let $\mathcal{B}$ be a Σ^0_α class for $\alpha \geq \omega$.
   (a) The set $\mathcal{U}(\mathcal{B}, \sigma)$ is an upward closed $\Sigma^0_1(C_{\alpha-1} \oplus \emptyset^{(\alpha)})$ open set if α is successor and an upward closed $\Sigma^0_1(\emptyset^{(\alpha)})$ open set if α is limit.
   (b) The relation $\sigma \mathrel{?{\vdash}} \mathcal{B}$ is $\Pi^0_1(C_\alpha \oplus \emptyset^{(\alpha+1)})$.

This is uniform in σ and a code for the class $\mathcal{B}$.

Proof. This is done by induction on the effective Borel codes. We start with $\alpha = 0$. Let $\mathcal{V}$ be a Σ^0_1 class and $\sigma \in 2^{<\omega}$. It is clear that

$$\mathcal{U}(\mathcal{V}, \sigma) = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ [\sigma \cup \tau] \subseteq \mathcal{V}\}$$

is an upward closed Σ^0_1 class. Then $\sigma \mathrel{?{\vdash}} \mathcal{V}$ iff $\mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}^{\mathcal{M}_0}_{C_0}$ is a largeness class, that is, by Lemma 2.2, iff for every finite set $F \subseteq C_0$, the class $\mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}^{\mathcal{M}_0}_F$ is a largeness class. By

Lemma 2.3, for each $F \subseteq C_0$, the statement is $\Pi^0_2(M_0)$ uniformly in F, and thus $\Pi^0_1(M_0')$ uniformly in F. It is then $\Pi^0_1(\emptyset')$ uniformly in F. Thus the whole statement is $\Pi^0_1(C_0 \oplus \emptyset')$.

Let $1 < \alpha < \omega_1^{ck}$. Suppose (1a)(1b) and (2a)(2b) are true for every σ and every $\beta < \alpha$. Let $\sigma \in 2^{<\omega}$ and let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ be a Σ^0_α class. Let

$$\mathcal{U}(\mathcal{B}, \sigma) = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \ ?\not\Vdash 2^\omega - \mathcal{B}_{\beta_n}\}$$

Suppose first $\alpha = m$ with $1 < m < \omega$. Let us show (1a). For each $n \in \omega$, the class $2^\omega - \mathcal{B}_{\beta_n}$ is a Σ^0_{m-1} class uniformly in $\sigma \cup \tau$ and in a code for $\mathcal{B}_{\beta_n}$. By induction hypothesis, the relation $\sigma \cup \tau \ ?\not\Vdash 2^\omega - \mathcal{B}_{\beta_n}$ is $\Sigma^0_1(C_{m-2} \oplus \emptyset^{(m-1)})$. It follows that $\mathcal{U}(\mathcal{B}, \sigma)$ is an upward closed $\Sigma^0_1(C_{m-2} \oplus \emptyset^{(m-1)})$ class.

Let us now show (1b). By Lemma 2.2, $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_{m-1}}_{C_{m-1}}$ is a largeness class if for all $F \subseteq C_{m-1}$, the class $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_{m-1}}_F$ is a largeness class. By Lemma 2.3, it is a $\Pi^0_2(M_{m-1})$ statement uniformly in F and then a $\Pi^0_1(M'_{m-1})$ statement uniformly in F and then a $\Pi^0_1(\emptyset^{(m)})$ statement uniformly in F. It follows that the statement "$\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_{m-1}}_{C_{m-1}}$ is a largeness class" is $\Pi^0_1(C_{m-1} \oplus \emptyset^{(m)})$.

Suppose now α is limit. Let us show (2a). For each $n \in \omega$, the class $2^\omega - \mathcal{B}_{\beta_n}$ is a $\Sigma^0_{\beta_n}$ class uniformly in $\sigma \cup \tau$ and in a code for $\mathcal{B}_{\beta_n}$. By induction hypothesis, the relation $\sigma \cup \tau \ ?\not\Vdash 2^\omega - \mathcal{B}_{\beta_n}$ is, in any case, $\Sigma^0_1(\emptyset^{(\beta_n+2)})$ and thus $\Sigma^0_1(\emptyset^{(\alpha)})$. It follows that $\mathcal{U}(\mathcal{B}, \sigma)$ is an upward-closed $\Sigma^0_1(\emptyset^{(\alpha)})$ open set.

Suppose now $\alpha \geq \omega$ with $\alpha = \beta + 1$. Let us show (2a). For each n we have that $2^\omega - \mathcal{B}_{\beta_n}$ is a Σ^0_β class uniformly in $\sigma \cup \tau$ and in a code for $\mathcal{B}_{\beta_n}$. By induction hypothesis, the relation $\sigma \cup \tau \ ?\not\Vdash 2^\omega - \mathcal{B}_{\beta_n}$ is $\Sigma^0_1(C_\beta \oplus \emptyset^{(\beta+1)})$. It follows that $\mathcal{U}(\mathcal{B}, \sigma)$ is an upward closed $\Sigma^0_1(C_{\alpha-1} \oplus \emptyset^{(\alpha)})$ class.

Suppose $\alpha \geq \omega$ successor or limit. Let us show (2b). Then $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_\alpha}_{C_\alpha}$ is a largeness class if for all $F \subseteq C_\alpha$, the class $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_\alpha}_F$ is a largeness class. It is a $\Pi^0_2(M_\alpha)$ statement uniformly in F and then a $\Pi^0_1(M'_\alpha)$ statement uniformly in F and then a $\Pi^0_1(\emptyset^{(\alpha+1)})$ statement uniformly in F. It follows that the statement $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_\alpha}_{C_\alpha}$ is a largeness class is $\Pi^0_1(C_\alpha \oplus \emptyset^{(\alpha+1)})$. □

Definition 5.15. Let $(\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. Let $\mathcal{U}$ be a Σ^0_1 class. We define

$$\begin{array}{rcccl} (\sigma, X) & \Vdash & \mathcal{U} & \leftrightarrow & [\sigma] \subseteq \mathcal{U} \\ (\sigma, X) & \Vdash & 2^\omega - \mathcal{U} & \leftrightarrow & \forall \tau \subseteq X \ [\sigma \cup \tau] \not\subseteq \mathcal{U} \end{array}$$

Then inductively for Σ^0_α classes $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$, we define:

$$\begin{array}{rcccl} (\sigma, X) & \Vdash & \mathcal{B} & \leftrightarrow & \exists n \ (\sigma, X) \Vdash \mathcal{B}_{\beta_n} \\ (\sigma, X) & \Vdash & 2^\omega - \mathcal{B} & \leftrightarrow & \forall n \ \forall \tau \subseteq X \ \sigma \cup \tau \ ?\Vdash 2^\omega - \mathcal{B}_{\beta_n} \end{array}$$

Lemma 5.16 Let $p \in \mathbb{P}_{\omega_1^{ck}}$. Let $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$ be a Π^0_α class. Then $p \Vdash \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$ iff for every $n \in \omega$ and every $q \leq p$, $q \ ?\Vdash \mathcal{B}_{\beta_n}$.

Proof. Trivial. □

Proposition 5.17 Let $p \in \mathbb{P}_{\omega_1^{ck}}$. Let $\mathcal{B}$ be an effectively Borel set. If $p \Vdash \mathcal{B}$ and $q \leq p$ then $q \Vdash \mathcal{B}$.

Proof. It is clear for Σ^0_1 and Π^0_1 classes. We proceed by induction for $\alpha > 1$ and suppose $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ is a Σ^0_α class. By definition, there is some $n \in \omega$ such that $p \Vdash \mathcal{B}_{\beta_n}$. As $\mathcal{B}_{\beta_n}$ is a $\Pi^0_{\beta_n}$ and $\beta_n < \alpha$, by induction hypothesis, $q \Vdash \mathcal{B}_{\beta_n}$ and thus $q \Vdash \mathcal{B}$.

Suppose now $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$ is a Π^0_α class. By Lemma 5.16, for all $n \in \omega$ and all $r \leq p$, $r \ ?\Vdash \mathcal{B}_{\beta_n}$. Then if $q \leq p$, then for all n and all $r \leq q$, $r \ ?\Vdash \mathcal{B}_{\beta_n}$. It follows that $q \Vdash \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$. □

Proposition 5.18 Let $p \in \mathbb{P}_{\omega_1^{ck}}$. Let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ be a Σ^0_α class for $0 < \alpha < \omega_1^{ck}$.

(1) Suppose $p \mathrel{?}\!\vdash \mathcal{B}$. Then there exists $q \leq p$ such that $q \Vdash \mathcal{B}$.
(2) Suppose $p \mathrel{?}\!\nvdash \mathcal{B}$. Then there exists $q \leq p$ such that $q \Vdash 2^\omega - \mathcal{B}$.

Proof. Let $p \in \mathbb{P}_{\omega_1^{ck}}$. We start with $\alpha = 1$. Let $\mathcal{V}$ be a Σ^0_1 class and suppose $p \mathrel{?}\!\vdash \mathcal{V}$. Let

$$\mathcal{U}(\mathcal{V}, \sigma) = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ [\sigma \cup \tau] \subseteq \mathcal{V}\}$$

The class $\mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}_{C_0}^{\mathcal{M}_0}$ is a largeness class. As $\mathcal{U}_{C_0}^{\mathcal{M}_0}$ is $\mathcal{M}_0$-cohesive, then $\langle \mathcal{U}_{C_0}^{\mathcal{M}_0} \rangle \subseteq \mathcal{U}(\mathcal{V}, \sigma)$. As $X \in \mathcal{S} \subseteq \langle \mathcal{U}_{C_0}^{\mathcal{M}_0} \rangle \subseteq \mathcal{U}(\mathcal{V}, \sigma)$, there is $\tau \subseteq X - \{0, \dots, |\sigma|\}$ such that $[\sigma \cup \tau] \subseteq \mathcal{V}$. As $\mathcal{S}$ contains only infinite sets, $X - \{0, \dots, \sigma \cup \tau\} \in \mathcal{S}$. Then $(\sigma \cup \tau, X - \{0, \dots, \sigma \cup \tau\})$ is a valid extension of (σ, X) such that $(\sigma \cup \tau, X - \{0, \dots, \sigma \cup \tau\}) \Vdash \mathcal{U}$.

Suppose now that $\sigma \mathrel{?}\!\nvdash \mathcal{U}$. The class $\mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}_{C_0}^{\mathcal{M}_0}$ is not a largeness class. It follows that there is a k-cover $Y_0 \cup \dots \cup Y_{k-1} \supseteq \omega$ such that $Y_i \notin \mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}_{C_0}^{\mathcal{M}_0}$ for each $i < k$. As $\mathcal{S}$ is partition regular and as $X \in \mathcal{S}$ we have some $i < k$ such that $Y_i \cap X \in \mathcal{S} \subseteq \mathcal{U}_{C_0}^{\mathcal{M}_0}$. It follows that $Y_i \cap X \notin \mathcal{U}(\mathcal{V}, \sigma)$. Note that $(\sigma, Y_i \cap X)$ is a valid extension of (σ, X) for which $(\sigma, Y_i \cap X) \Vdash 2^\omega - \mathcal{V}$.

Suppose now $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ be a Σ^0_α class for $1 < \alpha < \omega_1^{ck}$. Suppose $\sigma \mathrel{?}\!\vdash \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$. Let

$$\mathcal{U}(\mathcal{B}, \sigma) = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_n}\}$$

If $\alpha < \omega$ let $\beta = \alpha - 1$, otherwise let $\beta = \alpha$. By definition, the class $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}$ is a largeness class. As $\mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}$ is $\mathcal{M}_\beta$-cohesive and as, by Proposition 5.14, $\mathcal{U}(\mathcal{B}, \sigma)$ is a $\Sigma^0_1(Z)$ for some $Z \in \mathcal{M}_\beta$, then $\langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta} \rangle \subseteq \mathcal{U}(\mathcal{B}, \sigma)$. As $X \in \mathcal{S} \subseteq \langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta} \rangle \subseteq \mathcal{U}(\mathcal{B}, \sigma)$, there is $\tau \subseteq X - \{0, \dots, |\sigma|\}$ such that $\sigma \cup \tau \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_n}$ for some n. Note that as $\mathcal{S}$ contains only infinite sets we have $X - \{0, \dots, |\sigma \cup \tau|\} \in \mathcal{S}$. Also $(\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\})$ is a valid extension of (σ, X) such that $(\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\}) \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_n}$. By induction hypothesis we have some $q \leq (\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\})$ such that $q \Vdash \mathcal{B}_{\beta_n}$. It follows that $q \Vdash \mathcal{B}$.

Suppose now $\sigma \mathrel{?}\!\nvdash \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$. It follows that $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}$ is not a largeness class. It follows that there is a k-cover $Y_0 \cup \dots \cup Y_{k-1} \supseteq \omega$ such that $Y_i \notin \mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}$ for each $i < k$. As $\mathcal{S}$ is partition regular and as $X \in \mathcal{S}$, there is some $i < k$ such that $Y_i \cap X \in \mathcal{S} \subseteq \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}$. It follows that $Y_i \cap X \notin \mathcal{U}(\mathcal{B}, \sigma)$. It means that for every $\tau \subseteq Y_i \cap X$ and every $n \in \omega$, $\sigma \cup \tau \mathrel{?}\!\vdash 2^\omega - \mathcal{B}_{\beta_n}$. It follows that $(\sigma, Y_i \cap X) \Vdash \bigcap_{n<\omega} 2^\omega - \mathcal{B}_{\beta_n}$. □

Definition 5.19. Let $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ be a sufficiently generic filter. Then there is a unique set $G_\mathcal{F} \in 2^\omega$ such that for every $(\sigma, X) \in \mathcal{F}$ we have $\sigma \prec G_\mathcal{F}$.

Theorem 5.20 Let $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ be a generic enough filter. Let $p \in \mathcal{F}$. Let $\mathcal{B}_\alpha = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ be a Σ^0_α class for $0 < \alpha < \omega_1^{ck}$. Suppose $p \Vdash \mathcal{B}_\alpha$. Then $G_\mathcal{F} \in \mathcal{B}_\alpha$. Suppose $p \Vdash 2^\omega - \mathcal{B}_\alpha$. Then $G_\mathcal{F} \in 2^\omega - \mathcal{B}_\alpha$.

Proof. We show the following by induction on α. Let $p \in \mathbb{P}_{\omega_1^{ck}}$ with $p = (\sigma, X)$. We start with $\alpha = 1$. Let $\mathcal{U}$ be a Σ^0_1 class. Suppose $p \Vdash \mathcal{U}$, that is $[\sigma] \subseteq \mathcal{U}$. Then clearly $G_\mathcal{F} \in \mathcal{U}$. Suppose now $p \Vdash 2^\omega - \mathcal{U}$, that is, $[\sigma \cup \tau] \nsubseteq \mathcal{U}$ for all $\tau \subseteq X$. Then also $G_\mathcal{F} \in 2^\omega - \mathcal{U}$.

Let now $\mathcal{B}$ be a Σ^0_α class. Suppose $p \Vdash \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$. Then there exists n such that $p \Vdash \mathcal{B}_{\beta_n}$. By induction hypothesis we have if $\mathcal{F}$ is sufficiently generic, then $G_\mathcal{F} \in \mathcal{B}_{\beta_n} \subseteq \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$.

Let now $\mathcal{B}$ be a Π^0_α class. Suppose $p \Vdash \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$. Then by Lemma 5.16 for every n and every $q \leq p$, $q \mathrel{?}\!\vdash \mathcal{B}_{\beta_n}$. From Proposition 5.18, for every $n \in \omega$ and every $q \leq p$, there is some $r \leq q$ such that $r \Vdash \mathcal{B}_{\beta_n}$. It follows that for every n, the set $\{r \ : \ r \Vdash \mathcal{B}_{\beta_n}\}$ is dense below p. If $\mathcal{F}$ is sufficiently generic, for every $n \in \omega$, there is some $r \in \mathcal{F}$ such that $r \Vdash \mathcal{B}_{\beta_n}$. By induction hypothesis, if $\mathcal{F}$ is sufficiently generic, then for every $n \in \omega$, $G_\mathcal{F} \in \mathcal{B}_{\beta_n}$. It follows that $G_\mathcal{F} \in \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$. □

5.3. **Preservation of hyperarithmetic reductions**

We now prove that the infinite pigeonhole principle admits strong cone avoidance for hyperarithmetic reductions.

Theorem 5.21 Let $\alpha \leq \omega_1^{ck}$ be a limit ordinal. Suppose Z is not $\Delta_1^0(\emptyset^{(\beta)})$ for every $\beta < \alpha$. Let $\mathcal{F}$ be a sufficiently generic filter. Then for every $\beta < \alpha$, Z is not $\Delta_1^0(G_{\mathcal{F}}^{(\beta)})$.

Proof. Let Φ be a functional and $\beta < \alpha$. Let $\mathcal{B}^n = \{X \ : \Phi(X^{(\beta)}, n) \downarrow\}$. We want to show that $Z \neq \{n \ : G_{\mathcal{F}}^{(\beta)} \in \mathcal{B}^n\}$. From Proposition 5.2, $\mathcal{B}^n$ is a $\Sigma^0_{\beta+1}$ set for each $n \in \omega$ (Σ^0_β if $\beta \geq \omega$ and $\Sigma^0_{\beta+1}$ if $\beta < \omega$).

Let $p \in \mathbb{P}_{\omega_1^{ck}}$ be a condition. From Proposition 5.14, the set $\{n \ : \ p \mathrel{?}\!\vdash \mathcal{B}^n\}$ is $\Pi_1^0(\emptyset^{(\beta+3)})$. As Z is not $\Pi_1^0(\emptyset^{(\beta+3)})$, then there is some $n \in Z$ such that $p \mathrel{?}\!\nvdash \mathcal{B}^n$ or some $n \notin Z$ such that $p \mathrel{?}\!\vdash \mathcal{B}^n$. In the first case, there is an extension $q \leq p$ such that $q \Vdash 2^\omega - \mathcal{B}^n$ for some $n \in Z$. In the second case, there is an extension $q \leq p$ such that $q \Vdash \mathcal{B}^n$ for some $n \notin Z$. By Theorem 5.20, in the first case $\Phi(G_{\mathcal{F}}^{(\beta)}, n) \uparrow$ holds for some $n \in Z$, and in the second case, $\Phi(G_{\mathcal{F}}^{(\beta)}, n) \downarrow$ holds for some $n \notin Z$.

If $\mathcal{F}$ is sufficiently generic, this is true for any $\beta < \alpha$ and any functional Φ. It follows that for any ordinal β the set Z is not $\Sigma_1^0(G_{\mathcal{F}}^{(\beta)})$ and thus not $\Delta_1^0(G_{\mathcal{F}}^{(\beta)})$. □

This shows in particular cone avoidance for arithmetic degrees. In order to show cone avoidance for hyperarithmetic degrees, one should additionally argue that if $\mathcal{F}$ is sufficiently generic, then $\omega_1^{G_{\mathcal{F}}} = \omega_1^{ck}$. The remainder of this section is devoted to the proof of this fact.

Definition 5.22. A largeness class $\mathcal{A}$ is Γ-minimal, where Γ is a class of complexity, if for every Γ-open set $\mathcal{U}$ we have $\mathcal{A} \cap \mathcal{U}$ large implies $\mathcal{A} \subseteq \mathcal{U}$.

Proposition 5.23 The class $\bigcap_{\alpha < \omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is Δ_1^1-minimal.

Proof. For every $\alpha < \omega_1^{ck}$ we have that $\emptyset^{(\alpha)} \in \mathcal{M}_\alpha$ and $\bigcap_{\alpha < \omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \langle \mathcal{M}_\alpha \rangle$ where $\langle \mathcal{M}_\alpha \rangle$ is $\mathcal{M}_\alpha$-minimal. As $\emptyset^{(\alpha)} \in \mathcal{M}_\alpha$ we also have that $\langle \mathcal{M}_\alpha \rangle$ is minimal for $\Sigma_1^0(\emptyset^{(\alpha)})$ open sets. It follows that $\bigcap_{\alpha < \omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is Δ_1^1-minimal. □

Proposition 5.24 There is a set $C \in \bigcap_{\alpha < \omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ such that C is Δ_1^1-cohesive and $\omega_1^C = \omega_1^{ck}$

Proof. Let us argue that for any upward closed partition regular class $\bigcap_{n<\omega} \mathcal{U}_n$ where each $\mathcal{U}_n$ is open, not necessarily effectively of uniformly, there is a Δ_1^1-cohesive C in $\bigcap_{n<\omega} \mathcal{U}_n$. This is done by Mathias forcing with conditions (σ, X) such that $X \cap \{0, \dots, |\sigma|\} = \emptyset$ and such that X is Δ_1^1 with $X \in \bigcap_{n<\omega} \mathcal{U}_n$. Given a condition (σ, X) and n we can force the generic to be in $\mathcal{U}_n$ as follows : As $X \in \mathcal{U}_n$ we must have that $\sigma \cup X \in \mathcal{U}_n$ because $\mathcal{U}_n$ is upward closed. Thus there must be $\tau \subseteq X \cap \{0, \dots, |\sigma|\}$ such that $[\sigma \cup \tau] \subseteq \mathcal{U}_n$. As $\bigcap_{n<\omega} \mathcal{U}_n$ contains only infinite set we must have $X - \{0, \dots, \sigma \cup \tau\} \in \bigcap_{n<\omega} \mathcal{U}_n$. Thus $(\sigma \cup \tau, X - \{0, \dots, \sigma \cup \tau\})$ is a valid extension. Let now Y be Δ_1^1. We can force the generic to be included in Y or $\omega - Y$ up to finitely many elements as follow : We have $X \cap Y \in \bigcap_{n<\omega} \mathcal{U}_n$ or $X \cap (\omega - Y) \in \bigcap_{n<\omega} \mathcal{U}_n$. Then $(\sigma, X \cap Y)$ or $(\sigma, X \cap (\omega - Y))$ is a valid extension.

We have that the set $\bigcap_{\alpha < \omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is a Σ_1^1 class which is also upward closed and partition regular. We also have that the class of Δ_1^1-cohesive sets is a Σ_1^1 class. By the previous argument their intersection is non-empty. By the Σ_1^1-basis theorem it must contains C with $\omega_1^C = \omega_1^{ck}$. □

Lemma 5.25 Suppose C is Δ_1^1-cohesive with $C \in \bigcap_{\alpha < \omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$. Let $\mathcal{U}$ be a Δ_1^1 open set. If $\mathcal{L}_C \cap \mathcal{U}$ is a largeness class, then $\bigcap_{\alpha < \omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \mathcal{U}$

Proof. Suppose $\mathcal{L}_C \cap \mathcal{U}$ is a largeness class. Let us show that $\mathcal{U} \cap \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is a largeness class. Suppose first for contradiction that it is not. Then there is a Δ^1_1 cover $Y_0 \cup \dots \cup Y_{k-1} \supseteq \omega$ together with a Δ^1_1 open largeness class $\mathcal{V} \supseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ such that $Y_i \notin \mathcal{U} \cap \mathcal{V}$ for every $i < k$. As each Y_i is Δ^1_1, there is some $i < k$ such that $C \subseteq^* Y_i$. Note also that since $C \in \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$, then $C \in \mathcal{L}(\mathcal{V})$ and thus $\mathcal{L}_C \cap \mathcal{V}$ is a largeness class. It follows that $Y_j \in \mathcal{L}_C \cap \mathcal{V}$ for some $j < k$. As $j \neq i$ implies $|Y_j \cap C| < \infty$, then $Y_i \in \mathcal{L}_C \cap \mathcal{V}$ and thus $Y_i \in \mathcal{V}$. As $\mathcal{L}_C \cap \mathcal{U}$ is a largeness class then by a similar argument, $Y_i \in \mathcal{L}_C \cap \mathcal{U}$ and thus $Y_i \in \mathcal{U}$. It follows that $Y_i \in \mathcal{U} \cap \mathcal{V}$, contradicting our hypothesis. Thus $\mathcal{U} \cap \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is a largeness class.

Now from Proposition 5.23 we have that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is minimal for Δ^1_1 open sets, then $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \mathcal{U}$. ☐

Definition 5.26. Let $\mathcal{B} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ be a $\Sigma^0_{\omega_1^{ck}}$ class. Let $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. We define $p \,?\!\vdash \mathcal{B}$ if the set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists \alpha < \omega_1^{ck} \ \sigma \cup \tau \,?\!\nvdash 2^\omega - \mathcal{B}_\alpha\} \cap \mathcal{L}_C$$

is a largeness class.

Given a $\Sigma^0_{\omega_1^{ck}}$ class $\mathcal{B} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ the following set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists \alpha < \omega_1^{ck} \ \sigma \cup \tau \,?\!\nvdash 2^\omega - \mathcal{B}_\alpha\}$$

is a Π^1_1 open set, that is an open set $\bigcup_{\sigma\in B}[\sigma]$ where $B = \bigcup_{\alpha<\omega_1^{ck}} B_\alpha$ is a Π^1_1 set of strings. We also suppose that each B_α is $\emptyset^{(\alpha)}$-computable and that $\{B_\alpha\}_{\alpha<\omega_1^{ck}}$ is increasing. Given such sets we write $\mathcal{U}_\alpha$ for the Δ^1_1 open set $\bigcup_{\sigma\in B_\alpha}[\sigma]$.

Proposition 5.27 Let $\mathcal{U}$ be an upward-closed Π^1_1 open set. The class $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class iff there exists some $\alpha < \omega_1^{ck}$ such that $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class.

Proof. Suppose $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class. Then clearly $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. Suppose now that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. For each n let $\mathcal{U}_n^C$ be the $\Sigma^0_1(C)$ open set such that $\mathcal{L}_C = \bigcap_n \mathcal{U}_n^C$. We have

$$\forall n \ \forall k \ \exists \alpha \ \forall Y_0 \cup \dots \cup Y_{k-1} \ \exists i < k \ \exists \sigma \subseteq Y_i \ [\sigma] \subseteq \mathcal{U}_\alpha \cap \mathcal{U}_n^C$$

Note that given k and α the predicate $P_\alpha^{n,k} \equiv \forall Y_0 \cup \dots \cup Y_{k-1} \ \exists i < k \ \exists \sigma \subseteq Y_i \ [\sigma] \subseteq \mathcal{U}_\alpha \cap \mathcal{U}_n^C$ is $\Sigma^0_1(C \oplus \emptyset^{(\alpha+1)})$ uniformly in n, k and α. Thus the function $f : \omega^2 \to \omega_1^{ck}$ which to n, k associates the smallest α such that $P_\alpha^{n,k}$ is true is a total $\Pi^1_1(C)$ function. By Σ^1_1-boundedness we have $\beta = \sup_{n,k} f(n,k) < \omega_1^C = \omega_1^{ck}$. It follows that

$$\forall n \ \forall k \ \forall Y_0 \cup \dots \cup Y_{k-1} \ \exists i < k \ \exists \sigma \subseteq Y_i \ [\sigma] \subseteq \mathcal{U}_\beta \cap \mathcal{U}_n^C$$

Also $\mathcal{U}_\beta \subseteq \mathcal{U}$ is such that $\mathcal{U}_\beta \cap \mathcal{L}_C$ is a largeness class. ☐

Corollary 5.28 Let $\mathcal{B} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ be a $\Sigma^0_{\omega_1^{ck}}$ class. Let $(\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. The relation $p \,?\!\vdash \mathcal{B}$ is $\Sigma^0_{\omega_1^{ck}}(C)$

Proof. The relation $p \,?\!\vdash \mathcal{B}$ is equivalent to

$$\exists \alpha < \omega_1^{ck} \ \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \sigma \cup \tau \,?\!\nvdash 2^\omega - \mathcal{B}_\alpha\} \cap \mathcal{L}_C$$

is a largeness class ☐

Corollary 5.29 The class $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is minimal for Π^1_1 open sets $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class.

Proof. Given a Π^1_1-open set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$, there must be $\alpha < \omega_1^{ck}$ such that $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class. By Lemma 5.25 it must be that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \mathcal{U}_\alpha$. ☐

Definition 5.30. Let $\mathcal{B} = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ be a $\Pi^0_{\omega_1^{ck}}$ class. Let $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. We define $p \Vdash \mathcal{B}$ if for every $\tau \subseteq X - \{0, \dots, |\sigma|\}$ and for every $\alpha < \omega_1^{ck}$ we have $\sigma \cup \tau \mathrel{?\vdash} \mathcal{B}_\alpha$

Proposition 5.31 Let $\mathcal{B} = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ be a $\Pi^0_{\omega_1^{ck}}$ class. Let $\mathcal{F}$ be sufficiently generic with $p \in \mathcal{F}$. If $p \Vdash \mathcal{B}$, then $G_\mathcal{F} \in \mathcal{B}$.

Proof. Using Proposition 5.18, for every α and every $q \leq p$, there is some $r \leq q$ such that $r \Vdash \mathcal{B}_\alpha$. Thus for every α the set $\{r \ : \ r \Vdash \mathcal{B}_\alpha\}$ is dense below p. It follows from Theorem 5.20 that if $\mathcal{F}$ is sufficiently generic, $G_\mathcal{F} \in \mathcal{B}$. □

Definition 5.32. Let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ be a $\Sigma^0_{\omega_1^{ck}+1}$ class where each $\Pi^0_{\omega_1^{ck}}$ set $\mathcal{B}_n = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$. We define $p \mathrel{?\vdash} \mathcal{B}$ if the set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?\nvdash} 2^\omega - \mathcal{B}_n\} \cap \mathcal{L}_C$$

is a largeness class.

Given a $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ with $\mathcal{B}_n = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$, the following set

$$\mathcal{U} = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?\nvdash} 2^\omega - \mathcal{B}_n\}$$

is a $\Sigma^1_1(C)$ open set, that is an open set $\mathcal{U} = \bigcup_{\sigma\in B}[\sigma]$ where $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ is a $\Sigma^1_1(C)$ set of strings. We furthermore assume that $\{B_\alpha\}_{\alpha<\omega_1^{ck}}$ is decreasing. We then write $\mathcal{U}_\alpha$ for the $\Delta^1_1(C)$-open set $\bigcup_{\sigma\in B_\alpha}[\sigma]$.

Computability theorists have a strong habits of working with enumerable open sets. With that respect, Σ^1_1-open sets, that is, co-enumerable along the computable ordinals, are strange objects to consider. Note that given such an open set we have $\mathcal{U} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha$, but not necessarily equality. However the elements X of $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha - \mathcal{U}$ are all such that $\omega_1^X > \omega_1^{ck}$. It is in particular a meager and nullset.

Let us detail a little bit the set $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ that we can consider so that $\mathcal{U} = \bigcup_{\sigma\in B}[\sigma]$. To ease the notation we introduce the following definition, in the same spirit as $\mathcal{U}(\mathcal{B}, \sigma)$ defined above:

Definition 5.33. Let $\mathcal{B}$ be a Σ^0_α class. We define $\mathcal{V}(\mathcal{B}, \sigma)$ to be the set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \sigma \cup \tau \mathrel{?\nvdash} \mathcal{B}\}$$

Given a $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ with $\mathcal{B}_n = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$, given

$$\mathcal{U} = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?\nvdash} 2^\omega - \mathcal{B}_n\}$$

we have by Corollary 5.28 that $\mathcal{U}$ equals:

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \forall \alpha < \omega_1^{ck} \ \mathcal{V}(2^\omega - \mathcal{B}_{n,\alpha}, \sigma \cup \tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

Let

$$B = \{\tau \ : \ \exists n \ \forall \alpha < \omega_1^{ck} \ \mathcal{V}(2^\omega - \mathcal{B}_{n,\alpha}, \sigma \cup \tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

Let

$$B_\alpha = \{\tau \ : \ \exists n \ \forall \beta < \alpha \ \mathcal{V}(2^\omega - \mathcal{B}_{n,\beta}, \sigma \cup \tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

By Σ^1_1-boundedness we have that $B = \bigcap_\alpha B_\alpha$. We also have $\mathcal{U} = \bigcup_{\sigma\in B}[\sigma]$.

We now show the core lemma that will be used to show $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$ for $\mathcal{F}$ a sufficiently generic filter:

Lemma 5.34 Let $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ be a $\Sigma^1_1(C)$ set of strings where each B_α is $\Delta^1_1(C)$ uniformly in α and where $\beta < \alpha$ implies $B_\alpha \subseteq B_\beta$. Let $\mathcal{U} = \bigcup_{\sigma\in B}[\sigma]$ be a $\Sigma^1_1(C)$ upward closed open set with $\mathcal{U}_\alpha = \bigcup_{\sigma\in B_\alpha}[\sigma]$ be a $\Delta^1_1(C)$ upward closed open set. We have $\mathcal{U} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha$. Furthermore, $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class iff for every $\alpha < \omega_1^{ck}$, $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class.

Proof. It is clear that $\mathcal{U} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha$. Also it is clear that if $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class, then also $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class.

Suppose $\mathcal{U} \cap \mathcal{L}_C$ is not a largeness class. Then there is a cover $Y_0 \cup \dots \cup Y_{k-1} \supseteq \omega$ with $Y_i \notin \mathcal{U} \cap \mathcal{L}_C$ for every $i < k$. There must be a $\Sigma^0_1(C)$ open set $\mathcal{V}$ such that $Y_i \notin \mathcal{U} \cap \mathcal{V}$ for every $i \leq k$.

Let $f : \omega \to \omega_1^{ck}$ be the function which on n finds a cover $\sigma_0 \cup \dots \cup \sigma_k \supseteq \{0, \dots, n\}$ and α such that for $i < k$ and every $\tau \preceq \sigma_i$ we have $[\tau] \subseteq \mathcal{V}$ implies $\tau \notin B_\alpha$. As $\mathcal{U} \cap \mathcal{V}$ is not a largeness class, f is a total $\Pi^1_1(C)$ function. By Σ^1_1-boundedness, $\beta = \sup_n f(n) < \omega_1^C = \omega_1^{ck}$. By compactness, there is a cover $Y_0 \cup \dots \cup Y_{k-1}$ such that for every $i < k$ if $Y_i \in \mathcal{V}$ then for every $\tau \prec Y_i$, $\tau \notin B_\beta$ and thus $Y_i \notin \mathcal{U}_\beta$.

It follows that $\mathcal{U}_\beta \cap \mathcal{L}_C$ is not a largeness class. □

Corollary 5.35 $\mathcal{L}_C$ contains a unique largeness subclass, which is minimal for both Π^1_1 and $\Sigma^1_1(C)$-open sets $\mathcal{U}$.

Proof. Suppose $\mathcal{U}_0, \mathcal{U}_1$ are two $\Sigma^1_1(C)$ open sets with $\mathcal{U}_i = \bigcup_{\sigma \in B_i} [\sigma]$ and $\mathcal{U}_{i,\alpha} = \bigcup_{\sigma \in B_{i,\alpha}} [\sigma]$. for $i < 2$. Suppose also $\mathcal{U}_0 \cap \mathcal{L}_C$ and $\mathcal{U}_1 \cap \mathcal{L}_C$ are largeness classes. By Lemma 5.34 it follows that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{0,\alpha} \cap \mathcal{L}_C$ and $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{1,\alpha} \cap \mathcal{L}_C$ are largeness classes. By Lemma 5.25 it follows that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{0,\alpha}$ and $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{1,\alpha}$.

Thus $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{0,\alpha} \cap \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{1,\alpha} = \bigcap_{\alpha<\omega_1^{ck}} (\mathcal{U}_{0,\alpha} \cap \mathcal{U}_{1,\alpha})$ is a largeness class and thus by Lemma 5.34 we have that $\mathcal{U}_0 \cap \mathcal{U}_1$ is a largeness class.

It follow that the intersection $\mathcal{I}$ of every $\Sigma^1_1(C)$ open set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class, is a largeness class. Furthermore as $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \cap \mathcal{L}_C$ is a largeness class for every α, the class $\mathcal{I}$ must be included in $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$. Also from Corollary 5.29 the class $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is minimal for Π^1_1-open sets $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. It follows that the class $\mathcal{I} \cap \mathcal{L}_C$ is minimal for $\Sigma^1_1(C)$ and Π^1_1 open sets. □

We can now detail the class $\mathcal{S}$ involved in the definition of $\mathbb{P}_{\omega_1^{ck}}$: Let $\mathcal{S}$ be the unique largeness class included in $\mathcal{L}_C$ which is minimal for $\Sigma^1_1(C)$ and Π^1_1 open sets. Note that $\mathcal{S}$ must be partition regular.

Lemma 5.36 Consider a $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ with $\Pi^0_{\omega_1^{ck}}$ set $\mathcal{B}_n = \bigcap_{\alpha\in\omega_1^{ck}} \mathcal{B}_{n,\alpha}$. Let $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. Suppose $\sigma \mathrel{?{\vdash}} \mathcal{B}$. Then there is a condition $q \leq p$ together with some n such that $q \Vdash \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$

Proof. Let
$$\mathcal{U} = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_n\}$$
The class $\mathcal{U}$ is a $\Sigma^1_1(C)$-open set and $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. As $\mathcal{S}$ is minimal for $\Sigma^1_1(C)$-open sets, $\mathcal{S} \subseteq \mathcal{U}$. As $X \in \mathcal{S} \subseteq \mathcal{U}$. Then there is some $\tau \subseteq X - \{0, \dots, |\sigma|\}$ and some n such that $\sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_n$. Let now
$$\mathcal{V} = \{Y \ : \ \exists \rho \subseteq Y - \{0, \dots, |\sigma \cup \tau|\} \ \exists \alpha \ \sigma \cup \tau \cup \rho \mathrel{?{\nvdash}} \mathcal{B}_{n,\alpha}\}$$
As $\sigma \cup \tau \mathrel{?{\nvdash}} \bigcup_{\alpha\in\omega_1^{ck}} 2^\omega - \mathcal{B}_{n,\alpha}$ then $\mathcal{V} \cap \mathcal{L}_C$ is not a largeness class. Thus there is a cover $Y_0 \cup \dots \cup Y_{k-1} = \omega$ such that $Y_i \notin \mathcal{V} \cap \mathcal{L}_C$ for every $i < k$. As $\mathcal{V} \cap \mathcal{L}_C$ is upward-closed, $X \cap Y_i \notin \mathcal{V} \cap \mathcal{L}_C$ for every $i < k$. As $\mathcal{S} \subseteq \mathcal{L}_C$ is partition regular, there is some $i < k$ such that $X \cap Y_i \in \mathcal{S} \subseteq \mathcal{L}_C$. Therefore we must have $X \cap Y_i \notin \mathcal{V}$ and thus
$$\forall \rho \subseteq X \cap Y_i - \{0, \dots, |\sigma \cup \tau|\} \ \forall \alpha \ \sigma \cup \tau \cup \rho \mathrel{?{\vdash}} \mathcal{B}_{n,\alpha}$$
Thus $(\sigma \cup \tau, X \cap Y_i)$ is an extension of (σ, X) such that:
$$(\sigma \cup \tau, X \cap Y_i) \Vdash \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$$
□

Lemma 5.37 Consider a $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ with $\Pi^0_{\omega_1^{ck}}$ set $\mathcal{B}_n = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$. Let $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. Suppose $\sigma \,?\!\nvdash \mathcal{B}$. Then there is a condition $q \leq p$ together with some $\beta < \omega_1^{ck}$ such that $q \Vdash \bigcap_{n\in\omega} \bigcup_{\alpha<\beta} 2^\omega - \mathcal{B}_{n,\alpha}$

Proof. Let

$$\mathcal{U} = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \,?\!\nvdash 2^\omega - \mathcal{B}_n\}$$

The class $\mathcal{U}$ is a $\Sigma^1_1(C)$-open set and $\mathcal{U} \cap \mathcal{L}_C$ is not a largeness class. Let us recall Definition 5.33 together with the notation coming after it: $\mathcal{V}(\mathcal{B}, \sigma)$ is the set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \sigma \cup \tau \,?\!\nvdash \mathcal{B}\}$$

Together with

$$B = \{\tau \ : \ \exists n \ \forall \alpha < \omega_1^{ck} \ \mathcal{V}(\mathcal{B}_{n,\alpha}, \sigma \cup \tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

with $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ such that

$$B_\alpha = \{\tau \ : \ \exists n \ \forall \beta < \alpha \ \mathcal{V}(\mathcal{B}_{n,\beta}, \sigma \cup \tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

and with $\mathcal{U} = \bigcup_{\sigma\in B} [\sigma]$.

Using Lemma 5.34, there is some $\alpha < \omega_1^{ck}$ such that the set

$$\mathcal{U}_\alpha = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \forall \beta < \alpha \ \mathcal{V}(\mathcal{B}_{n,\beta}, \sigma \cup \tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

is such that $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is not a largeness class. Thus there is a cover $Y_0 \cup \dots \cup Y_{k-1} \supseteq \omega$ such that $Y_i \notin \mathcal{U}_\alpha \cap \mathcal{L}_C$ for every $i < k$. As $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is upward-closed, then also $X \cap Y_i \notin \mathcal{U}_\alpha \cap \mathcal{L}_C$ for every $i < k$. As $X \in \mathcal{S} \subseteq \mathcal{L}_C$ and as $\mathcal{S}$ is partition regular, there is some $i < k$ such that $X \cap Y_i \in \mathcal{S} \subseteq \mathcal{L}_C$. It follows that $X \cap Y_i \notin \mathcal{U}_\alpha$ and thus that:

$$\forall \tau \subseteq X \cap Y_i - \{0, \dots, |\sigma|\} \ \forall n \ \exists \beta < \alpha \ \mathcal{V}(\mathcal{B}_{n,\beta}, \sigma \cup \tau) \cap \mathcal{L}_C \text{ is a largeness class}$$

Let $\{\beta_m\}_{m\in\omega}$ be such that $\sup_m \beta_m = \alpha$. Let $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ and $n \in \omega$. We have for some m that $\mathcal{V}(\mathcal{B}_{n,\beta_m}, \sigma \cup \tau) \cap \mathcal{L}_C$ is a largeness class. Then the set

$$\{Y \ : \ \exists \rho \subseteq Y - \{0, \dots, |\sigma|\} \ \exists m \ \sigma \cup \tau \cup \rho \,?\!\nvdash \mathcal{B}_{n,\beta_m}\} \cap \mathcal{L}_C$$

is a largeness class and then

$$\{Y \ : \ \exists \rho \subseteq Y - \{0, \dots, |\sigma|\} \ \exists m \ \sigma \cup \tau \cup \rho \,?\!\nvdash \mathcal{B}_{n,\beta_m}\} \cap \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$$

is a largeness class and thus that $\sigma \cup \tau \,?\!\vdash \bigcup_m 2^\omega - \mathcal{B}_{n,\beta_m}$. As this is true for every n and every $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ it follows that $(\sigma, X \cap Y_i)$ is an extension of (σ, X) such that

$$(\sigma, X \cap Y_i) \Vdash \bigcap_{n\in\omega} \bigcup_{\beta<\alpha} 2^\omega - \mathcal{B}_{n,\beta}$$

$\square$

We now show that if $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ is sufficiently generic, then $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$. We use the following fact : If $\omega_1^G > \omega_1^{ck}$, then in particular some G-computable ordinal must code for ω_1^{ck}, that is, there must be a G-computable function Φ such that for every n, $\Phi(G, n)$ codes, relative to G, for an ordinal smaller than ω_1^{ck} and with $\sup_n |\Phi(G, n)| = \omega_1^{ck}$. We show that this never happens by forcing that for every functional Φ either for some n, $\Phi(G, n)$ does not code for an ordinal smaller than ω_1^{ck}, or there is an ordinal $\alpha < \omega_1^{ck}$ such that $\Phi(G, n)$ always codes for some ordinal smaller than α.

Given G and α let $\mathcal{O}^G_\alpha$ be the set of G-codes for ordinals smaller than α. For $\alpha < \omega_1^{ck}$, the class $\{G \ : \ n \in \mathcal{O}^G_\alpha\}$ is Δ^1_1 uniformly in α and n.

Theorem 5.38 Suppose $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ is sufficiently generic. Then $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$

Proof. Let $p \in \mathbb{P}_{\omega_1^{ck}}$ be a condition. Given a functional $\Phi : 2^\omega \times \omega \to \omega$, let

$$\mathcal{B} = \{X \ : \ \exists n \ \forall \alpha < \omega_1^{ck} \ \Phi(X, n) \notin \mathcal{O}_\alpha^X\}$$

Suppose $p \mathrel{?}\!\vdash \mathcal{B}$. Then from Lemma 5.36, there is an extension $q \leq p$ and some n such that

$$q \Vdash \{X \ : \ \forall \alpha < \omega_1^{ck} \ \Phi(X, n) \notin \mathcal{O}_\alpha^X\}$$

It follows from Proposition 5.31 that if $\mathcal{F}$ is sufficiently generic for every $\alpha < \omega_1^{ck}$, $\Phi(G_\mathcal{F}, n) \notin \mathcal{O}_\alpha^{G_\mathcal{F}}$. Suppose now $p \mathrel{?}\!\nvdash \mathcal{B}$. Then from Lemma 5.37, there is an extension $q \leq p$ and some $\alpha < \omega_1^{ck}$ such that

$$q \Vdash \{X \ : \ \forall n \ \Phi(X, n) \in \mathcal{O}_\alpha^X\}$$

It follows from Theorem 5.20 that if $\mathcal{F}$ is sufficiently generic, $\sup_n \Phi(G_\mathcal{F}, n) \leq \alpha$. □

5.4. **Tight α-jump cone avoidance**

In this section, we use a restriction of the forcing $\mathbb{P}_{\omega_1^{ck}}$ to give another proof of Turing cone avoidance. For a $\emptyset^{(\alpha)}$-computable set B, we will find a generic $G \in [A]^\omega$ such that B is not $G^{(\alpha)}$-computable. The difficulty is that the forcing question for $\mathbb{P}_{\omega_1^{ck}}$ (Definition 5.12) is more complex than the one of Definition 3.7. The proof of cone avoidance will then be more complicated. The advantage is that we do not need disjunctive requirements and we have a sufficient condition on any set A so that B is not $G^{(\alpha)}$-computable for some $G \in [A]^\omega$: we simply need $A \in \langle \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \rangle$, which we know happens for at least A or $\overline{A}$.

Let us first slightly modify the sets $\{\mathcal{M}_\gamma\}_{\gamma \leq \alpha}$ of our forcing conditions : In addition to the requirements of Proposition 5.7 we also make sure using the relative cone avoidance theorem for Π_1^0 classes that for any $\gamma \leq \alpha$, the set M_γ does not compute B. Let $\mathbb{P}_\alpha$ be the same forcing as $\mathbb{P}_{\omega_1^{ck}}$, except that for $(\sigma, X) \in \mathbb{P}_\alpha$ we only require $X \in \langle \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \rangle$ instead of $X \in \mathcal{S}$.

Theorem 5.39 Suppose that B is not $\Delta_1^0(\emptyset^{(\alpha)})$ for $1 \leq \alpha < \omega_1^{ck}$. Let $\mathcal{F} \subseteq \mathbb{P}_\alpha$ be a sufficiently generic filter. Then B is not $\Delta_1^0(G_\mathcal{F}^{(\alpha)})$.

Proof. Let Φ be a functional. Let $\mathcal{B}^{0,n} = \{X \ : \ \Phi(X^{(\alpha)}, n) \downarrow= 0\}$ and let $\mathcal{B}^{1,n} = \{X \ : \ \Phi(X^{(\alpha)}, n) \downarrow= 1\}$. We want to show that $B \neq \{n \ : G_\mathcal{F}^{(\alpha)} \in \mathcal{B}^{0,n}\}$ or $\omega - B \neq \{n \ : G_\mathcal{F}^{(\alpha)} \in \mathcal{B}^{1,n}\}$. From Proposition 5.2, for each n, $\mathcal{B}^{0,n}$ and $\mathcal{B}^{1,n}$ are Σ_α^0 classes whenever $\alpha \geq \omega$ and Σ_{m+1}^0 classes whenever $\alpha = m < \omega$.

Let $p = (\sigma, X)$ be a forcing condition. For each $n \in \omega$, let $\mathcal{B}^{0,n} = \bigcup_{a \in \omega} \mathcal{B}_{\beta_a}^{0,n}$ and $\mathcal{B}^{1,n} = \bigcup_{b \in \omega} \mathcal{B}_{\beta_b}^{1,n}$. Suppose first that $\mathcal{A} \cap \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is a largeness class, where

$$\mathcal{A} = \{Y \ : \ \exists \tau_0, \tau_1 \subseteq Y - \{0, \dots, |\sigma|\} \ \exists \langle n, a, b \rangle \ \sigma \cup \tau_0 \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_a}^{0,n} \wedge \sigma \cup \tau_1 \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_a}^{1,n}\}$$

As $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is $\mathcal{M}_\alpha$-cohesive we must have $\mathcal{A} \subseteq \langle \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \rangle$ and thus $X \in \mathcal{A} \subseteq \langle \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \rangle$. Then there is $\tau_0, \tau_1 \subseteq X$ and n, a, b such that $\sigma \cup \tau_0 \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_a}^{0,n}$ and $\sigma \cup \tau_1 \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_b}^{1,n}$. If $n \in B$ we let $q = (\sigma \cup \tau_1, X - \{0, \dots, |\sigma \cup \tau_1|\})$ and if $n \notin B$ we let $q = (\sigma \cup \tau_0, X - \{0, \dots, |\sigma \cup \tau_0|\})$. We have $q \leq p$. In the first case we have $q \Vdash \mathcal{B}^{1,n}$ and in the second case we have $q \Vdash \mathcal{B}^{0,n}$. In the first case we have $G_\mathcal{F} \in \mathcal{B}^{1,n}$ and then $G_\mathcal{F} \notin \mathcal{B}^{0,n}$ for $n \in B$. Then $B \neq \{n \ : \ G_\mathcal{F} \in \mathcal{B}^{0,n}\}$. Symmetrically in the second case we have $\omega - B \neq \{n \ : \ G_\mathcal{F} \in \mathcal{B}^{1,n}\}$.

Suppose now that $\mathcal{A} \cap \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is not a largeness class. For any $q \leq p$ let

$$B_0^q = \{n \ : \ \exists r \leq q \ r \Vdash \mathcal{B}^{0,n}\} \quad \text{and} \quad B_1^q = \{n \ : \ \exists r \leq q \ r \Vdash \mathcal{B}^{1,n}\}$$

Suppose first that for some $q \leq p$ we have $B_0^q \neq B$ or $B_1^q \neq \omega - B$. Suppose first $B_0^q \neq B$. If there is n such that $n \notin B$ and $n \in B_0^q$, then $r \Vdash \mathcal{B}^{0,n}$ for some $r \leq q$ and we have $B \neq \{n \ : \ G_\mathcal{F} \in \mathcal{B}^{0,n}\}$. If there is n such that $n \in B$ and $n \notin B_0^q$, then for all $r \leq q$ we have $r \nVdash \mathcal{B}^{0,n}$. Thus there must be $r \leq q$ such that $r \Vdash 2^\omega - \mathcal{B}^{0,n}$. It follows that $B \neq \{n \ : \ G_\mathcal{F} \in \mathcal{B}^{0,n}\}$.

Suppose now $B_1^q \neq \omega - B$. Symmetrically if there is n such that $n \notin \omega - B$ and $n \in B_1^q$, then $\omega - B \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{1,n}\}$. Still symmetrically if there is n such that $n \in \omega - B$ and $n \notin B_1^q$, we have $\omega - B \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{1,n}\}$. Suppose now for contradiction that:

(1) For all $q \leq p$ we have $B_0^q = B$ and $B_1^q = \omega - B$

As $\mathcal{A} \cap \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is not a largeness class, there must be a cover $Y_0 \cup \cdots \cup Y_{k-1} \in \mathcal{M}_\alpha$ such that $Y_i \notin \mathcal{A} \cap \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ for every $i < k$. Note that either $\alpha = m$ in which case $\mathcal{B}^{0,n}$ is a Σ_{m+1}^0 class which implies that $Y \in \mathcal{M}_m = \mathcal{M}_\alpha$, or $\alpha \geq \omega$ in which case $\mathcal{B}^{0,n}$ is a Σ_α^0 class which implies that $Y \in \mathcal{M}_\alpha$.

As $\langle \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \rangle$ is partition regular there must be $i < k$ such that $X \cap Y_i \in \langle \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \rangle$. We also have $Y_i \in \langle \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \rangle \subseteq \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ and then $Y_i \notin \mathcal{A}$. Thus

(2) for all n, a, b, for all $\tau_0, \tau_1 \subseteq Y - \{0, |\sigma|\}$ the following holds:

$$\sigma \cup \tau_0 \mathrel{?{\vdash}} 2^\omega - \mathcal{B}_{\beta_a}^{0,n} \text{ or } \sigma \cup \tau_1 \mathrel{?{\vdash}} 2^\omega - \mathcal{B}_{\beta_b}^{1,n}$$

We shall now argue that for all $n \in B$ there exists $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ together with a such that $\sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_a}^{0,n}$. Let $n \in B$. Suppose for contradiction that $\sigma \mathrel{?{\nvdash}} \bigcup_{a \in \omega} \mathcal{B}_{\beta_a}^{0,n}$. Then there is some $q \leq p$ such that $q \Vdash 2^\omega - \mathcal{B}^{0,n}$ which contradicts (1). Thus $\sigma \mathrel{?{\vdash}} \bigcup_{a \in \omega} \mathcal{B}_{\beta_a}^{0,n}$. It follows that there exists $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ together with a such that $\sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_a}^{0,n}$.

Symmetrically, we show that for all $n \in \omega - B$ there exists $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ together with b such that $\sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_b}^{1,n}$. Therefore, for every $n \in B$ we have using (2) that:

(1) There exists some $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ and $a \in \omega$ such that $\sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_a}^{0,n}$

(2) For all $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ and for all $b \in \omega$, $\sigma \cup \tau \mathrel{?{\vdash}} 2^\omega - \mathcal{B}_{\beta_b}^{1,n}$

Symmetrically, for every $n \notin B$ we prove, using (2), that:

(1) There exists some $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ and $b \in \omega$ such that $\sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_b}^{1,n}$

(2) For all $\tau \subseteq Y - \{0, \dots, |\sigma|\}$ and for all $a \in \omega$, $\sigma \cup \tau \mathrel{?{\vdash}} 2^\omega - \mathcal{B}_{\beta_a}^{0,n}$

We can now compute B as follows : For each $n \in \omega$, look for some $\tau \subseteq Y - \{0, |\sigma|\}$ and some $c \in \omega$ such that either $\sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_c}^{1,n}$ or $\sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_{\beta_c}^{0,n}$. This is a $\Sigma_1^0(M_\alpha)$ event. Thus B is $\Delta_1^0(M_\alpha)$, which is a contradiction.

□

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