Higher randomness and forcing with closed sets

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Abstract

Kechris showed in [8] that there exists a largest Π^1_1 set of measure 0. An explicit construction of this largest Π^1_1 nullset has later been given in [6]. Due to its universal nature, it was conjectured by many that this nullset has a high Borel rank (the question is explicitely mentioned in [3] and [15]). In this paper, we refute this conjecture and show that this nullset is merely $\mathbf{\Sigma}^0_3$. Together with a result of Liang Yu, our result also implies that the exact Borel complexity of this set is $\mathbf{\Sigma}^0_3$.

To do this proof, we develop the machinery of effective randomness and effective Solovay genericity, investigating the connections between those notions and effective domination properties.

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1 Introduction

We will study in this paper the notion of forcing with closed sets of positive measure and several variants of it. This forcing is generally attributed to Solovay, who used it in [14] to produce a model of $ZF+DC$ in which all sets of reals are Lebesgue measurable. Stronger and stronger genericity for this forcing coincides with stronger and stronger notions of randomness. It is actually possible to express most of the randomness definitions that have been made over the years by forcing over closed sets of positive measure.

In the first section we give a brief overview of the part of algorithmic randomness that we need in the paper. In the second section we make a modification to the usual definition of effective Solovay genericity directly inspired by a notion introduced by Jockusch in [7] about effective genericity for Cohen forcing. This new definition will reveal itself to be interesting for its connections with effective domination properties. In the third section we will give a quick description of what we need of higher computability theory and higher randomness to approach the last section. Finally in the last section we give higher analogues of the Solovay genericity notions studied in section two, and we show again their connections with randomness and higher effective domination properties. This will allow us to conclude with the Borel complexity of the largest Π^1_1 nullset.

2 General Background

In this paper, we will work in the space of infinite sequences of 0's and 1's, called the Cantor space, denoted by 2^ω. We will call **strings** finite sequences of 0's and 1's, **sequences**

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elements of the Cantor space and **sets** the sets of sequences. For a string σ, we will denote the set of sequences extending σ by $[\sigma]$.

The set of integers W_e will denote the domain of the computable function Φ_e, and $[W_e]$ will denote $\bigcup_{\sigma \in W_e} [\sigma]$, where W_e is seen as a set of strings. We will denote by $\langle , \rangle$ a fixed computable pairing function from $\omega \times \omega$ to ω.

We will consider computable functionals (computable functions using sequences as oracles) as functions from the Cantor space to the Baire space. Then a computable functional Φ is considered define on $X \in 2^\omega$ if $\forall n\ \Phi^X(n) \downarrow$ and we denote by $\operatorname{dom} \Phi$ the set $\{X \mid \forall n\ \Phi^X(n) \downarrow\}$. We say that a function f is computable relative to X or X-computable if there is a computable functional defined on X such that $\Phi^X = f$.

The topology on Cantor space is generated by the basic intervals $[\sigma] = \{X \in 2^\omega \mid X \succ \sigma\}$ for σ a string. For $A \subseteq 2^\omega$ Lebesgue-measurable, $\lambda(A)$ will denote the Lebesgue measure of A, which is the unique Borel measure such that $\lambda([\sigma]) = 2^{-|\sigma|}$ for all strings σ.

2.1 About the arithmetical complexity of sets

In the Cantor space, open sets can be described as countable unions of strings. We call an open set **effective** if it can be described as the union of a computably enumerable set of strings, i.e. if it is equal to $[W_e]$ for some e. Such a set is said to be Σ^0_1. On the other hand, when it is open but not necessarily effectively open, the set is said to be $\boldsymbol{\Sigma^0_1}$. However, a non-effective open set is always effective relatively to some oracle. If X is such an oracle, we say that the set is $\Sigma^0_1(X)$. A closed set is called effective if its complement is an effective open set, in which case we say that the closed set is a Π^0_1 set. We can then continue to describe the effective Borel sets through the arithmetical hierarchy as effective unions of effective Borel set of lower complexity and as their complements. So a Σ^0_{n+1} set will be an effective union of Π^0_n sets, and a Π^0_{n+1} set will be the complement of a Σ^0_{n+1} set. For example, a set A is Σ^0_4 if we have an integer e such that $A = \bigcup_{m_1 \in W_e} \bigcap_{m_2 \in W_{m_1}} \bigcup_{m_3 \in W_{m_2}} [W_{m_3}]^c$.

We have a canonical surjection from integers to Σ^0_1 sets (The one which associates to e the computably enumerable set $[W_e]$), but also from integers to Σ^0_n sets for a fixed n. In the above example, with $n = 4$ the integer e is associated to the Σ^0_4 set A. In this context e will be called an **index** for the set A.

Also for a computably enumerable set of integers W, we denote by $W[t]$ the enumeration of W up to stage t. We extend this definition to effective open sets: if $O = [W]$, then $O[t] = [W[t]]$. Similarly, if $F = O^c$, $F[t] = O[t]^c$.

2.2 About algorithmic randomness

In 1966, Martin-Löf gave in [10] a definition capturing elements of the Cantor space that can be considered 'random'. Many nice properties of the Martin-Löf random sequences make this notion of randomness one of the most interesting and one of the most studied.

Intuitively a random sequence should not have any atypical property. A property is here considered atypical if the set of sequences having it is of measure 0. It also makes sense to consider only properties which can be described in some effective way (because any X has the property of being in the set $\{X\}$ and thus nothing would be random).

▶ **Definition 1.** An intersection of measurable sets $\bigcap_n A_n$ is said to be **effectively of measure** 0 if the function which to n associates the measure of A_n is bounded by a decreasing computable function whose limit is 0. A **Martin-Löf** test is a Π^0_2 set $\bigcap_n O_n$

effectively of measure 0. We say that $X \in 2^\omega$ is **Martin-Löf random** if it is in no Martin-Löf test.

One can iterate this idea by considering Π^0_n sets effectively of measure 0 for any $n \geq 2$. Martin-Löf randomness is also called **1-randomness**, the use of Π^0_3 sets effectively of measure 0 gives us **2-randomness**, Π^0_4 sets give us **3-randomness**, and so on. The requirement for a Martin-Löf test to be effectively of measure 0 is important and leads to very nice properties. In particular there exists a universal Martin-Löf test, i.e. a test containing all the others (see [10]). This is not the case anymore if we drop the 'effectively of measure 0' condition. Instead we get a notion known as weak-2-randomness.

▶ **Definition 2.** We say that $X \in 2^\omega$ is **weakly-2-random** if it is in no Π^0_2 nullset.

As a randomness notion, weak-2-randomness is a strictly stronger than 1-randomness, but is strictly weaker than 2-randomness (see [12] section 3.6).

3 Solovay genericity and its variants

Cohen introduced in [4] the general technique of forcing by forcing with all dense open sets of the Cantor space (with the usual topology) in a countable model of ZFC. The most basic effective version of this would be to say that X is generic if it belongs to all dense Σ^0_1 sets, a notion introduced by Kurtz in [9]. Jockusch introduced and studied in [7] a slightly different notion.

▶ **Definition 3** (Kurtz, Jockusch)**.** We say that X is **weakly-1-generic** if it belongs to all dense Σ^0_1 sets. We say that X is **1-generic** if for any Σ^0_1 set U, either X belongs to U or X belongs to some other Σ^0_1 set U' disjoint from U.

We will apply Jockusch's idea behind 1-genericity to forcing with Π^0_1 sets. First note that by definition, the weakly-2-randoms are exactly the sequences which are in all Σ^0_2 sets of measure 1. If we consider the topology generated by Π^0_1 sets of positive measure, because Σ^0_2 sets of measure 1 are then dense open sets for this topology, we also get in some sense a genericity notion.

3.1 Forcing with Π^0_1 sets

Adding a measure requirement to the definition of genericity will always link us to randomness. We study what happens if we drop the measure requirement and if we consider instead the Σ^0_2 sets which are dense for the topology generated by the Π^0_1 sets, i.e. the Σ^0_2 sets which intersect all non-empty Π^0_1 set. It is clear that the Cantor space with this topology is a Baire space, i.e. has the property that an intersection of dense open sets is dense. This directly comes from the fact that a decreasing intersection of non-empty closed sets is non-empty. This justifies the following definition:

▶ **Definition 4.** Let $\{G_i\}_{i\in\omega}$ be the collection of all Σ^0_2 sets which intersect all the Π^0_1 sets. We say that X is **weakly-Π^0_1-generic** if it belongs to $\bigcap_i G_i$.

As the next proposition shows, weak-Π^0_1-genericity has nothing to do with randomness.

▶ **Proposition 5.** *No weakly-Π^0_1-generic sequence is 2-random.*

Proof. We construct uniformly in n a Σ^0_2 set intersecting all Π^0_1 sets and with measure smaller than 2^{-n}. Let $\{F_e\}_{e\in\omega}$ be an enumeration of the Π^0_1 sets. For each e we initialize σ_e to the first string (using lexicographic order) of length $n+e+1$. Our Σ^0_2 set will consist of a computably enumerable set A of indices of Π^0_1 sets. We now describe the algorithm to enumerate elements of A: At stage t, for each substage $e < t$ in increasing order, if the index of $F_e \cap [\sigma_e]$ has not been enumerated yet into A, then enumerate it. After that, if $(F_e \cap [\sigma_e])[t] = \emptyset$ then reset σ_e to be the string of length $n+e+1$ following σ_e in the lexicographic order. If σ_e is already the last such string, leave it unchanged.

Let us prove that the measure of the Σ^0_2 set represented by A is smaller than 2^{-n}. For each e, if $F_e \cap [\sigma_e] = \emptyset$ then by compactness $(F_e \cap [\sigma_e])[t] = \emptyset$ for some t. Thus at most one string σ_e of length $n+e+1$ such that $F_e \cap [\sigma_e] \neq \emptyset$ has been enumerated into A, and the measure of A is bounded by $\sum_e 2^{-n-e-1} \leq 2^{-n}$. Now our Σ^0_2 set is dense because if F_e is not empty then there exists a string σ_e of length $n+e+1$ such that $F_e \cap [\sigma_e]$ is not empty and then A will intersect F_e.

From this we can then construct a Π^0_3 set effectively of measure 0 and containing all the weakly-Π^0_1-generic sequences. ◀

Following Jockusch's 1-genericity idea we now define Π^0_1-genericity:

▶ **Definition 6.** A sequence X is **Π^0_1-generic** if for all Σ^0_2 sets G, either X is in G or there is a Π^0_1 set F disjoint from G such that X is in F.

We now establish a simple but surprising connection with computability theory, which appears to be previously unknown. We say that a sequence X is **computably dominated** if for every total function $f : \omega \to \omega$, computable relative to X, there exists a total computable function g such that g dominates f (i.e. $\forall n\ f(n) \leq g(n)$).

▶ **Proposition 7.** *A set X is Π^0_1-generic iff it is computably dominated.*

Proof. Suppose X is computably dominated and take any Σ^0_2 set $\bigcup_n F_n$. Suppose that X belongs to its complement, a Π^0_2 set $\bigcap_n O_n$. Let us define the X-computable function $f : \omega \to \omega$ which to n associates the smallest t so that $X \in O_n[t]$. As X is computably dominated, there is a computable function g which dominates f. Then $X \in \bigcap_n O_n[g(n)]$, an effectively closed set disjoint from $\bigcup_n F_n$.

Conversely suppose that X is Π^0_1-generic and consider a functional Φ, defined on X. We have that $\operatorname{dom}\Phi = \{X \mid \forall n\ \Phi^X(n)\downarrow\}$ is a Π^0_2 set containing X. But then as X is Π^0_1-generic, it is contained in a Π^0_1 set F contained in the domain of Φ. Let us now build[1] a computable function f such that $\forall X \in F\ \Phi^X < f$. To compute the value of $f(n)$ we find the smallest pair $\langle m, t\rangle$ such that for all strings σ of size m with $[\sigma] \subseteq F[t]$, the functional Φ halts on n in less than t steps with σ as an oracle (considering that if Φ needs to use bits of the oracle at positions bigger than $|\sigma|$, it does not halt). Then we set $f(n)$ to the sum of all those values plus one. All we need to show is that f is total. Fix n and let us prove there is a m so that for all $X \in F$ we have $\Phi^{X\upharpoonright_m}(n)\downarrow$. Suppose not, then for all m there is $X \in F$ with $\Phi^{\sigma_m}(n)\uparrow$ where $\sigma_m = X\upharpoonright_m$. As $\{\sigma_m\}_{m\in\omega}$ is infinite it has at least one limit sequence Y and as F is closed we have $Y \in F$. Also as $\Phi^{Y\upharpoonright_m}(n)\uparrow$ for all m we have that Φ is not defined

[1] One can also directly deduce the existence of such a function f using the fact that the supremum of a computable function, over an effectively compact set, is right-ce.

on Y which contradicts the hypothesis. Thus for some t we have that $F[t]$ is covered by a finite union $\bigcup_{i\leq k}[\sigma_i]$ such that $\Phi^{\sigma_i}(n)\downarrow$. It follows that for some t and some m we have that $\Phi^{\sigma}(n)$ halts in less than t steps for all strings σ of size m such that $[\sigma]\subseteq F[t]$. ◀

A direct computation shows that the set of computably dominated sequences is Π^0_4. The above proposition lowers down the Borel complexity to $\mathbf{\Pi^0_3}$: if for every set A we denote by A° the interior of A for the topology generated by Π^0_1 sets, i.e. the union of all Π^0_1 sets included in A, then the set of computably dominated sequences is the intersection over all the Π^0_2 sets P, of $P^\circ\cup P^c$. We now give a lower bound on the Borel complexity of the computably dominated sequences, however we do not know if it can be $\mathbf{\Sigma^0_3}$.

▶ **Proposition 8.** *The set of computably dominated sequences is neither $\mathbf{\Sigma^0_2}$ nor $\mathbf{\Pi^0_2}$.*

Proof. Let us show that it is not $\mathbf{\Pi^0_2}$. First note that for any $\mathbf{\Pi^0_2}$ set A, if A is dense (for the usual topology) in some $[\sigma]$ then it contains a weakly-1-generic sequence as defined by Kurtz. Indeed, the intersection of $A\cap[\sigma]$ with all dense Σ^0_1 sets will not be empty and will then contain weakly-1-generic sequences. But by a result of computability theory (see [9]), no weakly-1-generic is computably dominated. Thus a $\mathbf{\Pi^0_2}$ set containing only computably dominated sequences is nowhere dense. But as the set of computably dominated sequences is dense, being closed under finite change of prefixes, such a $\mathbf{\Pi^0_2}$ set cannot contain all of them.

To show that it is not $\mathbf{\Sigma^0_2}$, we adapt a technique that Liang Yu exposed in [15]. Suppose that the set of computably dominated sequences is described as $\bigcup_n F_n$ with each F_n closed. For each n let $B_n=\bigcup\{T\mid T\cap F_n=\emptyset$ and T is a Π^0_1 set with no computable member$\}$. Let us prove that the set B_n intersects any non-empty Π^0_1 set with no computable members. Take any non-empty Π^0_1 set G with no computable members. By a classical result of computability theory (see [12] proposition 1.5.12 combined with fact 1.8.36) G contains a non-computably dominated sequence. Thus G contains a sequence X which is not in F_n. Then as F_n is closed there is a string σ such that $X\in G\cap[\sigma]$ but $G\cap[\sigma]\cap F_n=\emptyset$. Thus $G\cap[\sigma]$ is a non-empty Π^0_1 set with no computable sequence, intersecting G and disjoint from F_n. Consequently we have $B_n\cap G\neq\emptyset$ and then each B_n is dense for the topology generated by Π^0_1 sets with no computable member. It follows that $\bigcap_n B_n$ is also dense for this topology. From Proposition 7 the set of computably dominated sequences is also dense for this topology. Then there is a computably dominated sequence in $\bigcap_n B_n$. But we also have by design of the B_n that $\bigcap_n B_n\cap\bigcup_n F_n=\emptyset$, which contradicts the fact that $\bigcup_n F_n$ contains all computably dominated sequences. ◀

3.2 Forcing with Π^0_1 sets of positive measure

We now introduce a notion of genericity which is a measure-theoretic variation of Π^0_1-genericity defined in the previous section. The notion will be interesting for its counterpart in Higher computability. Let us now come back to the topology generated by Π^0_1 sets of positive measure. To obtain weak-2-randomness we consider only Σ^0_2 sets of measure 1. We now consider all Σ^0_2 sets which intersect with positive measure every Π^0_1 set of positive measure.

▶ **Definition 9.** Let $\{G_i\}_{i\in\omega}$ be the collection of all Σ^0_2 sets A such that for any Π^0_1 set F of positive measure we have $\lambda(A\cap F)>0$. Then we say that X is **weakly-Π^0_1-Solovay-generic** if it belongs to $\bigcap_i G_i$.

▶ **Definition 10.** We say that X is **Π^0_1-Solovay-generic** if for any Σ^0_2 set A, either X is in it or there exists a Π^0_1 set F of positive measure and disjoint from A such that X is in it.

▶ **Proposition 11.** *A set X is Π^0_1-Solovay-generic iff it is weakly-2-random and computably dominated.*

Proof. Suppose that X is weakly-2-random and computably dominated. Take any Σ^0_2 set and suppose that X does not belong to it. By Proposition 7, as X is computably dominated, we have that X belongs to some Π^0_1 set disjoint from the Σ^0_2 set. Also as X is weakly-2-random this Π^0_1 set has positive measure.

Conversely, suppose that X is Π^0_1-Solovay-generic. In particular it is weakly-2-random and Π^0_1-generic. Then by Proposition 7 we have that it is computably dominated. ◀

3.3 A separation for weak and non weak-genericity

We will now prove that weak-genericity is not enough to obtain computable domination. For this we shall adapt a proof of a theorem in [1] saying that for any function f, there is a weakly-2-random X and an X-computable function g not dominated by f. Here we want weak-Π^0_1-Solovay-genericity instead of weak-2-randomness.

▶ **Proposition 12.** *For any function $f : \omega \to \omega$ there is an X weakly-Π^0_1-Solovay-generic computing a function $g : \omega \to \omega$ which is above f infinitely often.*

Due to its length, the proof is given in appendix. Using Proposition 12, we have some weakly-Π^0_1-Solovay-generics which are not computably dominated and so not Π^0_1-Solovay-generic. One can prove that weakly-Π^0_1-Solovay-genericity implies weakly-Π^0_1-genericity by showing that any Σ^0_2 set intersecting all the Π^0_1 sets also intersects with positive measure all Π^0_1 sets of positive measure. Take any Σ^0_2 set intersecting all the Π^0_1 sets. Take now a set F of positive measure and consider the Σ^0_2 set $\bigcup_n F_n$ of Martin-Löf randoms (the complement of the universal Martin-Löf test). As it has measure 1, there is some F_n such that $F \cap F_n$ has positive measure. But by hypothesis our Σ^0_2 set intersects $F \cap F_n$. The intersection contains only Martin-Löf random sequences and thus is necessarily of positive measure. Thus there is also some weakly-Π^0_1-generics which are not Π^0_1-generics.

4 Background on higher computability and higher randomness

We now give a few definitions of higher computability and higher randomness. The Turing reductions are replaced by hyperarithmetical reductions. One intuitive way to understand a hyperarithmetical computation is to think of a standard Turing computation, but with an infinite-time Turing machine. For those machines the computational time is not an integer anymore, but an ordinal. Tapes are infinite and pre-filled with 0's, at a successor stage everything happens as in a regular Turing machine. At a limit stage, the machine changes to a special 'limit' state, the head comes back to the first cell of the first tape and if the value of a cell of a tape does not converge, it is reset to 0 (otherwise it is set to the limit of its previous values). The rest works as usual.

For example, we can build the ordinal time Turing machine which on a tape, at finite computation time $t = \langle s, e\rangle$ write 1 on the cell number e of this tape if the program number e halts in less than s steps. At ordinal time ω we then have the halting problem on this tape. Then stages $\omega + n$ can be used to compute what one could compute with the halting problem. This can be iterated to compute anything that could be computed in a finite jump. But we can even go beyond a finite jump and continue through the ordinal jumps. To formalize this properly we need to fix the notion of notation for computable ordinals.

4.1 Computable ordinals

More details about this section can be found in [13]. An ordinal is defined as the order type of a well-ordered set. When the ordinal is infinite and countable it can be the order-type of a well-ordered set with domain ω. We say that a countable ordinal α is computable if we have a relation $R \subseteq \omega \times \omega$ which is a well-founded linear order of a subset of ω of order-type α and if there is some e such that $(n,m) \in R \leftrightarrow \langle n,m \rangle \in W_e$. In this case we say that e codes for α and we write $|e| = \alpha$. Let us denote by $\mathcal{W}$ the set of integers which code for computable ordinals and let us denote by $\mathcal{W}_\alpha$ the set of integers which code for computable ordinals strictly smaller than α.

As there are uncountably many countable ordinals, not all of them are computable. Moreover it is known that they form a strict initial segment of the countable ordinals. We denote by ω_1^{ck} the smallest non-computable ordinal. This notion can then be relativised. We say that e is an X-code for the ordinal α if we have a relation $R \subseteq \omega \times \omega$ which is a well-founded linear order of a subset of ω of order-type α and if $(n,m) \in R \leftrightarrow \langle n,m \rangle \in W_e^X$. We then write $|e|^X = \alpha$. We denote by $\mathcal{W}^X$ the set of X-codes for X-computable ordinals, and we denote by $\mathcal{W}_\alpha^X$ the set of X-codes for X-computable ordinals strictly smaller than α. Finally, we call ω_1^X the smallest ordinal which is non-computable relatively to X. Note that any countable ordinal is computable with a representation of itself as an oracle.

4.2 Second order definable sets

We say that a sequence X is hyperarithmetic if for some computable function f and some computable ordinal α we have $n \in X \leftrightarrow f(n) \in \mathcal{W}_\alpha$. One can define the hyperarithmetic sequences equivalently as the sequences we can Turing-compute with sufficiently many successive effective joins and iterations of the jump, constructed by induction over the computable ordinals. Also coming back to the analogy with infinite-time Turing machines we have in [5] a theorem saying that a sequence X is hyperarithmetic iff it can be computed by an infinite-time Turing-machine in a computable ordinal length of time. Similarly we define what is hyperarithmetic for sets. We say that $A \subseteq 2^\omega$ is hyperarithmetic if there exists e and α computable such that $X \in A \leftrightarrow e \in \mathcal{W}_\alpha^X$.

We now define Π_1^1 sequences. While hyperarithmetic sequences can be considered to be the higher counterpart of computable sequences, Π_1^1 sequences can be considered to be the higher counterpart of computably enumerable sequences. They are the sequences one can define with a formula of arithmetic containing arbitrary many first order quantifications and only universal second order quantifications (with no negations in front of them). We have another equivalent definition. A sequence X is Π_1^1 if for some computable function f we have $n \in X \leftrightarrow \exists \alpha < \omega_1^{ck}\ f(n) \in \mathcal{W}_\alpha$. Coming back to the analogy with infinite-time Turing machines, the Π_1^1 sequences also correspond to the sets of integers one can enumerate along computable ordinal length of time with such a machine (when we interpret sequences as sets of integers, considering that n in the set iff the n-th bit of the sequence is one). The Σ_1^1 sequences are their complements (again, when we see sequences as sets of integers), the higher equivalent of co-recursively enumerable sequences. Finally a set A is Π_1^1 if we have an integer e so that $X \in A \leftrightarrow \exists \alpha < \omega_1\ e \in \mathcal{W}_\alpha^X$. We also have a canonical surjection from integers to Π_1^1 sets, so like the arithmetical sets, they can be indexed (in the above example, e is an index for the Π_1^1 set A).

A set is called Δ_1^1 if it is both Σ_1^1 and Π_1^1. By a theorem of Kleene (see chapter 2 in [13]) they are exactly the hyperarithmetical sets. An index for a Δ^1_1 set will consist of a pair of two indices. One expressing it as a Π^1_1 predicate and one expressing its complement as a Π^1_1 predicate.

Note that for Π^1_1 sets, the existential quantification over the ordinals goes up to ω_1. Indeed, if $\omega_1^X > \omega_1^{ck}$ it is possible that $X \in A$ is witnessed by some X-code e for $\alpha \geq \omega_1^{ck}$. This leads us to a Π^1_1 set of great importance for this paper, the set $\{X \mid \omega_1^X > \omega_1^{ck}\}$ (the proof that this set if Π^1_1 can be found in section 9.1 of [12]). We now state two theorems that will be useful for the rest of the paper.

▶ **Theorem 13** (Sacks [13]). *Uniformly in ε and an index for a Δ^1_1 set A, one can compute an index for a Σ^1_1 closed set F so that $F \subseteq A$ and $\lambda(A - F) \leq \varepsilon$. Also one can uniformly from an index of a Δ^1_1 set obtain an index for the Δ^1_1 real being the measure of this set.*

▶ **Theorem 14** (Spector [13]). *If $f : \omega \to \mathcal{W}^X$ is a total $\Pi^1_1(X)$ functional predicate then* $\sup_n |f(n)| < \omega_1^X$.

4.3 Higher randomness

We now introduce notions of randomness which are higher effective variations of the usual randomness notions.

▶ **Definition 15** (Sacks). We say that $X \in 2^\omega$ is **Δ^1_1-random** if it is in no Δ^1_1 nullset.

Martin-Löf was actually the first to promote this notion (see [11]), suggesting that it was the appropriate mathematical concept of randomness. Even if his first definition undoubtedly became the most successful over the years, this other definition got a second wind recently on the initiative of Hjorth and Nies who started to study the analogy between the usual notions of randomness and their higher counterparts. In order to do so they created in [6] a higher analogue of Martin-Löf randomness.

▶ **Definition 16** (Hjorth, Nies). A Π^1_1-Martin-Löf test is given by an effectively null intersection of open sets $\bigcap_n O_n$, each O_n being Π^1_1 uniformly in n. A sequence X is **Π^1_1-ML-random** if it is in no Π^1_1-Martin-Löf test.

This definition is strictly stronger than Δ^1_1-randomness (see Corollary 9.3.5 in [12]). The higher analogue of weak-2-randomness has also been studied (see [3]).

▶ **Definition 17.** We say that X is **weakly-Π^1_1-random** if it belongs to no $\bigcap_n O_n$ with each O_n open set Π^1_1 uniformly in n and with $\lambda(\bigcap_n O_n) = 0$.

Earlier, Sacks gave an even stronger definition, made possible by a theorem of Lusin saying that even though Π^1_1 sets are not necessarily Borel, they remain all measurable.

▶ **Definition 18** (Sacks). We say that $X \in 2^\omega$ is **Π^1_1-random** if it is in no Π^1_1 nullset.

This last definition is of great importance. Kechris proved that there is a universal Π^1_1 nullset, in the sense that it contains all the others (see [8]). Later, Hjorth and Nies gave in [6] an explicit construction of this Π^1_1 nullset. Chong and Yu proved in [3] that weakly-Π^1_1-randomness is strictly stronger than Π^1_1-Martin-Löf-randomness, but it is still unknown whether Π^1_1-randomness coincides with weakly-Π^1_1-randomness.

To separate the two notions, the idea of showing they have different Borel complexity was promoted in [3]. In the next section we show that this will not be possible, by proving that the biggest Π^1_1 nullset has the surprisingly small Borel complexity of $\mathbf{\Sigma^0_3}$. Using results of [16] we will conclude that the Borel complexity of both the weakly-2-randoms and the Π^1_1-randoms, is strictly $\mathbf{\Pi^0_3}$. We now give some important results about higher randomness, that will be needed to achieve this:

▶ **Theorem 19** (Sacks). *The set $\{X \mid \omega_1^X > \omega_1^{ck}\}$ has measure 0.*

Thus no X such that $\omega_1^X > \omega_1^{ck}$ is Π^1_1-random. The following beautiful theorem of Chong, Yu and Nies (see [2]) strengthens Sacks' theorem:

▶ **Theorem 20** (Chong, Yu, Nies). *A sequence X is Π^1_1-random iff it is Δ^1_1-random and $\omega_1^X = \omega_1^{ck}$.*

One could also define the randomness notion introduced by Σ^1_1 nullsets, but this turns out to be equivalent to Δ^1_1-randomness.

▶ **Theorem 21** (Sacks). *A Δ^1_1-random sequence is in no Σ^1_1 nullset. Therefore Σ^1_1-randomness coincides with Δ^1_1-randomness.*

5 Higher Solovay genericity and its variants

▶ **Definition 22.** We say that X is **weakly-Σ^1_1-Solovay-generic** if it belongs to all sets of the form $\bigcup_n F_n$ which intersect with positive measure all the Σ^1_1 closed sets of positive measure, where each F_n is a Σ^1_1 closed set uniformly in n.

▶ **Definition 23.** We say that X is **Σ^1_1-Solovay-generic** if for any set of the form $\bigcup_n F_n$ where each F_n is a Σ^1_1 closed set uniformly in n, either X is in $\bigcup_n F_n$ or X is in some Σ^1_1 closed set of positive measure F, disjoint from $\bigcup_n F_n$.

As in the lower case, one could drop the measure requirement in the definition of Σ^1_1-Solovay-genericity and obtain interesting relations with domination properties. However we will focus in this paper only on (weakly-)Σ^1_1-Solovay-genericity.

Unlike in the lower case, we have that the set of weakly-Σ^1_1-Solovay-generics is of measure 1. We can actually prove easily that they coincide with the weakly-Π^1_1-randoms. Let $\bigcup_n F_n$ be a uniform union of Σ^1_1 closed sets with measure strictly smaller than 1. Let $\bigcap_n O_n$ be its complement. As it is a Π^1_1 set, we have e such that $X \in \bigcap_n O_n \leftrightarrow \exists \alpha < \omega_1 \quad e \in \mathcal{W}^X_\alpha$. But by Theorem 19 we have that $\{X \mid \exists \alpha \geq \omega_1^{ck} \quad e \in \mathcal{W}^X_\alpha\} \subseteq \mathcal{S}$ is of measure 0. Thus for some computable α we have that $\{X \mid e \in \mathcal{W}^X_\alpha\}$ has positive measure. As it is a Δ^1_1 set, we can find using Theorem 13 a Σ^1_1 closed set of positive measure contained in it. Thus $\bigcup_n F_n$ does not intersect all Σ^1_1 closed sets of positive measure. Conversely a uniform union of Σ^1_1 closed sets of measure 1 intersects with positive measure any Σ^1_1 closed set of positive measure. Then the weakly-Σ^1_1-Solovay-generics are exactly the weakly-Π^1_1-randoms.

We will now prove that the notion of Σ^1_1-Solovay-genericity is exactly the notion of Π^1_1-randomness. As explained at the end of the section (after Theorem 26), one can also consider this equivalence as the higher counterpart of Proposition 11.

We already know from Theorem 20 that if X is weakly-Π^1_1-random but not Π^1_1-random, then $\omega_1^X > \omega_1^{ck}$. We will show that if X is Σ^1_1-Solovay-generic then $\omega_1^X = \omega_1^{ck}$ which will prove the difficult part of the equivalence.

In order to prove this, we use a technique developed by Sacks and simplified by Greenberg, to show that the set of X with $\omega_1^X > \omega_1^{ck}$ has measure 0. First note that if $\omega_1^X > \omega_1^{ck}$ then there is $o \in \mathcal{W}^X$ such that $|o|^X = \omega_1^{ck}$. In particular for each n we can uniformly restrain the relation coded by o to all elements smaller than n. If $|o|^X$ is a limit ordinal this gives a set of X-codes for ordinals smaller than $|o|^X$ but cofinal (i.e. unbounded) in $|o|^X$. Thus if $\omega_1^X > \omega_1^{ck}$, there is a function $f : \omega \to \mathcal{W}^X$ computable in X such that $\sup_n |f(n)|^X = \omega_1^{ck}$. The idea is the following. Suppose that for some X we have a computable function Φ_e such that:

$$\forall n \ \ \exists \alpha < \omega_1^{ck} \ \ \Phi_e^X(n) \in \mathcal{W}_\alpha^X$$

Suppose also that X is Σ_1^1-Solovay-generic. Then we will show that the supremum of $|\Phi_e^X(n)|$ over $n \in \omega$ is strictly smaller than ω_1^{ck}. To show this we need an approximation lemma, which can be seen as an extension of Theorem 13.

▶ **Lemma 24.** *For a Σ_1^1 predicate $S(X) \leftrightarrow \forall \alpha < \omega_1^{ck}\ e \notin \mathcal{W}_\alpha^X$, uniformly in e and n one can find a Σ_1^1 closed set $F \subseteq S$ with $\lambda(S - F) \leq 2^{-n}$.*

Proof. One can equivalently write $S(X) \leftrightarrow \forall o \in \mathcal{W} \ \ e \notin \mathcal{W}_{|o|}^X$. Let S_o be the predicate $e \notin \mathcal{W}_{|o|}^X$. If $o \in \mathcal{W}$ one can uniformly in o and e obtain an index for the Δ_1^1 predicate S_o. The Π_1^1 index for it corresponds to the property : "There exists no bijection from $|e|$ to a strict initial segment of $|o|^X$", and a Π_1^1 index for its complement is : "There exists no infinite backward sequence in $|e|$, and there exists no bijection from $|o|^X$ to an initial segment of $|e|$." Note that if $o \notin \mathcal{W}$, the index is still well defined but does not correspond to anything specific.

Then uniformly in an index for S_o and in n we can find using Theorem 13 a Σ_1^1 closed set F_o such that $F_o \subseteq S_o$ with $\lambda(S_o - F_o) \leq 2^{-o}2^{-n}$. Now let us define $F(X) \leftrightarrow \forall o \in \mathcal{W} \ \ X \in F_o$. As an intersection of closed sets, the set F is closed. And as $\mathcal{W}$ is Π_1^1 and F_o is Σ_1^1 uniformly in o, we have that F is Σ_1^1. To conclude we also we have that:

$$\begin{aligned} \lambda(S - F) &= \lambda(\textstyle\bigcup_{o \in \mathcal{W}} S - F_o) \\ &\leq \lambda(\textstyle\bigcup_{o \in \mathcal{W}} S_o - F_o) \\ &\leq \textstyle\sum_{o \in \mathcal{W}} \lambda(S_o - F_o) \leq 2^{-n}. \end{aligned}$$

◀

We can now prove the desired theorem:

▶ **Theorem 25.** *If Y is Σ_1^1-Solovay-generic then $\omega_1^Y = \omega_1^{ck}$.*

Proof. Suppose that Y is Σ_1^1-Solovay-generic. For any functionnal Φ, consider the set

$$P = \{X \mid \forall n \ \ \exists \alpha < \omega_1^{ck} \ \ \Phi^X(n) \in \mathcal{W}_\alpha^X\}.$$

Let $P_n = \{X \mid \exists \alpha < \omega_1^{ck} \ \ \Phi^X(n) \in \mathcal{W}_\alpha^X\}$ and $P_{n,\alpha} = \{X \mid \Phi^X(n) \in \mathcal{W}_\alpha^X\}$, so $P = \bigcap_n P_n$ and $P_n = \bigcup_{\alpha < \omega_1^{ck}} P_{n,\alpha}$.

From Lemma 24 we can find uniformly in n a uniform union of Σ_1^1 closed sets included in P_n^c with the same measure as P_n^c. From this we can find a uniform union of Σ_1^1 closed sets included in P^c with the same measure as P^c. Suppose that Y is in P, as it is Σ_1^1-Solovay-generic we have a Σ_1^1 closed set F of positive measure containing Y which is disjoint from P^c up to a set of measure 0, formally $\lambda(F \cap P^c) = 0$. In particular for each n we have $\lambda(F \cap P_n^c) = 0$ and then $\lambda(F^c \cup P_n) = 1$. Then let f be the total function which to each pair $\langle n, m \rangle$ associates the smallest code $o_{n,m} \in \mathcal{W}$ such that:

$$\lambda(F_{|o_{n,m}|}^c \cup P_{n,|o_{n,m}|}) > 1 - 2^{-m}$$

where F^c_α is the Δ^1_1 set of strings which are witnessed to be in F^c via an ordinal smaller than α. Using second part of Theorem 13 one can prove that f is Π^1_1. Let $\alpha^* = \sup_{n,m} |f(n,m)|$. By Theorem 14 we have that $\alpha^* < \omega_1^{ck}$. Then we have:

$$\begin{array}{rlcl} & \forall n \ \ \lambda(F^c_{\alpha^*} \cup \bigcup_{\alpha<\alpha^*} P_{\alpha,n}) & = & 1 \\ \rightarrow & \forall n \ \ \lambda(F_{\alpha^*} \cap \bigcap_{\alpha<\alpha^*} P^c_{\alpha,n}) & = & 0 \\ \rightarrow & \forall n \ \ \lambda(F - \bigcup_{\alpha<\alpha^*} P_{\alpha,n}) & = & 0 \\ \rightarrow & \lambda(F - \bigcap_n \bigcup_{\alpha<\alpha^*} P_{\alpha,n}) & = & 0 \end{array}$$

As X is Σ^1_1-Solovay-generic it is in particular weakly-Σ^1_1-Solovay-generic and then weakly-Π^1_1-random. Thus by Theorem 21 it belongs to no Σ^1_1 set of measure 0. Then as $F - \bigcap_n \bigcup_{\alpha<\alpha^*} P_{\alpha,n}$ is a Σ^1_1 set of measure 0 we have that X belongs to $\bigcap_n \bigcup_{\alpha<\alpha^*} P_{\alpha,n}$ and then $\sup_n \Phi^X(n) \leq \alpha^* < \omega_1^{ck}$. ◀

We can now prove the equivalence:

▶ **Theorem 26.** *The set of Σ^1_1-Solovay-generics is exactly the set of Π^1_1-randoms.*

Proof. Using Theorem 20 we have that the Σ^1_1-Solovay-generics are included in the Π^1_1-randoms. We just have to prove the reverse inclusion.

Suppose Y is not Σ^1_1-Solovay-generic. If $\omega_1^Y > \omega_1^{ck}$ then Y is not Π^1_1-random. Otherwise $\omega_1^Y = \omega_1^{ck}$ and in this case there is a sequence of Σ^1_1 closed sets $\bigcup_n F_n$ of positive measure such that X is not in $\bigcup_n F_n$ and such that any Σ^1_1 closed set of positive measure which is disjoint from $\bigcup_n F_n$ does not contain Y. The complement of $\bigcup_n F_n$ is a Π^1_1 set P containing Y. Let e be so that $P(X) \leftrightarrow \exists \alpha < \omega_1 \ e \in \mathcal{W}^X_\alpha$. As $\omega_1^Y = \omega_1^{ck}$ and $P(Y)$, we have that $\exists \alpha < \omega_1^{ck} \ e \in \mathcal{W}^Y_\alpha$. But then Y is in a Δ^1_1 set that one can approximate using Theorem 13 by an effective union of Σ^1_1 closed sets of the same measure. Thus as X can be in none of them it is in a Π^1_1 set of measure 0 and then not Π^1_1-random. ◀

The previous theorem gives an interesting corollary, making a connection with another domination property. We say that a sequence X is hyp-dominated if for every total function $f : \omega \rightarrow \omega$, Δ^1_1 relative to X, there exists a total Δ^1_1 function g such that g dominates f (i.e. $\forall n \ f(n) \leq g(n)$). Chong, Yu and Nies proved in [2] that all Π^1_1-random sequences are hyp-dominated. It follows from that and from the previous theorem that a sequence X is Σ^1_1-Solovay-generic iff it is weakly-2-random and hyp-dominated. This can be seen as the higher counterpart of Proposition 11.

We have a second corollary, giving a higher bound on the Borel complexity of the Π^1_1-randoms, and then on the biggest Π^1_1 nullset.

▶ **Corollary 27.** *The set of Π^1_1-randoms is $\mathbf{\Pi^0_3}$.*

The $\mathbf{\Pi^0_3}$ set is obtained exactly the same way we obtain the $\mathbf{\Pi^0_3}$ set of computably dominated sequences. The following result of Liang Yu (see [16]) can be used to prove that the set of Π^1_1-randoms is not $\mathbf{\Sigma^0_3}$.

▶ **Theorem 28** (Liang Yu). *Let $\bigcap_n O_n$ be a $\mathbf{\Pi^0_2}$ sets contaning only weakly-Π^1_1-randoms. Then the set $\{F \mid F$ is a Σ^1_1 closed set and $\bigcap_n O_n \cap F = \emptyset\}$ intersects with positive measure any Σ^1_1 closed sets of positive measure.*

It follows that the set of weakly-Π^1_1-randoms cannot be $\mathbf{\Sigma^0_3}$ but also that the set of Π^1_1-randoms cannot be $\mathbf{\Sigma^0_3}$. Indeed, suppose that the set of Π^1_1-randoms is equal to $\bigcup_n \bigcap_m O_{n,m}$ each $O_{n,m}$ being open. For each n let $A_n = \{F \mid F$ is a Σ^1_1 closed set and $\bigcap_m O_{n,m} \cap F = \emptyset\}$. We have $\bigcap_n A_n \cap \bigcup_n \bigcap_m O_{n,m} = \emptyset$, and from Theorem 28 we have that $\bigcap_n A_n$ contains some Solovay-Σ^1_1-generic elements, which contradicts that $\bigcup_n \bigcap_m O_{n,m}$ contains all of them.

The question whether it is possible for X to be weakly-Solovay-Σ^1_1-generic but not Solovay-Σ^1_1-generic (equivalently weakly-Π^1_1-random but not Π^1_1-random) is still open. The technique that we use in the lower case to separate weak genericity from non weak genericity does not seem to work here.

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References

1 George Barmpalias, Rod Downey, and Keng Meng Ng. Jump inversions inside effectively closed sets and applications to randomness. *Journal of Symbolic Logic*, 76(2):491–518, 2011.
2 Chi Tat Chong, André Nies, and Liang Yu. Lowness of higher randomness notions. *Israel Journal of Mathematics*, 166:39–60, 2008.
3 Chi Tat Chong and Liang Yu. Randomness in the higher setting. *under refereeing*, `http://math.nju.edu.cn/~yuliang/cy1.pdf`.
4 Paul J Cohen. *Set theory and the continuum hypothesis.* Dover Publications, 1966.
5 Joel David Hamkins and Andy Lewis. Infinite time Turing machines. *Journal of Symbolic Logic*, pages 567–604, 2000.
6 Greg Hjorth and André Nies. Randomness via effective descriptive set theory. *Journal of the London Mathematical Society*, 75(2):495–508, 2007.
7 Carl G Jockusch Jr. Simple proofs of some theorems on high degrees of unsolvability. *Canadian Journal of Mathematics*, 29(5):1072–1080, 1977.
8 Alexander S Kechris. The theory of countable analytical sets. *Transactions of the American Mathematical Society*, 202:259–297, 1975.
9 S Kurtz. Randomness and genericity in the degrees of unsolvability. phd diss., University of Illinois. *Urbana*, 1981.
10 Per Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.
11 Per Martin-Löf. On the notion of randomness. *Studies in Logic and the Foundations of Mathematics*, 60:73–78, 1970.
12 Andre Nies. *Computability and Randomness.* Oxford University Press, 2009.
13 Gerald E Sacks. *Higher recursion theory.* Springer, 2010.
14 Robert M Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *The Annals of Mathematics*, 92(1):1–56, 1970.
15 Liang Yu. Descriptive set theoretical complexity of randomness notions. *Fundamenta Mathematicae*, 215:219–231, 2011.
16 Liang Yu. LogicBlog2013, Higher randomness, `http://dl.dropboxusercontent.com/u/370127/Blog/Blog2013.pdf`, 2013.

A Proof of Theorem 12

For this proof we will use the Kučera-Gács theorem, saying that within any Π^0_1 set of positive measure, we can computably encode any real by some sequence of this Π^0_1 set. However for both the encoding and the decoding we need to know the same lower bound on the measure of the set. We give here the exact theorem that we use to do our proof:

▶ **Theorem 29** (Kučera-Gács). *There is a computable function Φ such that uniformly in an integer n, a rational q and an index e for a Π^0_1 set F with $\lambda(F) > q$, one can find, relatively to the halting problem, a string σ so that $[\sigma] \cap F$ is still of positive measure and so that the function Φ with σ as an oracle and applied to $\langle e, q\rangle$, outputs n.*

One can find a detailed proof of this theorem in section 3.3.2 of [12]. Let us say that a Σ^0_2 set has the $(*)$ property if it intersects with positive measure all Π^0_1 sets of positive measure. The weakly-Π^0_1-Solovay-generic sequence X will be defined as the limit of a sequence of strings $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \dots$ of longer and longer length. We first give a naive version of our argument: In the first Π^0_1 component F of the first Σ^0_2 set with the (*) property, we use the Kučera-Gács theorem to encode $f(0)+1$ using an index e for F and a lower bound q for the measure of F. It means that as specified in Theorem 29 we find a string σ_0 so that the function Φ with σ_0 as an oracle and applied to $\langle q, e\rangle$ will output $f(0)+1$. As $[\sigma_0] \cap F$ still has positive measure there is a Π^0_1 set enumerated in the second Σ^0_2 set with the $(*)$ property, which intersects $[\sigma_0] \cap F$ with positive measure. We then encode in this second Π^0_1 set the value $f(1)+1$. We can continue like this for all Σ^0_2 sets with the (*) property.

There are two obstacles here. During the decoding, we do not know what Π^0_1 sets and what lower bound on their measure have been used for the encoding. So we do not just encode in each Π^0_1 set the value $f(n)+1$, but also the index of the next Σ^0_2 set with the $(*)$ property. But even if we have the right Σ^0_2 set, we still do not know which of its Π^0_1 sets have been used. We fix this by doing something special in both the encoding and the decoding. Fix in advance an enumeration $\{\langle n_i, q_i\rangle\}_{i<\omega}$ where n_i is an integer and q_i a rational. During the encoding, pick the Π^0_1 set number n_i such that the pair $\langle n_i, q_i\rangle$ is the first in the enumeration with the property that q_i is a lower bound of its measure. During the decoding, we will pick Π^0_1 sets in the order given by the same enumeration. If at some point the measure goes below the corresponding rational we will know it in a finite time. Then we restart the decoding with the next Π^0_1 set in the enumeration. We know that at some point we will have the right one.

However, a last problem is that by the time we have the right Π^0_1 set, we might have decided a lot of values of the function f and maybe the one that we have coded is already taken. The trick is to design the encoding in a way that we know in advance the time it will take to reach the right Π^0_1 set. The value of f we encode has to be chosen accordingly. We now give the details.

The encoding:

Without loss of generality we can suppose f strictly increasing. Let $\{S_i\}_{i\in\omega}$ be an enumeration of all the Σ^0_2 sets. For each S_i and each n let us define the Π^0_1 set $F_{i,n}$ so that $S_i = \bigcup_n F_{i,n}$. Now let $\{e_i\}_{i<\omega}$ be a list of indices for all the Σ^0_2 sets S_{e_i} having the $(*)$ property. Let $\{\langle n_i, q_i\rangle\}_{i<\omega}$ be an effective list of all pairs of integers and rationals.

Let us define a string $\sigma_0 = \epsilon$ (the empty string), a Π^0_1 set $T_0 = F_{e_0,0}$, and an integer representing time with $t_0 = 0$. Start by encoding $f(0)+1$ and e_1 into $T_0 \cap [\sigma_0]$, assuming without loss of generality that it has measure bigger than some q that we will reuse in the decoding. Let σ_1 be the string encoding those values. Suppose now that for $i \leq k+1$

the strings σ_i have been defined, and for $i \leq k$ the integers t_i and the Π^0_1 sets T_i have been defined. Let us define σ_{k+2}, t_{k+1} and T_{k+1}. Let i be the smallest integer such that $\lambda(F_{e_{k+1},n_i} \cap T_k \cap [\sigma_{k+1}]) \geq q_i$. Let t be the computational time necessary to decode the two values encoded in $T_k \cap [\sigma_k]$, which is also the computational time necessary to find σ_{k+1} during the decoding, assuming we already know X, T_k, σ_k, and the lower bound on the measure of $T_k \cap [\sigma_k]$ that has been used for the encoding. Let t' be the smallest computational time s so that for all $j < i$ we have $\lambda(F_{k+1,n_j} \cap T_k \cap [\sigma_{k+1}])[s] < q_j$, and let $t_{k+1} = t_k + t + t'$. let T_{k+1} be $F_{e_{k+1},n_i} \cap T_k$. Then we encode $f(t_{k+1}+1)+1$ and e_{k+2} into $T_{k+1} \cap \sigma_{k+1}$, using q_i as a lower bound. Finally let σ_{k+2} be the string encoding those values.

The decoding:

We set e_0 to be the same as the one used in the encoding, $n_0 = 0$, q_0 to be the measure used in the encoding of the first two values in $T_0 \cap [\sigma_0]$, and $T_0 = F_{e_0,n_0}$. For each $k > 0$ we initialize the Π^0_1 set $T_k = \emptyset$, the integers $e_k = 0$ and the pairs $\langle n_k, q_k\rangle$ to be the first element in the list $\{\langle n_i, q_i\rangle\}_{i<\omega}$. Then we set for each $k \geq 0$ the string $\sigma_k = \epsilon$.

At stage t for each substage $k \leq t$ in increasing order, if $T_k = \emptyset$ go to the stage $t+1$. Otherwise check whether $\lambda(T_k \cap [\sigma_k])[t] \geq q_k$.

Case 1 : $\lambda(T_k \cap [\sigma_k])[t] \geq q_k$. Then perform the t first computation steps of the decoding with $T_k \cap [\sigma_k]$ as the Π^0_1 set and q_k as the lower bound on the measure. If in t steps or less we get two values a and b, set all the unassigned values of $g(m)$ for $m \leq t$ to be a and set e_{k+1} to be b. Also set σ_{k+1} to be the prefix used in the decoding and T_{k+1} to be $T_k \cap F_{e_{k+1},n_{k+1}}$. Then go to next substage, or next stage if it is the last substage.

Case 2 : $\lambda(T_k \cap [\sigma_k])[t] < q_k$. Then move $\langle n_k, q_k\rangle$ to the next element in the list $\{\langle n_i, q_i\rangle\}_{i<\omega}$. Set $T_k = T_{k-1} \cap F_{e_k,n_k}$. Also for all $k' > k$ reset $T_{k'}$ to be $\emptyset$ and $\langle n_{k'}, q_{k'}\rangle$ to the first element in the list $\{\langle n_i, q_i\rangle\}_{i<\omega}$. Then restart at the same stage t and the same substage k.

Verification:

Let the stages t_k be those that we defined during the encoding process. Let us prove that at stage t_k and substage k of the decoding process we have the right Π^0_1 set T_k, the right string σ_k and the right value q_k used for the encoding.

It is obviously true for t_0. Suppose this is true up to k and let us show this is true for $k+1$. Let i be the index so that $\langle n_i, q_i\rangle$ is the first pair in the enumeration with $\lambda(F_{e_{k+1},n_i} \cap T_k \cap [\sigma_{k+1}])[t] \geq q_i$. Recall that $t_{k+1} = t_k + t + t'$, where t' is the smallest time such that for all $j < i$ we have $\lambda(F_{e_{k+1},n_j} \cap T_k \cap [\sigma_{k+1}])[t] < q_j$ and where t is the computational time necessary to find σ_{k+1} during the decoding, assuming we already know T_k, σ_k and the lower bound on the measure of $T_k \cap [\sigma_k]$ that have been used for the encoding. By the induction hypothesis we already have those three inputs at time t_k. Then at time $t_k + t$ we know T_k, σ_{k+1}, and using σ_{k+1}, we know e_{k+1}. Obviously by the time $t_k + t + t'$ we could eliminate all the F_{e_{k+1},n_j} for $j < i$ and then we have the right T_{k+1}.

Now let us prove that g is infinitely often bigger than f. When we just moved to the right T_k, no more values of g will be decided until the two values are decoded inside $T_k \cap [\sigma_k]$. The reason is that for all $k' > k$ the set $T_{k'}$ is reset to $\emptyset$ in the algorithm. Then until this is done the values of g are decided at most up to t_k. And once this is done we have the right value of $f(t_k+1)+1$ that we will have assigned to some $g(s)$ for at least one $s \leq t_k + 1$. As f is strictly increasing we have $g(s) > f(s)$.