GENERICITY AND RANDOMNESS WITH ITTMS

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Abstract. We study genericity and randomness with respect to ITTMs, continuing the work initiated by Carl and Schlicht. To do so, we develop a framework to study randomness in the constructible hierarchy. We then answer several of Carl and Schlicht's question. We also ask a new question one the equality of two classes of randoms. Although the natural intuition would dictate that the two classes are distinct, we show that things are not as simple as they seem. In particular we show that the categorical analogues of these two classes coincide, in contradiction with the natural intuition. Even though we are not able to answer the question for randomness in this paper, we delineate and sharpen its contour and outline.

§1. Introduction.

1.1. Background. The study of Infinite-Time Turing machines, ITTMs for short, goes back to a paper by Hamkins and Lewis [14]. Informally these machines work like regular Turing machines, with the addition that the time of computation can be any ordinal. Special rules are then defined to specify what happens at a limit step of computation.

This simple computational model yields several new non-trivial classes of objects, the first one being the class of objects which are computable using some ITTM. These classes have been later well understood and characterized by Welch [24]. ITTMs are not the first attempt of extending computability notions. This was done previously for instance with α-recursion theory, an extension of recursion theory to Σ_1-definability of subsets of ordinals, within initial segments of the Gödel constructible hierarchy. Even though α-recursion theory is defined in a rather abstract way, the specialists have a good intuition of what "compute" means in this setting, and this intuition relies on the rough idea of "some" informal machine carrying computation times through the ordinal. ITTMs appeared all the more interesting, as they consist of a precise machine model that corresponds to part of α-recursion theory. It is worthy to note that there is now an exact characterization of α-recursion via machine models due to Koepke and Seyfferth, using variants of ITTMs with an ordinal tape, in [17].

Recently Carl and Schlicht used the ITTM model to extend algorithmic randomness [6] and effective genericity notions [5] in this setting. Genericity and randomness are two different approaches to study typical objects, that is, objects having "all the typical properties" for some notion of typicality. For randomness, a property is typical if the class of reals sharing it is of measure 1, whereas for genericity, a property is typical if the class of reals sharing it is co-meager. In both cases, for any countable collection of typical properties, it is still a typical property to share all of them: the intersection of countably many measure-one sets is still a measure-one set, and the intersection of countably many co-meager sets is still a co-meager set. Depending on the countably many properties we consider, the reals that share all of them may be of great interest, in forcing constructions or to study various notions of degrees, from Turing to α-degrees.

Algorithmic randomness has known an impressive development during the past twenty years. A very rich theory emerged, as a complex and beautiful answer to the original philosophical question of what are random objects. Just like recursion theory had been extended to higher recursion theory, to α-recursion theory and to the theory of ITTMs, algorithmic randomness is meant to know a similar development. This has been started with Higher randomness by Hjorth and Nies [16], Chong and Yu [7] [8] and Bienvenu, Greenberg and Monin [13] [3] [21]. Recently this was extended to ITTMs by Carl and Schlicht [6]. The goal of this paper is first to pursue their work.

1.2. About this paper. We answer in this paper several open questions of Carl and Schlicht, and we ask new ones. This paper also aims at developing a framework that can be used in general to study randomness and genericity within Gödel's constructible hierarchy. For this reason, the first half of the paper focuses on recalling the main results (in Section 2) and on developing this framework (in Section 3). We include these rather long sections to the paper, in order to make it as self-contained as possible: it is meant to be readable by the trained logician, not necessarily familiar with ITTMs or constructibility. However some formal details of Section 2 and Section 3 may be a bit tedious to read, and there is no way around that. Any recursion theorist may have struggled in its early days to read all the technical details on the equivalence between various models of computations, and developed after that a very solid intuition of what is computable, without the necessity of coming back every time to the formal definitions. Also the reader who is not familiar with constructibility will certainly need to furnish a similar effort with some proof of Section 3 and maybe also Section 2, whereas the reader who is used to it will certainly have no problem admitting these theorems without reading the proofs. Despite the difficulties inherent to the material presented here, we tried as much as possible never to confuse rigor and formalism, by ensuring the former without getting trapped in the latter.

In Section 4, even though we answer several questions of [6], we feel that this section's main achievement is not there, but more in a new question (Question 4.9) that we ask on the separation of two randomness notions defined by Carl and Schlicht. It seems so clear at first that the two notions should be different, that the question was not asked so far. The reason is certainly that the analogues of this two notions in Higher randomness actually differ for simple reasons. We emphasize here that things are not so simple in the settings of ITTMs, and we show that the two notions are much closer than we think, even though we are not able to settle the question.

This question was the original motivation for Section 5: In order to argue that it is not absurd to think that these two randomness notions may actually coincide, we show that it is the case for their categorical analogues. Note that the versions of these analogues with Higher genericity are also known to differ for simple reasons, like it is the case with randomness. In some sense, Theorem 5.13 that shows equality of these notions, may actually be the most important of this paper: it uses the new phenomenons that occur within some levels of the constructible hierarchy to show that two classes collapse in a very unexpected way. Despite that, we decided to leave this section at the end, so that the paper follows the logic exposed so far, that we now sum up:

In Section 2 we expose the background (with a few original results such as Theorem 2.29 to Theorem 2.31), in Section 3 we develop a general framework to study randomness in any limit level of the constructible hierarchy, in Section 4 we study randomness notions with respect to ITTMs, focusing first on the question we mentioned above and proving several results meant to delineate and sharpen the contour and outline of this question. In this same section we then answer several questions of [6], the most interesting theorem about that being maybe Theorem 4.22. In Section 5 we then define and study, in the setting of ITTMs, the categorical analogues of the studied randomness notions. The section focuses on answering for categoricity the question that is still too hard in the randomness case.

§2. Background. Ordinals will be denoted by letters $\alpha, \beta, \gamma, \delta$. Ordinals will sometimes be seen as computation times, in which case they might also be denoted by letters r, s, t. Reals will be denoted by letters x, y, z, which will also denotes sometimes constructible sets. Integers will be denoted by letters n, m, k, i, j, e.

2.1. Infinite-Time Turing machines. We first briefly recall the three-tapes ITTM model as it was introduced in [14]. We then argue that this model is *essentially* equivalent, to a one-tape machine model that is the one we are going to use in this paper.

2.1.1. *The three-tape machine model.* In the three-tapes ITTM model, machines have an input tape, a working tape and an output tape. The input tape is meant to contain an oracle the machine can use during its computation, the working tape is where the machine is meant to perform its computation, and the output tape is where the machine is meant to write the result of its computation. Each tape is a sequence of cells indexed by ω.

The head of the machine reads simultaneously the n-th value of the three tapes altogether. At a successor step of computation, the behavior is as the one of a standard Turing machine: Given the current state of the machine and given the values read by the head, it may write something where it is, then goes left or right, and the machine may change state, all according to a finite set of transition rules fixed in advance and which determines the machine.

At a limit step of computation, the machine enters a special "limit" state, the head goes back to the first cell, and the value of each cell is the lim inf of its previous values: if the value of a cell converges, it is set to its value of convergence, otherwise it is set to 0. Every machine also has a halting state. Whenever a machine enters this state, it stops.

2.1.2. *The one-tape machine model.* It will be convenient here to consider a one tape infinite-time Turing machine instead of a three-tapes infinite-time

Turing machine: it is not very hard to see that any three-tape ITTM M can be simulated by a one-tape ITTM N, where the n-th cell of M's input tape corresponds to $3n$-th cell of N, the n-th cell of M's working tape corresponds to the $(3n+1)$-th cell of N, and the n-th cell of M's output tape corresponds to $(3n+2)$-th cell of N.

This is done formally in [15, Lemma 1.1]. Note that if one starts with the tape fully filled with the oracle, instead of filled only the part corresponding to the input tape of the three-tapes model, the equivalence between the one-tape model and the three-tapes model breaks. This is studied in [15]. Here this is not the case, and we work with the one-tape machine model as if it was the three-tape one, but with all the tapes condensed in one.

2.1.3. *Notations.* Given an ITTM M, we write $M(x) \downarrow [\alpha]$ if the machine M starts with the real x on its input tape, and if M enters its halting state at stage $\alpha + 1$. Furthermore if at stage α the real y is written on the output tape, we write $M(x) \downarrow [\alpha] = y$. We also write $M(x)[\alpha] = y$ if at stage α the real y is written on M's output tape (and the machine does not necessarily halt). If the machine M starts with its input tape empty, we simply write $M \downarrow [\alpha] = y$ and $M[\alpha] = y$.

Given an ITTM M, we denote by $C_M(n)[\alpha]$ the value (0 or 1) of the cell number n of M at stage α (when M starts with the empty set on its input tape). We denote by $C_M[\alpha]$ the sequence of values of all the cells at stage α. The value $C_M[\alpha]$, together with the state of the machine and the position of its head, at stage α, is called the *configuration* of the ITTM at stage α. Note that at a limit stage α, the configuration of an ITTM depends only of $C_M[\alpha]$ (the head always being at the first cell and the state always being the limit state).

Finally it is straightforward, though tedious, to show that there is a universal ITTM, that simulates in parallel all the ITTMs (see [14] or [24]). This universal ITTM will be denoted by U.

2.1.4. *Main theorems.*

Definition 2.1 (Hamkins, Lewis [14]). Let $y \in 2^\omega$.

- The real y is *writable* if there is an ITTM M such that $M \downarrow [\alpha] = y$.
- The real y is *eventually writable* if there is an ITTM M and an ordinal α such that $\forall \beta \geq \alpha$ we have $M[\beta] = y$.
- The real y is *accidentally writable* if there is an ITTM M and a stage α such that $M[\alpha] = y$.

For $x \in 2^\omega$ we define the notions of x-writable, x-eventually writable and x-accidentally writable similarly, but with the ITTMs starting with x on their input tape.

Hamkins and Lewis introduced the three previous analogues of being computable by an ITTM. They used these notions to study the ordinals that are computable by an ITTM, with respect to these definitions. In what follows, we use the following coding system of countable ordinals by reals: $x \in 2^\omega$ codes for α if the order type of $<_x$ is the order-type of α, where $n <_x m$ iff $\langle n, m\rangle \in x$, where $\langle , \rangle : \omega^2 \to \omega$ is fixed a computable bijection. We say that such a real x is a code for α.

Definition 2.2 (Hamkins, Lewis [14]). An ordinal α is *writable*, resp. *eventually writable*, resp. *accidentally writable*, if it has a writable code, resp. an eventually writable code, resp. an accidentally writable code. For $x \in 2^\omega$ we define analogously the notion of x-writable, x-eventually writable and x-accidentally writable.

It is clear that the writable, eventually writable, and accidentally writable ordinals, are all initial segments of the ordinals. Hamkins and Lewis showed that the supremum of the writable ordinals was eventually writable and that the supremum of the eventually writable ordinals was accidentally writable.

Definition 2.3 (Hamkins, Lewis [14]). We define the following ordinals:

- λ is the supremum of the writable ordinals.
- ζ is the supremum of the eventually writable ordinals.
- Σ is the supremum of the accidentally writable ordinals.

λ^x, ζ^x, Σ^x are defined the same way but relative to x.

Hamkins and Lewis also defined the clockable ordinals: an ordinal α is clockable if it is the halting time of some ITTM M, that is, $M \downarrow [\alpha]$. It is clear that the supremum of the clockable ordinals is at least λ: if an ordinal α is writable, one can design the machine that writes α and then "counts down α" in at least α steps [1]. The question of equality between λ and the supremum of the clockable ordinals was one of the main question in Hamkins and Lewis [14]. It was later solved by Welch:

Theorem 2.4 (Welch [24]). *Let M be an ITTM.*

1. *If $\{C_M(n)[\alpha]\}_{\alpha<\lambda}$ converges to $i \in \{0,1\}$, then $C_M(n)[\alpha] = i$ for every $\alpha \geq \lambda$.*
2. *If $\{C_M(n)[\alpha]\}_{\alpha<\zeta}$ converges to $i \in \{0,1\}$, then $C_M(n)[\alpha] = i$ for every $\alpha \geq \zeta$.*
3. *If $\{C_M(n)[\alpha]\}_{\alpha<\zeta}$ diverges iff $\{C_M(n)[\alpha]\}_{\alpha<\Sigma}$ diverges.*

We have in particular $C_M[\zeta] = C_M[\Sigma]$. Also ζ, Σ is the lexicographically smallest pair of ordinals such that $C_U[\zeta] = C_U[\Sigma]$ for the universal machine U.

Note that, while it is standard to state the theorem in this way, we could also have said that each entry is minimal:

Corollary 2.5 (Welch [24]). *The ordinal λ is also the supremum of ITTMs' halting time.*

Proof. By Theorem 2.4, we have that if an ITTM M has not halted before stage Σ, then it will never halt, because the configuration of an ITTM at stage Σ is the same as the configuration of an ITTM at stage ζ, and every 1 in the tape at stage ζ will stay a 1 at every stage between ζ and Σ. Thus the computation will loop forever, and if an ITTM halts it must halt before stage Σ. We can then run an ITTM which looks for all the accidentally writable ordinals α (using some universal ITTM) and for each of them, which runs M for α steps. When

[1] For instance one can search for the smallest element of the order written on the tape, remove it and repeat that until the order is empty, then halts. This takes at least α steps of computation.

the machine finds an accidentally writable ordinal α such that $M[\alpha]\downarrow$, then it writes α and halts. By hypothesis on M our ITTM will write α and halt at some point. Thus α is a writable ordinal, which implies that M halts at a writable step. ⊣

Welch's theorem and proof provided a clear understanding of ITTMs allowing us, as we will see it soon, to cut ourselves off the machine model, and to reason within the constructible hierarchy.

2.2. The constructibles.

2.2.1. *Notations.* We denote by $\mathrm{tc}(x)$ the transitive closure of x. We recall that a formula of set theory is said to be Δ_0 if it has only bounded quantifiers, that is, of the form $\exists x \in y$ or $\forall x \in y$. The Σ_n and Π_n formulas are then built like their analogue in the language of arithmetic, but with the quantification done over all the sets of the model we consider.

Definition 2.6. Let M be an $\mathcal{L}$-structure for some language $\mathcal{L}$, and $p \in M$. We say that $P \subseteq M^k$ is Σ_n^M definable (or Σ_n definable in M) with parameter p if there is a Σ_n formula Φ such that $M \models \Phi(x_1, \ldots, x_k, p)$ iff $(x_1, \ldots, x_k) \in P$. The Π_n^M definable subsets of M are defined similarly, but with Π_n formulas.

We say that $P \subseteq M^k$ is Δ_n^M definable (or Δ_n definable in M) with parameter p if it is both Σ_n^M and Π_n^M definable with parameter p.

We sometimes write $\Sigma_n^M(p)$ (resp. $\Pi_n^M(p)$) to mean Σ_n^M-definable with parameter p (resp. Π_n^M-definable with parameter p).

2.2.2. *The constructible universe.* The study of ITTM is closely related to the study of α-recursion theory, restricted to the three special ordinals λ, ζ and Σ. One difference is that whereas in α-recursion theory, we study subsets of ordinals, with ITTMs, we study subsets of integers. It involves a manipulation of initial segments of the constructible universe. We recall here the main definitions and theorems that will be used in the paper. This section is mainly for the computability theorist not yet comfortable with extending notions of computations to Σ_1-definability within initial segments of the constructible hierarchy. We will also focus on Σ_1-definability inside L_α even when α is not admissible (we will see that we need to do so because the smallest non-accidentally writable ordinal, Σ, is not admissible).

The constructible universe is usually defined starting with $L_\emptyset = \emptyset$. When using some oracle $x \in 2^\omega$, it starts with $L_\emptyset(x) = \mathrm{tc}(x)$ (which equals $\{x\} \cup \omega$ when x is infinite). In order to keep some consistency between the constructible universe defined with and without oracle, we start with $L_\emptyset = \omega$.

Definition 2.7. The constructible universe is defined by induction over the ordinals as follow:

- $L_\emptyset = \omega$
- $L_{\alpha+1} = \{X \subseteq L_\alpha : X \text{ is first order definable in } L_\alpha \text{ with parameters in } L_\alpha\}$
- $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma$ when α is limit.

Let $x \in 2^\omega$. The constructible universe starting with x as an oracle is defined by induction over the ordinals as follow:

- $L_\emptyset(x) = \{x\} \cup \omega$

- $L_{\alpha+1}(x) = \{X \subseteq L_\alpha(x) : X$ is first order definable in $L_\alpha(x)$ with parameters in $L_\alpha(x)\}$
- $L_\alpha(x) = \bigcup_{\gamma<\alpha} L_\gamma(x)$ when α is limit.

For $a \in L$, the rank (or L-rank when confusion is possible) of a, denoted by $\mathrm{rk}(a)$, is the smallest α such that $a \in L_{\alpha+1}$.

2.2.3. *Admissibility and definition by induction.* In order to safely conduct Σ_1 inductions, we normally need to be in a model of KP: a weakening of set theory in which we have extensionality, pairing, union, Cartesian product, induction over the $\in$ relation (suppose for all a we have $[\forall b \in a\ \Phi(b)] \to \Phi(a)$, then for all a we have $\Phi(a)$), Δ_0-comprehension and Σ_1-collection.

For any α limit, we have that L_α is a model of all these axioms, except Σ_1-collection.

Definition 2.8. Let $\mathcal{A} = (A, \epsilon)$ be an $\mathcal{L}$-structure for the language of set theory. We say that $\mathcal{A}$ is a Σ_n-admissible structure if $\mathcal{A}$ is a model of extensionality, pairing, union, Cartesian product, induction over the $\in$ relation, Δ_n-comprehension and Σ_n-collection.

Definition 2.9. We say that α is admissible if L_α is a model of KP, that is, if α is limit and L_α is a model of Σ_1-collection and Δ_1-comprehension.

More generally we say that α is Σ_n-admissible if L_α is a Σ_n admissible structure.

Dealing with ITTMs, we will have to work with ordinals which are not necessarily admissible. We will see for instance that Σ, the smallest non-accidentally writable ordinal, is not admissible.

Fortunately, we can already define a lot of things in models L_α for α simply limit (and not necessarily admissible). Working with the constructibles involves constantly working with Σ_1-inductive definitions. Whereas these are perfectly safe in L_α for α admissible, some additional care needs to be taken when α is not admissible. Let us determine what we need:

Let $E \in L_\alpha$ and $< \in L_\alpha$ be a well-founded order on elements of E. We define by induction the $<$-rank of elements $a \in E$, denoted by $\mathrm{rk}_<(a)$, to be the smallest ordinal β such that for every $b < a$, b has $<$-rank less than β. Let E_β be the elements of E of $<$-rank strictly smaller than or equal to β, let $E_{<\beta}$ be the elements of E of $<$-rank smaller than β and $E_{=\beta}$ the elements of E of $<$-rank exactly β. Let γ be the supremum of the $<$-rank of elements of E and suppose $\gamma \leq \alpha$.

Suppose we have a Δ_0 formula $F(a, f, r)$ such that for any $a \in E$, with $\mathrm{rk}_<(a) = \beta$ whenever $f \in L_\alpha$ is defined on $E_{<\beta}$, then there is a unique $r \in L_\alpha$ such that $L_\alpha \models F(a, f, r)$. The classical theorem of set theory, that justifies definition by induction, says that we then have a unique function f defined on E and such that the Δ_1 formula $\Phi(\gamma, f)$ is true, where:

$$\Phi(\gamma, f) \equiv \text{For every } \beta < \gamma, \text{ for every } a \in E_{=\beta}, \text{ we have } F(a, f\restriction_{E_{<\beta}}, f(a))$$

Indeed the function f, if it exists, must be unique and Δ_1-recognizable by the formula $\Phi(\gamma, f)$ (using parameter γ). Also by induction one show that whenever $f\restriction_{E_{<\beta}}$ exists, then $f\restriction_{E_{<\beta+1}}$ must exists as it is Δ_1-definable by F with $f\restriction_{E_{<\beta}}$ as a parameter (see Proposition 2.10 below). This uses the axiom of Σ_1 collection: if for all $a \in E_{=\beta}$ there exists a unique $r \in L_\alpha$ such that $F(a, f\restriction_{E_{<\beta}}, r)$, then the corresponding function f' defined on $E_{=\beta}$ must exists. However if the ranks of the r's are unbounded in L_α, the function f' will not exist in L_α. Fortunately most of the time, for simple tasks, the rank will be bounded in L_α by something independent of $a \in E_{=\beta}$, but dependent only on β.

The axiom of Σ_1-collection also needs to be used at a limit step: If for any $\gamma < \beta$, there exists a unique function f_γ defined on E_γ and such that $\Phi(\gamma, f_\gamma)$, then by Σ_1-collection there exists a unique function f_β such that $\Phi(\beta, f_\beta)$ (and the function f_β is simply the union of the functions f_γ). Here again, this argument works within L_α as long as the rank of each function f_γ is bounded in L_α. We sum up in the following proposition conditions in which definitions by induction can be conducted in L_α for α limit:

Proposition 2.10 (Δ_0 Induction with bounded rank replacement). *Let E be a class well-ordered by $<$. Let $f : E \mapsto L$ be Δ_0-definable by induction on $<$, such that for any β there exists $k < \omega$ for which:*

1. *E_β is $\Delta_1^{L_{\beta+k}}$-definable uniformly in β, in particular $E_{<\alpha} \subseteq L_\alpha$ for α limit.*
2. *For any $a \in E_\beta$, $\mathrm{rk}_<(a)$ is $\Delta_1^{L_{\beta+k}}$-definable uniformly in β.*
3. *For any $a \in E_\beta$ we have $\mathrm{rk}(f(a)) < \beta + k$.*

Then f is $\Delta_1^{L_\alpha}$-definable uniformly in any limit ordinal α. By this we mean that there are single Π_1 and Δ_1 formulas that define $f\restriction_{E_{<\alpha}}$ when interpreted in L_α.

Proof. Let $\Phi(\beta, f)$ be the Δ_1 formula defined in the discussion above. We shall show that for any α limit we have:

(a) For any $\beta < \alpha$, the function $f\restriction_{E_\beta}$ belongs, as a set, to $L_{\beta+m}$ for some $m < \omega$.
(b) The function $f\restriction_{E_{<\alpha}}$ is $\Delta_1^{L_\alpha}$-definable by the formulas:

$$\begin{aligned} f\restriction_{E_{<\alpha}}(a) = r \quad &\equiv \quad \exists f\ \Phi(\mathrm{rk}_<(a), f) \wedge f(a) = r \\ &\equiv \quad \forall f\ \Phi(\mathrm{rk}_<(a), f) \to f(a) = r \end{aligned}$$

It is clear that for any α limit we have (a) implies (b). Suppose now $\alpha = 0$ or α limit and (b) is true for α, and let us show that (a) is true for $\alpha + \omega$. If $\alpha = 0$ we clearly have $f\restriction_{E_{<\alpha}} \in L_{\alpha+1}$. If α is limit and (b) is true for α, thus also it is clear by definition of L that $f\restriction_{E_{<\alpha}} \in L_{\alpha+1}$. Now from $f\restriction_{E_{<\alpha}} \in L_{\alpha+1}$ together with (1) (2) and (3), by iterating inductively the same argument for $n \in \omega$, we easily obtain that $f\restriction_{E_{\alpha+n}}$ is $\Delta_1^{L_{\alpha+(n+1)k}}$-definable and thus belongs to $L_{\alpha+(n+1)k+1} \subseteq L_{\alpha+\omega}$. Thus (b) is true for $\alpha + \omega$.

Suppose now that α is limit of limit and that for any $\beta < \alpha$ limit we have that (a) is true. Thus clearly (a) is true for α, and therefore also (b). This concludes the proposition. ⊣

We end by one last thing one needs to be careful about when working in L_α for α not admissible. In case α is admissible, formulas of the form $\forall n \in \omega\ \exists \beta\ \Phi(n, \beta)$ where $\Phi(n, \beta)$ is Δ_0, can be considered to be Σ_1-formulas, precisely because if the formula is true in L_α, there must exists $B \in L_\alpha$ such that $\forall n \in \omega\ \exists \beta \in B\ \Phi(n, \beta)$. This is of course not the case for α not admissible, and one has to be careful about keeping Σ_n formulas truly Σ_n.

2.2.4. *Theorems on definability.* Using induction with bounded rank replacement, it is possible to show that the function $\beta \mapsto L_\beta$ is absolute already in L_α for α limit. This is done formally in [10].

In order to show that the function $\beta \mapsto L_\beta$ is absolute in any model L_α for α limit, the author of [10] uses a bounded rank argument as sketched above. In this case, this requires to be a bit careful with the encoding one uses for ZF formulas by sets (hereditary finite sets in case the formula has no parameter). In particular, it is worth noting that one uses partial function from n to $\{p_1, \dots, p_n\}$ to encode finite sequences. This way, as long as $P \in L_\alpha$, for any n, a function from n into P has its rank bounded by some $\alpha + k$, where k is an integer independent of n (even in L_ω: recall that we start with $L_\emptyset = \omega$).

Using such an encoding of formulas, we write $\ulcorner \Phi \urcorner$ for the code of Φ. We have the following:

Theorem 2.11 (Lemma I.9.10 of [10]). *The predicate $M \models \Phi(p_1, \dots, p_n)$ is $\Delta_1^{L_\alpha}$ uniformly in any α limit, in M, in $\ulcorner \Phi \urcorner$ and in the sequence $\langle p_1, \dots, p_n \rangle$.*

By the above, we formally mean the following: there is a Σ_1 formula $\Phi(M, e, p)$, and a Π_1 formula $\Psi(M, e, p)$, such that for any α limit, as long as we take $M, \langle p_1, \dots, p_n \rangle$ in L_α, we have:

$$\begin{array}{rl} & M \models \phi(p_1, \dots, p_n) \\ \leftrightarrow & L_\alpha \models \Phi(M, \ulcorner \phi \urcorner, \langle p_1, \dots, p_n \rangle) \\ \leftrightarrow & L_\alpha \models \Psi(M, \ulcorner \phi \urcorner, \langle p_1, \dots, p_n \rangle) \end{array}$$

We will also sometimes use the following version of the above: in case Φ is a Δ_0 formula, then Φ is true in L_α iff Φ is true in the model being the transitive closure of all the parameters involved in the formula. Using that such a model can be obtained uniformly and that satisfaction is absolute in any L_α for α limit, we also have:

Corollary 2.12. *The predicate $L_\alpha \models \Phi(p_1, \dots, p_n)$ is $\Delta_1^{L_\alpha}$ uniformly in any α limit and in the code of any Δ_0 formula $\ulcorner \Phi \urcorner$.*

Using that satisfaction is absolute in any L_α for α limit, we also have:

Theorem 2.13 (Lemma II.2.8 of [10]). *The function $\beta \mapsto L_\beta$ is $\Delta_1^{L_\alpha}$ uniformly in any α limit.*

It is also well-known that L is well-ordered in L, that is, there is a well order $<_L$ on elements of L, which is definable in L. Again, one can show that this order is absolute in any L_α for α limit.

Theorem 2.14 (Lemma II.3.5 of [10]). *The relation $<_L$ and the function $a \mapsto \{b : b <_L a\}$, are $\Delta_1^{L_\alpha}$, uniformly in any α limit.*

We end this section by showing that in the special case of Σ_n-admissibility in the constructible hierarchy, only the axiom of Σ_n-collection is needed when α is limit.

Proposition 2.15. *Suppose L_α is a model of Σ_n-collection for α limit. Then, L_α is a model of Δ_n-comprehension.*

PROOF. The proof goes by induction on n. For $n = 0$ as α is limit we always have that L_α is a model of Δ_0-comprehension. Suppose the result is true for n and let us show it is true for $n+1$. Let L_α be model of Σ_{n+1}-collection. Let $\Phi(a, b)$ and $\Psi(a, b)$ be Π^0_n formulas with parameters in L_α. Let $A \in L_\alpha$ and $E \subseteq A$ be such that:

$$\begin{array}{lcl} a \in E & \leftrightarrow & L_\alpha \models \exists b\ \Phi(a, b) \\ a \notin E & \leftrightarrow & L_\alpha \models \exists b\ \Psi(a, b) \end{array}$$

We have in particular that $L_\alpha \models \forall a \in A\ \exists \beta\ \exists b \in L_\beta\ \Phi(a, b) \vee \Psi(a, b)$

By Σ_{n+1}-collection there exists $\beta < \alpha$ such that we have:

$$L_\alpha \models \forall a \in A\ \exists b \in L_\beta\ \Phi(a, b) \vee \Psi(a, b)$$

Note that we then have

$$\begin{array}{lcl} a \in E & \leftrightarrow & L_\alpha \models \exists b \in L_\beta\ \Phi(a, b) \\ a \notin E & \leftrightarrow & L_\alpha \models \exists b \in L_\beta\ \Psi(a, b) \end{array}$$

It follows that we have :

$$\begin{array}{lcl} a \notin E & \leftrightarrow & L_\alpha \models \forall b \in L_\beta\ \neg\Phi(a, b) \\ a \in E & \leftrightarrow & L_\alpha \models \forall b \in L_\beta\ \neg\Psi(a, b) \end{array}$$

As L_α is a model of Σ_n-collection, formulas $\forall b \in L_\beta\ \neg\Phi(a, b)$ and $\forall b \in L_\beta\ \neg\Psi(a, b)$ are both equivalent in L_α to Σ_n formulas. Therefore E is in fact defined by a Δ_n formula. By induction hypothesis we have that $E \in L_\alpha$.

$\dashv$

2.2.5. *Theorems on stability.* When lifting up notions of computability to various ordinals, new phenomenons start to appear, one of them central to the study of ITTMs is the notion of stability.

DEFINITION 2.16. For $\alpha \leq \beta$ we say that L_α is Σ_n-stable in L_β, and we write $L_\alpha \prec_n L_\beta$ if for every Σ_n formula Φ with parameters in L_α we have $L_\alpha \models \Phi$ iff $L_\beta \models \Phi$. Without confusion, we will also write $\alpha \prec_n \beta$ for $L_\alpha \prec_n L_\beta$.

The notion of n-stability is the same as the notion of elementary substructure for Σ_n formulas in model theory. The following proposition is easy and will be used in various places of the paper:

PROPOSITION 2.17. *Suppose $L_\alpha \prec_n L_\beta$. Let $\Phi(a_1, \dots, a_n)$ be a Π_{n+1} formula and let $p_1, \dots, p_n \in L_\alpha$. If $L_\beta \models \Phi(p_1, \dots, p_n)$ then $L_\alpha \models \Phi(p_1, \dots, p_n)$.*

PROOF. The formula $\Phi(a_1, \dots, a_n)$ is of the form $\forall x\ \Psi(x, a_1, \dots, a_n)$ for Ψ a Σ_n formula. Also for every $x \in L_\alpha$ we have $L_\beta \models \Psi(x, a_1, \dots, a_n)$ and thus $L_\alpha \models \Psi(x, a_1, \dots, a_n)$ by Σ_n stability. It follows that $L_\alpha \models \forall x\ \Psi(x, a_1, \dots, a_n)$. $\dashv$

PROPOSITION 2.18. *For β limit and $\alpha < \beta$, the predicate $L_\alpha \prec_n L_\beta$ is $\Pi^{L_\beta}_n$ uniformly in β and α.*

PROOF. We start with Σ_1-stability. We have $L_\alpha \prec_1 L_\beta$ iff

$$\begin{array}{lcl} L_\beta & \models & \text{For all } \Delta_0 \text{ formulas } \ulcorner\Phi(b, a_1, \dots, a_k)\urcorner\ \forall p_1, \dots, p_k \in L_\alpha \\ & & \quad \left[\forall x\ \neg\Phi(x, p_1, \dots, p_k) \text{ or } L_\alpha \models \exists y\ \Phi(y, p_1, \dots, p_k)\right] \end{array}$$

which is $\Pi^{L_\beta}_1$ by Proposition 2.11 and 2.12. Suppose now that the predicate $L_\alpha \prec_n L_\beta$ is $\Pi^{L_\beta}_n$. To show that $L_\alpha \prec_{n+1} L_\beta$ is $\Pi^{L_\beta}_{n+1}$, we write first the formula which says $L_\alpha \prec_n L_\beta$, in order to express that if L_α satisfies a Σ_{n+1} formula, then also L_β satisfies this formula (see Proposition 2.17). This formula is $\Pi_n^{L_\beta}$. We then combine it with the following $\Pi_{n+1}^{L_\beta}$ formula, which expresses that if L_β satisfies a Σ_{n+1} formula, then also L_α satisfies this formula:

$$L_\beta \models \text{For all } \Delta_0 \text{ formulas } \ulcorner\Phi(b_1,\dots,b_{n+1},a_1,\dots,a_k)\urcorner,\ \forall p_1,\dots,p_k \in L_\alpha, \begin{cases} \forall x_1\ \exists x_2 \cdots Qx_{n+1},\ \neg\Phi(x_1,\dots,x_{n+1},p_1,\dots,p_k) \\ \text{or } L_\alpha \models \exists y_1\ \forall y_2 \cdots Qy_{n+1}\ \Phi(y_1,\dots,y_{n+1},p_1,\dots,p_k) \end{cases}$$

where $Q \in \{\exists;\forall\}$ depends on the parity of n. This concludes the proof. ⊣

When dealing with the constructibles, stability presents additional features to the notion of elementary substructures in model theory. For instance, given that α is limit, the set of elements which are Σ_1-definable in L_α with no parameters is necessarily of the form L_β, and β is the smallest such that $L_\beta \prec_1 L_\alpha$ [2, Theorem 7.8]. We also have the following:

Theorem 2.19. *Suppose $\alpha < \beta$ for β limit, and $L_\alpha \prec_n L_\beta$. Then α is Σ_n-admissible.*

Proof. The proof is easy for Σ_1-admissibility, but does not lift straightforwardly to Σ_n-admissibility.

We first show the theorem for Σ_1-admissibility. Suppose α is not Σ_1-admissible. Then there exists $a \in L_\alpha$ and a Σ_1 formula $\Phi(x,y) = \exists z\ \Phi_0(x,y,z)$ with parameters in L_α witnessing the failure of Σ_1-admissibility, that is:

$$\begin{array}{rcccccc} & L_\alpha \models & & \forall p \in a & \exists r & \exists z & \Phi_0(p,r,z) \\ \text{and} & L_\alpha \not\models & \exists\gamma & \forall p \in a & \exists r \in L_\gamma & \exists z \in L_\gamma & \Phi_0(p,r,z) \end{array}$$

As $\alpha < \beta$ it is however clear that we have:

$$L_\beta \models \exists\alpha\ \forall p \in a\ \exists r \in L_\alpha\ \exists z \in L_\alpha\ \Phi_0(p,r,z)$$

In particular the above Σ_1 formula is satisfied in L_β but not in L_α, so we do not have $L_\alpha \prec_1 L_\beta$.

We continue by induction: suppose α is not Σ_{n+1}-admissible. Then if L_α is not Σ_n-stable in L_β, it is in particular not Σ_{n+1}-stable in L_β and the proposition is verified. Otherwise we have $L_\alpha \prec_n L_\beta$. Let $a \in L_\alpha$ and let $\Phi_0(x,y,z_1,\dots,z_{n+1})$ be a Δ_0 formula (where $Q \in \{\exists;\forall\}$ depends on the parity of n) such that:

$$\begin{array}{rcccccc} & L_\alpha \models & & \forall p \in a & \exists r & \exists z_1 \forall z_2 \cdots Qz_{n+1} & \Phi_0(p,r,z_1,\dots,z_{n+1}) \\ \text{and} & L_\alpha \not\models & \exists\gamma & \forall p \in a & \exists r \in L_\gamma & \exists z_1 \forall z_2 \cdots Qz_{n+1} & \Phi_0(p,r,z_1,\dots,z_{n+1}) \end{array}$$

Note that unlike with the Σ_1-case, we cannot necessarily bound the variables $z_1,\dots,z_{n+1}$ by L_γ. Indeed, it might be the case for every p in a there exists some r in L_γ which is Σ_{n+1}-definable in L_γ, even though it is not Σ_{n+1}-definable in L_α. We need to use that $L_\alpha \prec_n L_\beta$. In particular we have:

$$L_\beta \models \exists\alpha \prec_n \beta\ \forall p \in a\ \exists r \in L_\alpha \text{ s.t. } L_\alpha \models \exists z_1\ \forall z_2 \dots\ z_{n+1}\ \Phi_0(p,r,z_1,\dots,z_{n+1})$$

First let us note that by Proposition 2.18 the above formula is Σ_{n+1}. It is also clear that L_α cannot be a model of this formula, because then, using Proposition 2.17, it would also be a model of:

$$\exists\gamma\ \forall p \in a\ \exists r \in L_\gamma\ \exists z_1\ \forall z_2 \dots z_{n+1}\ \Phi_0(p,r,z_1,\dots,z_{n+1})$$

⊣

2.2.6. *Theorems on projectibles.* Another central notion in α-recursion theory is the notion of projectible ordinal. We are in particular able to lift most of the work done in algorithmic randomness and genericity, in the case α is projectible into ω.

Definition 2.20 (Projectum). We say that α is *projectible* in $\beta \leq \alpha$ if there is a one-one function Σ_1-definable (with parameters) in L_α, from α into β. We call *projectum* and write α^* for the smallest ordinal such that α is projectible into α^*. If $\alpha^* < \alpha$ we say that α is projectible. Otherwise we say that α is not projectible.

This notion of projectibility is very useful to lift proofs from lower to higher recursion. This has been done in particular in the hyperarithmetic setting, for instance in [3], using the fact that ω_1^{CK} is projectible into ω. We will later see what is the projectum of the three ordinals λ, ζ and Σ, associated with ITTMs. To do so, we give a general theorem on projectums. This theorem can be found in a similar form in [2], but we still include the proof for completeness.

Theorem 2.21. *Let α be admissible. We have that α^* is the smallest ordinal such that L_α is not a model of Σ_1-comprehension for subsets of α^*. If the Σ_1 formula Φ is a witness of this failure, then the projectum is definable with the same parameters as the ones used in Φ.*

Proof. We first show that L_α satisfies Σ_1-comprehension for subsets of ordinals smaller than α^*. Let $\delta < \alpha^*$ be an ordinal, and $A \subseteq \delta$ be such that $x \in A \Leftrightarrow L_\alpha \models \exists y\ \Phi(x, y)$ where Φ is Δ_0. Let f be the function defined on A, such that $f(a) = \delta \times \gamma + a$ where γ is the smallest ordinal such that $L_\gamma \models \exists y\ \Phi(a, y)$. Obviously f is 1-1. We then collapse $f[A]$ by defining $g(\gamma)$ to be the first $\beta \in f[A]$ that we find which is not in $\{g(\gamma') : \gamma' < \gamma\}$. Formally, let $\exists y\ \Psi(a, \beta, y)$ with Ψ Δ_0 be the Σ_1 formula defining f. Then we define the function g by $g(\gamma) = \beta$ if there exists η for which $\langle \beta, \eta \rangle$ is the smallest pair such that $L_\eta \models \exists y\ \exists a\ \Psi(a, \beta, y)$ and $\beta \notin \{g(\gamma') : \gamma' < \gamma\}$. We have that $f^{-1} \circ g$ is a Σ_1-definable bijection from an initial segment of α, onto A. Also the domain of $f^{-1} \circ g$ cannot be α otherwise α would be projectible into $\delta < \alpha^*$. Therefore the domain of $f^{-1} \circ g$ is a strict initial segment of α and thus the range of $f^{-1} \circ g$, which is A, is an element of L_α.

We now exhibit a Σ_1-definable subset of α^* which is not in L_α. If p is a projection into α^*, we have that $p[\alpha] = A \subseteq \alpha^*$ is a subset of α^* which is Σ_1 definable in L_α. This subset is not in L_α, as otherwise the function $g : \alpha^* \to \alpha$ defined by $g(\beta) = \sup_{x \in A \wedge x \leq \beta}(p^{-1}(x))$ would contradict the admissibility of α. ⊣

2.3. The ordinals λ, ζ, Σ. We state in this section results regarding the three ordinals λ, ζ, Σ, and their relative versions λ^x, ζ^x and Σ^x. These ordinals first allow to establish a clear connection between ITTMs and constructibles, summed up in the two following theorems. We introduce the following coding of hereditary countable sets before we mention the first one: Let $0 \in 2^\omega$ be a code for the empty set. Suppose that $A = \{a_n\ :\ n \in \omega\}$ where a_n is coded by $x_n \in 2^\omega$ for each n. Then $\oplus_n x_n$ is a code for A.

THEOREM 2.22 (Welch [24]). *The set L_λ (resp. L_ζ, resp.L_Σ) is the set of all sets with a writable (resp. eventually writable, resp. accidentally writable) code.*

The following theorem is similar to Theorem 2.4, but with the constructible hierarchy in place of ITTM's tapes.

THEOREM 2.23 (Welch [24]). *The triplet of ordinals (λ, ζ, Σ) is the lexicographically smallest triplet such that*

$$L_\lambda \prec_1 L_\zeta \prec_2 L_\Sigma$$

By relativization, $(\lambda^x, \zeta^x, \Sigma^x)$ is the lexicographically smallest triplet such that

$$L_{\lambda^x}[x] \prec_1 L_{\zeta^x}[x] \prec_2 L_{\Sigma^x}[x]$$

COROLLARY 2.24. *The ordinal ζ is Σ_2-admissible, and there are cofinally in ζ many eventually writable Σ_2-admissible ordinals.*

PROOF. As $L_\zeta \prec_2 L_\Sigma$, we deduce from Theorem 2.19 that ζ is Σ_2-admissible. In particular for any eventually writable α we have that L_Σ is a model of "there exists $\beta > \alpha$ which is Σ_2-admissible". It follows that L_ζ is also a model of that and thus that there are cofinally many writable Σ_2-admissible ordinals. ⊣

COROLLARY 2.25 (Hamkins, Lewis [14]). *The ordinal λ is Σ_1-admissible, and in λ there are cofinally many writable Σ_1-admissible ordinals.*

An important question of [14] was to determine whether Σ was admissible or not. Welch's proof that ITTMs halt only at ordinals smaller than λ provides insight on the way ITTMs work, and helped to solve the question. The proof can also be found in [25].

THEOREM 2.26 (Welch). *There is a function $f : \omega \mapsto \Sigma$ which is Σ_1-definable in L_Σ with ζ as a parameter and such that* $\sup_n f(n) = \Sigma$.

PROOF. Let U be the universal ITTM, which simulates every other ITTM. In particular we have by Theorem 2.4 that Σ is the smallest ordinal greater than ζ such that $C_U[\zeta] = C_U[\Sigma]$. For every n let us define the function f_n such that $f_n(0) = \zeta$ and $f_n(m+1)$ is the smallest ordinal bigger than $f_n(m)$ such that $\forall i \leq n$ for which $\{C_U(i)[\beta]\}_{\beta<\zeta}$ does not converge, we have that $C_U(i)$ has changed at least once in the interval $[f_n(m), f_n(m+1)]$.

If there was some n such that $\sup_m f_n(m) = \Sigma$, this would prove the theorem already. It is actually possible to show, by combining Σ_2-stability of L_ζ in L_Σ, together with admissibility of L_ζ, that this cannot happen for any n. Let us then define the function f as follow: $f(n)$ is the smallest ordinal α greater than ζ such that $C_U[\zeta]\restriction n = C_U[\alpha]\restriction n$. As for every m we have $\sup_m f_n(m) < \Sigma$, then $f(n) < \Sigma$ and thus f is Σ_1-definable in L_Σ with ζ as a parameter. It is clear that $f(n) \leq f(n+1)$. It is also clear that $f(n) < \sup_n f(n)$ as otherwise we would have $C_U[\zeta] = C_U[\alpha]$ for some $\alpha < \Sigma$.

Let $\alpha = \sup_n f(n)$ and let us show $\alpha = \Sigma$. Let $n \in \omega$. If $\{C_U(n)[\beta]\}_{\beta<\zeta}$ converges then by (2) of Theorem 2.4 we have $C_U(n)[\zeta] = C_U(n)[\alpha]$. If we have that $\{C_U(n)[\beta]\}_{\beta<\zeta}$ diverges then $C_U(n)[\zeta] = 0$. Then either $\{C_U(n)[\beta]\}_{\beta<\alpha}$ converges to $C_U(n)[\zeta] = 0$ or $\{C_U(n)[\beta]\}_{\beta<\alpha}$ diverges and then $C_U(n)[\alpha] = 0$. In both cases we have $C_U(n)[\alpha] = C_U(n)[\zeta]$. This implies that $C_U[\alpha] = C_U[\zeta]$ which implies $\alpha = \Sigma$. ⊣

Corollary 2.27 (Welch). *The ordinal Σ is not admissible.*

The function of Theorem 2.26 will be used in various places of this paper. We however have the following:

Theorem 2.28 (Welch). *The ordinal Σ is a limit of admissible ordinals.*

Proof. By Lemma 2.24, ζ is a limit of admissible ordinals. By Σ_2-stability, Σ must also be a limit of admissible ordinals.

⊣

We now study what effect the increase of one of the three main ordinals has on the others.

Theorem 2.29. *The following are equivalent:*

1. $\zeta^x > \zeta$
2. $\Sigma^x > \Sigma$
3. $\lambda^x > \Sigma$

Proof. Suppose $\zeta^x > \zeta$. In particular ζ is eventually writable in x. Let $\{\zeta_s\}_s$ be the successive approximations of ζ using an ITTM that eventually writes ζ using x. We can run an ITTM $M(x)$ which does the following: at step s, it uses ζ_s as a parameter in the function $f : \omega \mapsto \Sigma$ of Theorem 2.26, and whenever it has found values for every $f(n)$ (and no new version of ζ_s has arrived so far), it writes $\Sigma_s = \sup_n f(n)$ on the output tape. At some stage s we will have $\zeta_s = \zeta$ and thus $\Sigma_s = \Sigma$ will be on the output tape. It follows that $\Sigma^x > \Sigma$.

Suppose now that we have $\Sigma^x > \Sigma$. We can run the ITTM $M(x)$ which searches for two x-accidentally writable ordinals $\alpha < \beta$ such that $L_\alpha \prec_2 L_\beta$, then writes β and halts. As $\zeta < \Sigma$ is the smallest such pair of ordinals and as $\Sigma^x > \Sigma$, the ITTM will write an ordinal equal to or bigger than Σ at some point and halt. We then have $\lambda^x > \Sigma$.

Finally if $\lambda^x > \Sigma$ it is clear that $\zeta^x > \zeta$. ⊣

Theorem 2.30. *For every $\lambda \leq \alpha < \zeta$, there exists $x \in 2^\omega$ such that $\alpha \leq \lambda^x$, such that $\zeta^x = \zeta$ and $\Sigma^x = \Sigma$.*

Proof. Let α be such that $\lambda \leq \alpha < \zeta$. Let $x \in 2^\omega$ be an eventually writable code for α. It is clear that $\alpha \leq \lambda^x$. Let us show $\zeta^x = \zeta$. Let α be any x-eventually writable ordinal, via some ITTM M. Let N be the ITTM which starts to eventually write x and in the same time uses the current version x_s of x to run $M(x_s)$ and copy at every time the output tape of M on the output tape of N. There is some stage s such that for every stage $t \geq s$ we will have $x_s = x_t = x$ together with $M(x)[s] = M(x)[t] = \alpha$. This implies also $N[s] = N[t] = \alpha$. Thus α is eventually writable. Here, we essentially used the fact from [23] that eventually writable reals are closed under eventually writability.

From Theorem 2.29 we have $\Sigma^x = \Sigma$, as $\zeta^x = \zeta$. ⊣

We now study the projectibility of the three ordinals λ, ζ and Σ. Intuitively λ is projectible into ω, by the function which to $\alpha < \lambda$ associates the code of the first ITTM which is witnessed to write α. Such a thing is of course not possible to achieve with ζ, which indeed is not projectible.

Theorem 2.31.

1. λ *is projectible into* ω *without parameters.*
2. ζ *is not projectible.*

Proof. A direct proof of (1) would be possible. It is also a direct consequence of Theorem 2.21: it is well known that the set $\{e \in \omega : \text{the } e\text{-th ITTM halts}\}$ is not writable (by a standard diagonalization, see for instance [14]) and thus does not belong to L_λ. It is however Σ_1-definable in L_λ and thus L_λ is not a model of Σ_1-comprehension for subsets of ω. It follows that $\lambda^* = \omega$, with no parameters.

To prove (2), we will show that L_ζ satisfies Σ_1-comprehension for any set in L_ζ. We shall first argue that for every $\alpha < \zeta$, there exists $\beta \geq \alpha$ such that $L_\beta \prec_1 L_\zeta$. For every $\alpha < \zeta$, there exists by Theorem 2.30 some $x \in 2^\omega$ such that $\lambda^x > \alpha$ and such that $\zeta^x = \zeta$ and $\Sigma^x = \Sigma$. In particular we have $L_{\lambda^x} \prec_1 L_{\zeta^x} = L_\zeta$. Now suppose that for $\alpha < \zeta$ we have that $A \subseteq L_\alpha$ is Σ_1-definable in L_ζ with parameters in L_α. Let $\beta \geq \alpha$ be such that $L_\beta \prec_1 L_\zeta$. In particular $A \subseteq L_\alpha$ is Σ_1-definable in L_β. It follows that $A \in L_\zeta$. ⊣

We now study the projectibility of L_Σ. We will show that it is projectible into ω with parameter ζ, in a strong sense, that is, with a bijection. To do so we first need to argue that L_Σ is a model of "everything is countable". It is clear intuitively: if x belongs to L_Σ then it has an accidentally writable code, and this code gives the bijection between x and ω. Friedman showed a bit more:

Lemma 2.32 (Friedman and Welch, [12]). *Let α be limit. Suppose there exists $x \in L_\alpha$ such that $L_\alpha \models$ "x is uncountable". Then there exists $\gamma < \delta < \alpha$ such that $L_\gamma \prec L_\delta$ (that is, $L_\gamma \prec_n L_\delta$ for every n).*

Proof. Let us first argue that there must be a limit ordinal $\delta < \alpha$ such that $L_\alpha \models$ "L_δ is uncountable". If α is limit of limits this is clear, because there must be a limit ordinal δ such that L_δ contains an x which is uncountable in L_α. As L_δ is transitive, it must be itself uncountable in L_α. If α is not limit of limits, let δ be limit such that $\alpha = \delta + \omega$. Suppose that L_δ is countable in L_α. Thus also by definition of L, every element of $L_{\delta+1}$ must be countable in L_α. We can continue inductively to show that every element of $L_{\delta+\omega} = L_\alpha$ must be countable in L_α, contradicting our hypothesis.

Thus there must be a limit ordinal $\delta < \alpha$ such that $L_\alpha \models$ "L_δ is uncountable". We then conduct within L_α the Löwenheim-Skolem proof to find a countable set $A \subseteq L_\delta$ such that $A \prec L_\delta$. The Mostowski collapse A' of A is transitive, as $A' \prec L_\delta$ and as L_δ is a model of "everything is constructible" together with "for all β the set L_β exists" [2], then A' must be of the form L_γ for some $\gamma \leq \delta$. As L_γ is countable in L_α we must have $\gamma < \delta$. ⊣

Corollary 2.33. *For any limit ordinal $\alpha \leq \Sigma$, we have that $L_\alpha \models$ "everything is countable".*

Proof. It is immediate using that Σ is the smallest ordinal such that $L_\alpha \prec_2 L_\Sigma$ for some $\alpha < \Sigma$. ⊣

Theorem 2.34. *Suppose $L_\alpha \models$ "everything is countable" and α is not admissible. Then there is a bijection $b : \omega \to L_\alpha$ which is Σ_1-definable in L_α with the*

[2]Note that this is where we use that δ is limit, using Theorem 2.13

same parameters than the ones used in a witness of non-admissibility of α. In particular α is projectible into ω, with these parameters (using b^{-1} restricted to ordinals).

PROOF. We first show that there is a Σ_1-definable surjection from ω onto L_α. As α is not admissible, there is a set $a \in L_\alpha$ and a function $g : a \mapsto \alpha$ which is Σ_1-definable over L_α with some parameter $p \in L_\alpha$, and such that $\sup_{x\in a} g(x) = \alpha$. Note that as $L_\alpha \models$ "everything is countable", there is a bijection in L_α between a and ω. Using this bijection there is then a function $f : \omega \to \alpha$ which is Σ_1-definable over L_α with parameters p, a, and such that $\sup_{n\in\omega} f(n) = \alpha$ (the bijection does not need to be a parameter, as the smallest such can be Σ_1-defined). Let $\Psi(n, \beta)$ be the Σ_1 functional formula with parameters p, a, which defines f.

We now define a Σ_1 formula (with parameter p, a) $\Phi(n, m, z)$ such that for every n, m there is a unique $z \in L_\alpha$ for which $L_\alpha \models \Phi(n, m, z)$, and such that for every $z \in L_\alpha$, there exists n, m such that $L_\alpha \models \Phi(n, m, z)$. We define:

$$\begin{aligned}\Phi(n,m,z) \quad \equiv \quad & \exists g\, \exists \beta \text{ s.t.} \\ & \Psi(n,\beta) \text{ and} \\ & g \text{ is a bijection between } \omega \text{ and } L_\beta \text{ s.t. } g(m) = z \text{ and} \\ & \text{every } g' <_L g \text{ is not a bijection between } \omega \text{ and } L_\beta\end{aligned}$$

Note that Φ is Σ_1. It is clear that for every n, m, there is at most one z such that $\Phi(n, m, z)$. The fact that every $z \in L_\alpha$ is defined by Φ for some n, m is clear because $L_\alpha \models$"everything is countable".

It follows that there is a surjection f from ω onto L_α, which is Σ_1-definable in L_α with parameters p, a. To obtain a bijection, we define the function $h : \omega \to \omega$ such that $h(0) = 0$ and $h(n+1) = \min\{m \in \omega \ : \ \forall n' \leq n\ f(h(n')) \neq f(m)\}$. Note that h is defined by Σ_1-induction. As α is not admissible, we should make sure we can do so. This relies on the fact that h is defined only on integers: we can then essentially rely on the admissibility of ω. Indeed, to decide $h(n+1) = m$, we only need the finite function $h\restriction_n$ and the finite function $f\restriction_m$. In particular only finitely many witnesses for values of f are needed and they then all belongs to some L_β for $\beta < \alpha$. Formally we can define h in L_α as follow:

$$\begin{aligned}h(n) = m \quad \equiv \quad & \exists \beta\, \exists h'\restriction_n\ \forall k < n \\ & h'(k+1) > h'(k) \wedge L_\beta \models \forall i < k\ f(h'(k)) \neq f(i) \wedge \\ & \forall j \text{ with } h'(k) < j < h'(k+1)\ L_\beta \models \exists i < j\ f(h'(j)) = f(i) \\ & \text{and } h'(n) = m\end{aligned}$$

The bijection is then given by $b(n) = f(h(n))$. ⊣

COROLLARY 2.35. *There is a bijection $b : \omega \to L_\Sigma$ which is Σ_1-definable in L_Σ with ζ as a parameter. In particular Σ is projectible into ω, with parameter ζ.*

PROOF. From Theorem 2.26 there is a function $f : \omega \mapsto \Sigma$ which is Σ_1-definable over L_Σ with parameter ζ and such that $\sup_n f(n) = \Sigma$. From Corollary 2.33: we have that $L_\Sigma \models$"everything is countable". The result follows. ⊣

§3. Forcing in the constructibles. Algorithmic randomness normally deals with Borel sets of positive measure. Working in the constructibles will make this task a little bit harder, and requires to go into usual naming and forcing in L.

We will however not formally define a forcing relation. Instead we go around the need of defining one, by directly dealing with Borel sets. The reason we do so is to stick with what is traditionally done with algorithmic randomness: the manipulation of Borel sets. We believe that for the purpose of this paper, it is a bit more clear to use Borel sets rather than a formal forcing relation.

3.1. Borel codes. In order to be able to speak about sets of reals in L_α, we need to code them into elements of L_α. We do that with the notion of ∞-Borel codes and Borel codes. In this paper, due to technical reasons that will be made clear later, we need to be careful about the L-rank of our Borel codes. In particular, if $\{c_n\}_{n\in\omega}$ are Borel codes for $\Sigma^0_{\alpha+k}$ sets $\mathcal{B}_n$ such that each c_n has L-rank, say β, we need a code of $\bigcap_{n\in\omega}\mathcal{B}_n$ also to have L-rank β. In particular we cannot for instance define a code of $\bigcap_{n\in\omega}\mathcal{B}_n$ to be a set containing the codes c_n.

In what follows the coding trick is achieved with (3) and (4), by coding sequences of sequences of codes to be a partial function defined in $F \subseteq \omega$, using the usual bijection between ω and ω^2. This way the L-rank of a sequence of code stay at the same level.

Definition 3.1 (∞-Borel codes and Borel codes). We define, by induction, ∞-Borel codes together with their rank r, type $t = \Sigma_r$ or Π_r and interpretation ι:

1. The set $c = \langle 2, \{\sigma_i\}_{i<k}\rangle$, for any finite sequence $\{\sigma_i\}_{i<k}$ with each $\sigma_i \in 2^{<\omega}$, is an ∞-Borel code, with rank $r(c) = 0$, type $\Sigma_0 = \Pi_0 = \Delta_0$ and interpretation $\iota(c) = \bigcup_{i<k}[\sigma_i]$
2. Suppose that for some set I, there exists a function $f : i \in I \mapsto c_i$ such that c_i is an ∞-Borel code for every $i \in I$. Then $d_0 = \langle 0, f\rangle$ and $d_1 = \langle 1, f\rangle$ are ∞-Borel codes, with rank $r(d_0) = r(d_1) = \sup_{i\in I}(r(c_i) + 1)$, type respectively Σ^0_r and Π^0_r and interpretation $\iota(d_0) = \bigcup_{i\in I}\iota(c_i)$ and $\iota(d_1) = \bigcap_{i\in I}\iota(c_i)$.
3. Suppose for some set I, there is $k \in \omega$ and a function $i \in I \mapsto c_i$ where for every $i \in I$, the set $c_i = \langle 0, f_i : I^k \to L\rangle$ is an ∞-Borel code. Then we define $f : I^{k+1} \to L$ by $f(i, a_1 \dots, a_k) = f_i(a_1 \dots, a_k)$. The set $c = \langle 1, f : I^{k+1} \to L\rangle$ is an ∞-Borel code, with rank $r(c) = \sup_{i\in I}(r(c_i)+1)$, type Π_r and interpretation $\iota(c) = \bigcap_{i\in I}\iota(c_i)$.
4. Suppose for some set I, there is $k \in \omega$ and a function $i \in I \mapsto c_i$ where for every $i \in I$, the set $c_i = \langle 1, f_i : I^k \to L\rangle$ is an ∞-Borel code. Then we define $f : I^{k+1} \to L$ by $f(i, a_1 \dots, a_k) = f_i(a_1 \dots, a_k)$. The set $c = \langle 0, f : I^{k+1} \to L\rangle$ is an ∞-Borel code, with rank $r(c) = \sup_{i\in I}(r(c_i)+1)$, type Σ_r and interpretation $\iota(c) = \bigcup_{i\in I}\iota(c_i)$.

A Borel code is an ∞-Borel code where each set I involved equals ω. Note that a Borel code can be encoded by a real.

In order to lighten the notations, we will write $b = \bigvee_{i\in I} b_i$ if b is the ∞-Borel code of $\bigcup_{i\in I}\iota(b_i)$ and $b = \bigwedge_{i\in I} b_i$ if b is the ∞-Borel code of $\bigcap_{i\in I}\iota(b_i)$. Note that given a Borel code $b = \bigvee_{i \in I} b_i$ or $b = \bigwedge_{i \in I} b_i$, one can uniformly find I (using the domain of the function involved in the code), and find the code b_i uniformly in $i \in I$:

PROPOSITION 3.2. *We have:*

1. *The function which on an ∞-Borel code $b = \bigvee_{i \in I} b_i$ and some $i \in I$, associates the ∞-Borel code b_i, is $\Delta_1^{L_\alpha}$-definable uniformly in α limit. The same holds for $b = \bigwedge_{i \in I} b_i$.*
2. *The function which on an ∞-Borel code $b \in L_\gamma$ associates the ∞-Borel code d of $2^\omega - \iota(b)$ with $d \in L_\gamma$ and $r(b) = r(d)$, is $\Delta_1^{L_\alpha}$-definable uniformly in α limit.*
3. *The function which on ∞-Borel codes $b_0, \dots, b_k \in L_\gamma$ associates the ∞-Borel code $d \in L_\gamma$ with $r(d) = \max_{i \leq k}(r(b_i))$ and $\iota(d) = \bigcup_{i<k} \iota(b_i)$, is $\Delta_1^{L_\alpha}$-definable uniformly in α limit.*

PROOF. (1) is rather obvious: A code $b = \bigvee_{i \in I} b_i$ is of the form $\langle 0, f : I^{k+1} \to L \rangle$ for some $k \geq 0$. If $k = 0$ then b_i is given by $f(i)$. If $k > 0$ then b_i is given by $\langle 1, f_i : I^k \to L \rangle$ where f_i is defined by $f_i(a_1 \dots, a_k) = f(i, a_1 \dots, a_k)$. This is easily uniformly definable in L_α for any α limit. The same holds for $b = \bigwedge_{i \in I} b_i$.
(2) goes by propagating the complement into the ∞-Borel code, and (3) by propagating the finite union in the ∞-Borel code. Both (2) and (3) are straightforward by induction on γ, using bounded rank replacement of Proposition 2.10. ⊣

PROPOSITION 3.3. *The set of ∞-Borel codes and of Borel codes of L_α, are $\Delta_1^{L_\alpha}$-definable uniformly in any α limit.*

PROOF. We define by Δ_0-induction on the rank of sets of L_α, a total function $f : L_\alpha \to \{0, 1\}$. The function returns 1 iff its parameter is a Borel code. It is defined as follow:

$$\begin{aligned} f(c) &= 1 \text{ if } c \text{ is of the form } \langle 2, \{\sigma_i\}_{i<k} \rangle \text{ for a sequence of strings } \{\sigma_i\}_{i<k} \\ &= 1 \text{ if } c \text{ is of the form } \textstyle\bigvee_{i \in \omega} c_i \text{ or } \bigwedge_{i \in \omega} c_i \\ &\quad\ \text{and if for every } i \in \omega \text{ we have that } f(c_i) = 1 \\ &= 0 \text{ otherwise} \end{aligned}$$

Note that we are in the conditions of Proposition 2.10, with sets L_β in place of sets E_β. One easily see that (1) (2) and (3) of Proposition 2.10 are verified, which implies that f is well-defined in L_α for α limit, using bounded rank replacement.
The proof is similar for ∞-Borel codes. ⊣

3.2. The naming system. We use the naming system presented by Cohen in [9]: a name for a set $a \in L_\alpha(x)$ is given by the successive construction steps that lead to the construction of a, starting from an oracle x that we do not know.

We define P_0 as the set of names for elements of $L_0(x)$, that is, for $\{x\} \cup \omega$. The integer 0 is a name for x and the integer $n+1$ is a name for $n \in \omega$.

Suppose now by induction that for an ordinal α, the set of names P_α for elements of $L_\alpha(x)$ has been defined. We define the set of names $P_{\alpha+1}$ for elements of $L_{\alpha+1}(x)$. Let $b \in L_{\alpha+1}(x)$ be such that $b = \{a \in L_\alpha(x) : L_\alpha(x) \models \Phi(a, p_1, \dots, p_n)\}$, for $p_1, \dots, p_n \in L_\alpha(x)$. A name for b is given by the following $\dot{b} = \langle P_\alpha, \ulcorner \Phi \urcorner, \dot{p}_1, \dots, \dot{p}_n \rangle$, where $\dot{p}_1, \dots, \dot{p}_n \in P_\alpha$ are names for $p_1, \dots, p_n$.

Suppose now that the set of names P_β have been defined for $\beta < \alpha$. Then we define $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$.

In general if $a \in L_\alpha(x)$, its corresponding name is written $\dot{a}$. Note that the naming system allows us to speak about elements of $L_\alpha(x)$ without any requirement on x.

PROPOSITION 3.4. *The function $\beta \mapsto P_\beta$ is $\Delta_1^{L_\alpha}$-definable uniformly in α is limit.*

PROOF. We only sketch the proof here. It is straightforward by Δ_0-induction on ordinals, using bounded rank replacement of Proposition 2.10, where $E_{<\beta}$ is simply β. One should show that for any β, the set P_β belongs to $L_{\beta+k}$ for some $k \in \omega$. This ensures (3) of Proposition 2.10, whereas (1) and (2) are obvious. ⊣

We shall now argue that given a name $p \in P_\alpha$ and given $x \in 2^\omega$, we can, uniformly in p and x, define the set of $L_\alpha(x)$ that is coded by the name. Such a set will be denoted by $p[x]$, and is defined by induction on the rank of p as follows:

- If $p = 0$ then $p[x] = x$. If $p = n \in \omega - \{0\}$ then $p[x] = n - 1$.
- Suppose $p[x]$ has been defined for every name $p \in P_\alpha$. We define

$$P_\alpha[x] = \{p[x] \ : \ p \in P_\alpha\}$$

  Note that $P_\alpha[x]$ is intended to equal $L_\alpha(x)$. Let $p = \langle P_\alpha, \ulcorner\Phi\urcorner, \dot{p_1}, \ldots, \dot{p_n}\rangle$ be a name of $P_{\alpha+1}$. Then $p[x]$ is defined as:

$$p[x] = \{q[x] \ : \ q \in P_\alpha \ \text{ s.t. } P_\alpha[x] \models \Phi(q[x], \dot{p_1}[x], \ldots, \dot{p_n}[x])\}$$

It is clear by induction that for any ordinal α, for any $x \in 2^\omega$ and any $p \in L_\alpha(x)$, we have $\dot{p}[x] = p$.

Note that with the definition we gave, we do not have $P_\alpha \subseteq P_{\alpha+1}$. However for $\beta < \alpha$ and $p \in P_\beta$, one can uniformly obtain a name $q \in P_\alpha$ such that $p[x] = q[x]$.

PROPOSITION 3.5. *The function which to ordinals $\gamma < \beta$ and names $\dot{p_\gamma} \in P_\gamma$ of elements $p \in L_\gamma(x)$, associates names $\dot{p_\beta} \in P_\beta$ for the same element p, is $\Delta_1^{L_\alpha}$ uniformly in α limit.*

PROOF. This is again Δ_0-induction on ordinals, using bounded rank replacement of Proposition 2.10. If $\beta = 1$, given $\dot{p_0} \in P_0$, a name for some $p \in L_0(x)$, we let $\dot{p_\beta} = \langle P_0, \ulcorner a \in z\urcorner, \dot{p_0}\rangle \in P_1$. Note that z is a free variable in the formula, meant to be replaced by $\dot{p}_0$. It is clear that $\dot{p_\beta} \in L_k$ for some $k < \omega$ and that $\dot{p_\beta}$ is also a name for p.

Let β and let f be the function of the theorem defined on any $\gamma' < \beta' \leq \beta$ and names of $P_{\gamma'}$. Let us show that we can extend f on any $\gamma < \beta + 1$ and any name of P_γ. Let $\gamma < \beta+1$ and $\dot{p_\gamma} \in P_\gamma$ be a name for some $p \in L_\gamma(x)$. If $\gamma < \beta$, using f we can find $\dot{p_\beta} \in P_\beta$, a name for p. Thus we can work as in the case $\gamma = \beta$ and consider that we always have a name $\dot{p_\beta} \in P_\beta$. In particular we have that $\dot{p_\beta}$ equals $\langle P_\gamma, \ulcorner\Phi(a, \overline{z_i})\urcorner, \overline{p_i}\rangle$ for some $\ulcorner\Phi(a, \overline{z_i})\urcorner$, some $\overline{p_i} \in P_\gamma$, and some $\gamma < \beta$ (with $\gamma = \beta - 1$ if β is successor).

Using f one can find names $\overline{q_i} \in P_\beta$ corresponding to the names $\overline{p_i} \in P_\gamma$. Note that a name for L_γ is given by $\langle P_\gamma, \ulcorner a = a \urcorner \rangle$. Let $\Psi(a, r, \overline{p_i})$ be the conjunction of the formula $a \in r$, together with the formula $\Phi(a, \overline{p_i})$ where every instance of $\exists x$ (resp. $\forall x$) is replaced by $\exists x \in r$ (resp. $\forall x \in r$). The name $\dot{p_{\beta+1}} \in P_{\beta+1}$ is then given by:

$$\langle P_\beta, \ulcorner \Psi(a, r, \overline{q_i}) \urcorner, \langle P_\gamma, \ulcorner a = a \urcorner \rangle, \overline{q_i} \rangle$$

If is clear that $\dot{p_{\beta+1}} \in L_{\beta+k}$ for some k. Therefore we are in the conditions of Proposition 2.10 and the function of the proposition is $\Delta_1^{L_\alpha}$ uniformly in α limit.

Also for the limit case the induction is clear as for β limit we have $P_\beta = \bigcup_{\gamma < \beta} P_\gamma$. ⊣

3.3. The canonical Borel sets. We develop here the notations and the main theorem to deal with the canonical sets with ∞-Borel codes, that will be used in this paper. Let β be an ordinal. Let $p_1, \ldots, p_n \in P_\beta$. Let $\Phi(p_1, \ldots, p_n)$ be a formula. Then we write:

$$B_\Phi^\beta(p_1, \ldots, p_n) \text{ for the set } \{x \in 2^\omega \ : \ L_\beta(x) \models \Phi(p_1[x], \ldots, p_n[x])\}$$

The upcoming theorem makes sure that $B_\Phi^\beta(p_1, \ldots, p_n)$ truly has an ∞-Borel code, definable uniformly in $\ulcorner \Phi \urcorner$, β and $p_1, \ldots, p_n$.

We will sometimes write B_Φ or B_Φ^β when the ordinal β and/or parameters $p_1, \ldots, p_n$ are not specified. Also given an ∞-Borel set B_Φ for a Σ_n formula Φ, we say that B_Φ^β is a Σ_n^β ∞-Borel set. Note that a fixed formula Φ gives rise to many possible ∞-Borel sets depending on the model L_β that we consider.

The second part of the following theorem says that for α limit, if Φ is Δ_0, then an ∞-Borel code for $B_\Phi^\alpha(p_1, \ldots, p_n)$ belongs to L_α, and can be found uniformly. It follows that one can picture a Σ_n^α Borel set with similar intuitions one has with the usual Σ_n Borel sets used in the realm of computable objects and algorithmic randomness : The Σ_1^α Borel sets can be seen as increasing uniform unions of Δ_0^α Borel sets over the names of elements of L_α. Note that if α is limit we have $P_\alpha \subseteq L_\alpha$ and:

$$\begin{aligned} & \{x \in 2^\omega \ : \ L_\alpha(x) \models \exists z \ \Phi(z, p_1[x], \ldots, p_n[x])\} \\ = \ & \textstyle\bigcup_{\dot{z} \in P_\alpha} \{x \in 2^\omega \ : \ L_\alpha(x) \models \Phi(\dot{z}[x], p_1[x], \ldots, p_n[x])\} \end{aligned}$$

Similarly, Σ_2^α Borel sets are unions of intersections of Δ_0^α Borel sets. Indeed we have for α limit that:

$$\begin{aligned} & \{x \in 2^\omega \ : \ L_\alpha(x) \models \exists z_1 \ \forall z_2 \ \Phi(z_1, z_2, p_1[x], \ldots, p_n[x])\} \\ = \ & \textstyle\bigcup_{\dot{z_1} \in P_\alpha} \bigcap_{\dot{z_2} \in P_\alpha} \{x \in 2^\omega \ : \ L_\alpha(x) \models \Phi(\dot{z_1}[x], \dot{z_2}[x], p_1[x], \ldots, p_n[x])\} \end{aligned}$$

One easily sees how to continue for Σ_n^α Borel sets in general.

THEOREM 3.6. *Let α be limit. Then, we have the following:*

1. *A function which on β, $\ulcorner \Phi(x_1, \ldots, x_n) \urcorner$ and $p_1, \ldots, p_n \in P_\beta$, associates an ∞-Borel code for $B_\Phi^\beta(p_1, \ldots, p_n)$ is $\Delta_1^{L_\alpha}$ uniformly in α.*
2. *A function which on Δ_0 formulas $\ulcorner \Phi(x_1, \ldots, x_n) \urcorner$ and $p_1, \ldots, p_n \in P_\alpha$, associates an ∞-Borel code for $B_\Phi^\alpha(p_1, \ldots, p_n)$ is $\Delta_1^{L_\alpha}$ uniformly in α.*

PROOF. (1) is proved by a Δ_0-induction, using bounded rank replacement of Proposition 2.10, with the class of elements of the form $(\beta, \ulcorner \Phi(\overline{x}) \urcorner, \overline{p})$ in place of E: an ordinal β, a formula with n free variables, and n parameters of P_β. The induction is done only on the ordinal β. For a set F of formulas (for instance the atomic formulas) let $\mathcal{H}_\beta(F)$ be the induction hypothesis:

$(\mathcal{H}_\beta(F))$ The function f which on β, formulas $\ulcorner\Phi(\overline{x})\urcorner \in F$ and $\overline{p} \in P_\beta$ associates an ∞-Borel code for $B^\beta_\Phi(\overline{p})$, belongs to $L_{\beta+k}$ for some k

Let F_0 be the set of atomic formulas and F_∞ be the set of all formulas. We will show $\mathcal{H}_0(F_0)$. Then we will show $\mathcal{H}_\beta(F_0)$ implies $\mathcal{H}_\beta(F_\infty)$, then we will show $\mathcal{H}_\beta(F_\infty)$ implies $\mathcal{H}_{\beta+1}(F_0)$. Finally we will show $\bigwedge_{\gamma<\beta}\mathcal{H}_\gamma(F_\infty) \to \mathcal{H}_\beta(F_0)$, together with (2) of the Theorem.

Let us begin with $\mathcal{H}_0(F_0)$, Let $p_1, p_2 \in P_0$. Consider $B_= = \{x \in 2^\omega \ : \ L_0(x) \models p_1[x] = p_2[x]\}$ and $B_\in = \{x \in 2^\omega \ : \ L_0(x) \models p_1[x] \in p_2[x]\}$. Recall that p_1, p_2 must be integers, with 0 coding for x and $n+1$ coding for n. Therefore we have $B_= = 2^\omega$ if $p_1 = p_2$ and $B_= = \emptyset$ otherwise. We also have $B_\in = 2^\omega$ if $p_1, p_2 > 0$ and $p_1 \in p_2$ or if $p_1 \neq 0$, $p_2 = 0$ and $p_1 - 1 \in x$. Otherwise we have $B_\in = \emptyset$. It is clear that the two possible Borel codes (2^ω or $\emptyset$) belongs to L_k for some $k \in \omega$ and that the computable function which assign the right Borel code depending on the atomic formulas and parameters, also belongs to L_k for some $k \in \omega$ (recall that we start with $L_0 = \omega$).

Now we prove $\mathcal{H}_\beta(F_0) \Rightarrow \mathcal{H}_\beta(F_\infty)$. We proceed in 5 stages, first showing $\mathcal{H}_\beta(F_0) \Rightarrow \mathcal{H}_\beta(F_1)$, for F_1 the set of atomic formulas and their negations, then showing $\mathcal{H}_\beta(F_1) \Rightarrow \mathcal{H}_\beta(F_2)$, for F_2 the set of finite disjunctions of formulas of F_1, then showing $\mathcal{H}_\beta(F_2) \Rightarrow \mathcal{H}_\beta(F_3)$, for F_3 the set of finite conjunctions of formulas of F_2, then showing $\mathcal{H}_\beta(F_3) \Rightarrow \mathcal{H}_\beta(F_4)$ for F_4 the set of all formulas of F_3 closed by finitely many quantifications, and finally showing $\mathcal{H}_\beta(F_4) \Rightarrow \mathcal{H}_\beta(F_\infty)$.

The step $\mathcal{H}_\beta(F_0)$ implies $\mathcal{H}_\beta(F_1)$ simply follows from (2) of Proposition 3.2. The step $\mathcal{H}_\beta(F_1)$ implies $\mathcal{H}_\beta(F_2)$ then follows from (3) of Proposition 3.2, whereas the step $\mathcal{H}_\beta(F_2)$ implies $\mathcal{H}_\beta(F_3)$ follows from both (2) and (3) of Proposition 3.2. Let us now show the step $\mathcal{H}_\beta(F_3)$ implies $\mathcal{H}_\beta(F_4)$. Let $\overline{p} \in P_\beta$ and let $\Phi(\overline{a}) = \exists a_1 \ \forall a_2 \ \dots \Psi(a_1, a_2, \dots, \overline{a})$ be any formula of F_4 (that is in prenex normal form with its quantifier-free part in disjunctive normal form, in particular with Ψ in F_3). We then have:

$$\begin{aligned} B^\beta_\Phi(\overline{p}) &= \{x \in 2^\omega \ : \ L_\beta(x) \models \exists x_1 \ \forall x_2 \ \dots \Psi(x_1, x_2, \dots, \overline{p}[x])\} \\ &= \textstyle\bigcup_{q_1 \in P_\beta} \bigcap_{q_2 \in P_\beta} \dots \{x \in 2^\omega \ : \ L_\beta(x) \models \Psi(q_1[x], q_2[x], \dots, \overline{p}[x])\} \end{aligned}$$

Using (3) and (4) of Definition 3.1, and assuming we have the function given by $\mathcal{H}_\beta(F_3)$, it is easy to build the Borel code for $B^\beta_\Phi(\overline{p})$ whose rank does not increase with the number of quantification, and furthermore, to uniformly do so. In particular we obtain $\mathcal{H}_\beta(F_4)$. In order to obtain $\mathcal{H}_\beta(F_\infty)$, one simply has to use the computable function which transforms any formula into a formula in prenex normal form with its quantifier-free part in disjunctive normal form.

We continue by assuming $\mathcal{H}_\beta(F_\infty)$ and proving $\mathcal{H}_{\beta+1,0}$. We let $p_1, p_2 \in P_{\beta+1}$ with $p_1 = \langle P_\beta, \ulcorner\Phi_1\urcorner, a_1, \dots, a_n\rangle$ and $p_2 = \langle P_\beta, \ulcorner\Phi_2\urcorner, b_1, \dots, b_m\rangle$. For $q \in P_\beta$, let:

$$\begin{aligned} B_{\Phi_1}(q) &= \{x \in 2^\omega \ : \ L_\beta(x) \models \Phi_1(q[x], a_1[x], \dots, a_n[x])\} \\ B_{\Phi_2}(q) &= \{x \in 2^\omega \ : \ L_\beta(x) \models \Phi_2(q[x], b_1[x], \dots, b_m[x])\} \end{aligned}$$

Note that $q[x] \in p_1[x]$ iff $x \in B_{\Phi_1}(q)$ and $q[x] \in p_2[x]$ iff $x \in B_{\Phi_2}(q)$. Also by induction hypothesis, the function which on $q \in P_\beta$ and on any formula Ψ associates the code of $B_\Psi(q)$ belongs to $L_{\beta+k}$ for some $k \in \omega$. We have:

$$L_{\beta+1}(x) \models p_1[x] \in p_2[x] \quad \text{iff} \quad \exists q \in P_\beta,\ x \in B_{\Phi_2}(q) \wedge L_{\beta+1}(x) \models p_1[x] = q[x]$$

$$L_{\beta+1}(x) \models p_1[x] = p_2[x] \quad \text{iff} \quad \forall q \in P_\beta \text{ we have } x \in B_{\Phi_1}(q) \leftrightarrow x \in B_{\Phi_2}(q)$$

Thus we have:

$$\begin{array}{rcl} B_{=}^{\beta+1}(p_1, p_2) & = & \{x \in 2^\omega \ : \ L_{\beta+1}(x) \models p_1[x] = p_2[x]\} \\ & = & \bigcap_{q \in P_\beta} [B_{\Phi_1}(q) \cap B_{\Phi_2}(q)] \cup [(2^\omega - B_{\Phi_1}(q)) \cap (2^\omega - B_{\Phi_2}(q))] \end{array}$$

It is clear that a code for $B_{=}^{\beta+1}(p_1, p_2)$ can be obtained uniformly and belongs to $L_{\beta+1+k}$ for some k which is independent from p_1, p_2. It follows that we have $\mathcal{H}_{\beta+1,0}$ for equality. Also the set

$$\begin{array}{rcl} B_{\in}^{\beta+1}(p_1, p_2) & = & \{x \in 2^\omega \ : \ L_{\beta+1}(x) \models p_1[x] \in p_2[x]\} \\ & = & \bigcup_{q \in P_\beta} [B_{\Phi_2}(q) \cap \{x \in 2^\omega \ : \ L_{\beta+1}(x) \models p_1[x] = q[x]\}] \end{array}$$

Using Proposition 3.5 one can uniformly transform $q \in P_\beta$ into a name that belongs to $P_{\beta+1}$ and thus perform the induction given by the $=$ case just above. We thus have $\mathcal{H}_{\beta+1}(F_\infty)$.

We now deal with the limit case, together with (2) of the theorem. We shall show $\bigwedge_{\gamma<\beta} \mathcal{H}_\gamma(F_\infty) \to \mathcal{H}_\beta(F_0)$. We will actually show more in order to also show (2): We show $\bigwedge_{\gamma<\beta} \mathcal{H}_\gamma(F_\infty) \to \mathcal{H}_\beta(F_{\Delta_0})$ where F_{Δ_0} is the set of Δ_0 formulas.

For a Δ_0 formula $\Phi(p_1, \dots, p_n)$, let γ be the smallest such that $p_1, \dots, p_n \in P_\gamma$. Note that γ is $\Delta_1^{L_\beta}$-definable uniformly in $p_1, \dots, p_n$. We have that the Borel $B_\Phi^\beta(p_1, \dots, p_n)$ also equals the Borel $B_\Phi^\gamma(p_1, \dots, p_n) = \{x \in 2^\omega \ : \ L_\gamma(x) \models \Phi(p_1[x], \dots, p_n[x])\}$. By induction hypothesis the function which on Φ and $p_1, \dots, p_n \in P_\gamma$ gives the Borel code of $B_\Phi^\gamma(p_1, \dots, p_n)$ belongs to $L_{\gamma+k}$ for some k. As this function can be recognized with a Δ_0 formula uniformly in γ, this gives us (2). Note that the union of all these functions belongs to $L_{\beta+k}$ for some k, which gives us $\mathcal{H}_\beta(F_0)$. This concludes the proof. ⊣

Note that for the study of ITTMs, we can always assume that we work with Borel codes and not ∞-Borel codes. Indeed by Corollary 2.33, for every $\alpha \leq \Sigma$ limit the sets L_α is a model of "everything is countable". We can then uniformly transform any ∞-Borel codes into a Borel code, working in L_α for α limit, by searching inductively for the smallest (in the sense of $<_L$) bijection between elements of a Borel code, and ω.

§4. Randomness.

The main idea of algorithmic randomness is to define as random the elements of 2^ω which are in no set that is both of measure 0 and simple to define. Martin-Löf defined its eponymous randomness notion [20] using computability. Higher randomness has then been studied by working in $L_{\omega_1^{ck}}$ (see [13] [3] [7] [8] [16] [21] [1]). Recently Carl and Schlicht initiated the study of randomness with infinite-time Turing machines [6]. In this section, we pursue their work, solving some of their open questions.

We start with a lemma extending computable measure theory to levels of the constructible hierarchy. In what follows, μ denotes the Lebesgue measure on 2^ω.

Lemma 4.1.

1. *The function $b \mapsto \mu(\iota(b))$, defined on ∞-Borel codes b, is $\Delta_1^{L_\alpha}$ uniformly in any α limit.*
2. *We have the following, where b range over Borel codes and q over rationnals:*
   - *The function $b, q \mapsto u$ such that u is the Borel code of an open set with $\iota(b) \subseteq \iota(u)$ and $\mu(\iota(u) - \iota(b)) \leq q$*
   - *The function $b, q \mapsto c$ such that c is the Borel code of a closed set with $\iota(c) \subseteq \iota(b)$ and $\mu(\iota(b) - \iota(c)) \leq q$*

   are $\Delta_1^{L_\alpha}$ definable uniformly in any α limit.

Proof. Both (1) and (2) are proved by a Δ_0-induction on ranks of Borel sets (their rank as elements of L). This uses the bounded rank replacement of Proposition 2.10.

Proof of (1). For a Borel code b of rank 0, the measure is easily computable, as the measure of a clopen set. Let now $b = \bigvee_{n\in I} b_n$ and γ the smallest such that $b \in L_{\gamma+1}$. Note that each b_i belongs to L_γ. Let $P_f(I)$ be the set of finite subsets of I. We have:

$$\mu(\iota(b)) = \sup_{F \in P_f(I)} \left(\lambda \left(\bigcup_{b_i \in F} \iota(b_i) \right) \right) \tag{1}$$

Using (3) of Proposition 3.2 we can obtain an ∞-Borel code d_F such that $\iota(d_F) = \bigcup_{b_i \in F} \iota(b_i)$ and such that $d_F \in L_\gamma$. It is also clear that the function which to I associates $P_f(I)$ is $\Delta_1^{L_\alpha}$-definable uniformly in α limit. The function can then be defined by the Δ_0-induction with bounded rank replacement of Proposition 2.10.

To compute $\mu(\bigwedge_{n\in\omega} b_n)$, we can use (2) of Proposition 3.2 to take the complement to 1 of the measure of $2^\omega - \iota(c)$.

Proof of (2). The function is also defined by Δ_0-induction over γ, using bounded rank replacement of Proposition 2.10. In this first point, we could use ∞-Borel codes (and not Borel codes) but still compute the measure by considering all finite unions of codes of smaller complexity. In the second point, we do need to use Borel codes in order to associate a quantity 2^{-n} to each component of a Borel set.

For a Borel code b of rank 0, both the open and the clopen sets are given by b itself. Let now $b = \bigvee_{n\in\omega} b_n$ with γ^+ the smallest such that $b \in L_{\gamma^+}$. Note that each b_n belongs to L_γ. By induction, for each b_n we find codes u_n and c_n of respectively open and closed sets, such that $\mu(\iota(u_n) - \iota(b_n)) < 2^{-n}q$ and $\mu(\iota(b_n) - \iota(c_n)) \leq 2^{-n}q$. The code for the desired open set is then $\bigvee_{n\in\omega} u_n$. For the closed set, note that we have $\mu(\iota(\bigvee_n b_n) - \iota(\bigvee_n c_n)) \leq q$. It follows that there must be some m such that $\mu(\iota(\bigvee_n b_n) - \iota(\bigvee_{n<m} c_n)) \leq q$. The code for the closed set is then given by a code d equivalent to $\bigvee_{n<m} \iota(c_n)$, where we propagate the finite union using Proposition 3.2.

⊣

4.1. Main definitions. We give the first definition, which in full generality extends algorithmic randomness to every level of the constructible hierarchy.

DEFINITION 4.2. Let α be a countable ordinal. A set x is *random over* L_α if x is in no null set with a Borel code in L_α.

This can be seen as an extension, to any level of the constructible hierarchy, of Δ^1_1-randomness, which corresponds to randomness over $L_{\omega_1^{ck}}$ in the above definition.

The most famous and studied randomness notion is undoubtedly Martin-Löf randomness [20], whose counterpart for $L_{\omega_1^{ck}}$ was defined in [16]. We also extend the definition of Martin-Löf randomness to any level of the constructible hierarchy:

DEFINITION 4.3 (Carl, Schlicht [6]). An α-recursively enumerable open set $\mathcal{U}_n$ is an open set with a code Σ_1-definable in L_α (with parameters). A set x is *α-ML-random* if x is in no intersection $\bigcap_n \mathcal{U}_n$ where each set $\mathcal{U}_n$ is an α-recursively enumerable open set, uniformly in n, such that $\mu(\mathcal{U}_n) \leq 2^{-n}$.

We now turn to randomness notions which are specific to ITTMs. In order to do so, we first need the following definition:

DEFINITION 4.4 (Hamkins, Lewis [14]). A set $P \subseteq 2^\omega$ or $P \subseteq \omega$ is *ITTM-semi-decidable* if there is an ITTM M such that $x \in P \Leftrightarrow M(x)\downarrow$. A set $P \subseteq 2^\omega$ or $P \subseteq \omega$ is *ITTM-decidable* if it is both semi-decidable and co-semi-decidable, equivalently, there is an ITTM M such that $M(x)\downarrow = 1 \leftrightarrow x \in P$ and $M(x)\downarrow = 0 \leftrightarrow x \in P$.

If $x \subseteq \omega$, it is clear by admissibility of L_λ that x is ITTM-decidable iff $x \in L_\lambda$, and that x is ITTM-semi-decidable iff x is Σ_1-definable over L_λ.

DEFINITION 4.5 (Carl, Schlicht [6]). An ITTM-semi-decidable open set is an open set $\mathcal{U}$ with an ITTM-semi-decidable description $W \subseteq 2^{<\omega}$ such that we have $\bigcup_{\sigma \in W}[\sigma] = \mathcal{U}$. A set X is *ITTM-ML-random* if X is in no intersection $\bigcap_n \mathcal{U}_n$ where each set $\mathcal{U}_n$ is an ITTM-semi-decidable open set, uniformly in n, such that $\mu(\mathcal{U}_n) \leq 2^{-n}$.

Before we continue, we would like to make a small digression about the definition of ITTM-semi-decidable open sets. In the case of Turing machines, given an open set $\mathcal{U}$, it is equivalent to have a recursively enumerable set $W \subseteq 2^{<\omega}$ such that $\mathcal{U} = \bigcup_{\sigma \in W}[\sigma]$ and to have a functional Φ such that $\Phi(X)\downarrow \leftrightarrow X \in \mathcal{U}$.

In the case of computability over $L_{\omega_1^{ck}}$, the same holds: given an open set $\mathcal{U}$, it is equivalent to have a Π^1_1 set $W \subseteq 2^{<\omega}$ such that $\mathcal{U} = \bigcup_{\sigma \in W}[\sigma]$ and for the open set $\mathcal{U}$ to be Π^1_1 as a set of reals.

The corresponding fact with ITTMs does not hold:

PROPOSITION 4.6. *Every ITTM semi-decidable open set $\mathcal{U} \subseteq 2^\omega$ is also ITTM semi-decidable as a set of reals, but there is an open set that is ITTM-decidable as a set of reals and which is not ITTM-semi-decidable as a set of strings.*

PROOF. Suppose $\mathcal{U}$ has an ITTM semi-decidable code $W \subseteq 2^{<\omega}$. We can design another ITTM which on input x looks for some n such that $x_{\upharpoonright n} \in W$.

Whenever it find such an n it halts. It is clear that this other ITTM semi-decide $\mathcal{U}$ as a set of reals.

Let us now exhibit an open set $\mathcal{U}$ that is ITTM-decidable as a set of reals, but does not have an ITTM-semi-decidable code. Let c be given by the "Lost melody lemma" [14], that is $\{c\}$ is ITTM-decidable but c is not writable. Then, $A = 2^\omega - \{c\}$ is ITTM-decidable. However, no ITTM-semi-decidable set $W \subseteq 2^{<\omega}$ can be such that $\mathcal{U} = \bigcup_{\sigma \in W}[\sigma]$, as otherwise c would be writable by the following algorithm: if we know that $\sigma \prec c$, we compute a longer prefix of c by waiting for W to cover either $\sigma^\frown i$ (for $i = 0$ or 1), which will happen by compactness, and then extend our prefix to $\sigma^\frown(1-i) \prec c$. ⊣

We now turn to the most interesting randomness notions defined with ITTM.

Definition 4.7 (Carl, Schlicht [6]). A real x is ITTM-random if it is in no semi-decidable null set of reals. A real x is ITTM-decidable random if it is in no decidable null set of reals.

ITTM-decidable randomness can be seen as a counterpart of Δ^1_1-randomness, and indeed Carl and Schlicht showed that ITTM-decidable randomness coincides with randomness over L_Σ.

The notion of ITTM-randomness is more interesting and is in many regards an equivalent for the notion of Π^1_1-randomness. We try in this paper to provide a better understanding of this notion.

4.2. ITTM-randomness. It is not immediately clear that every ITTM semi-decidable set is measurable. A semi-decidable set has the form $\{x \in 2^\omega : L_{\Sigma^x}(x) \models \Phi\}$ for some Σ_1-formula Φ. Such sets need not to be Borel, but we can separate them into a Borel part and a non-Borel part always included in a Borel set of measure 0. In particular any such set is included in the set $\{x \in 2^\omega : L_\Sigma(x) \models \Phi\} \cup \{x \in 2^\omega : \Sigma^x > \Sigma\}$. The fact that every ITTM semi-decidable set is measurable follows from the fact that the set $\{x \in 2^\omega : \Sigma^x > \Sigma\}$ is included in a Borel set of measure 0. This will be a consequence of Corollary 4.10 together with Theorem 2.23.

ITTM-randomness is by many aspects the ITTM counterpart of Π^1_1-randomness. For instance, there is a greatest Π^1_1 null set, and Carl and Schlicht showed that there is a greatest ITTM-semi-decidable null set. We also have that x is Π^1_1-random iff x is Δ^1_1-random and $\omega_1^x = \omega_1^{ck}$. Carl and Schlicht proved the analogous statement for ITTM-randomness:

Theorem 4.8 (Carl, Schlicht [6]). *A real x is ITTM-random if and only if it is random over L_Σ and $\Sigma^x = \Sigma$.*

There are of course differences between ITTM-randomness and Π^1_1-randomness. It is for instance straightforward to build a sequence that is Δ^1_1-random but not Π^1_1-random: to do so one can show that the set of Π^1_1-randoms is included in the set of Π^1_1-ML-randoms, which is included in the set of Δ^1_1-randoms. One can then build a sequence which is Δ^1_1-random but not Π^1_1-ML-random, with a construction similar to the one given in the proof of (1) implies (3) in Theorem 4.22. The same thing is not possible with ITTM-randomness. We will see in particular that the Σ-ML-randoms are strictly included in the ITTM-randoms. Also, it is not clear that there are reals x which are randoms over L_Σ and such that $\Sigma^x > \Sigma$. We will for instance show later in Section 5 that the equivalent notions for genericity collapse: a real x is ITTM-generic iff x is generic over L_Σ iff x is generic over L_Σ and $\Sigma^x = \Sigma$. The question for ITTM-randomness remains open:

QUESTION 4.9. *Does ITTM-randomness coincide with randomness over L_Σ ?*

Although we are not able to answer the question here, we still can say meaningful things about ITTM-randomness. In [6] Carl and Schlicht proved the following:

THEOREM 4.10 (Carl, Schlicht [6]). *Suppose that α is countable and admissible or a limit of admissibles ordinals. Then:*

1. *If $L_\beta \prec_1 L_\alpha$ and z is random over L_α, then $L_\beta(z) \prec_1 L_\alpha(z)$*
2. *If $L_\beta \prec_n L_\alpha$ and z is random over L_γ where $L_\gamma \models$ "α is countable", then $L_\beta(z) \prec_n L_\alpha(z)$ for $n \geq 2$.*

In order to understand better Σ-randomness, we introduce a stronger notion that will be enough to obtain (2) in the previous theorem.

DEFINITION 4.11. A weak α-ML test is given by a set $\bigcap_{q \in L_\alpha} \mathcal{B}_q$ such that the function which to q associates a Borel code of $\mathcal{B}_q$ is $\Delta_1^{L_\alpha}$ and such that $\mu(\bigcap_{q \in L_\alpha} \mathcal{B}_q) = 0$.

A real x is captured by a weak α-ML test if $x \in \bigcap_{q \in L_\alpha} \mathcal{B}_q$. Otherwise we say that x passes the test. A real x which passes all the weak α-ML tests is weakly α-ML-random.

PROPOSITION 4.12. *Let α be admissible and $L_\alpha \models$ "everything is countable". Then weak α-ML-randomness coincides with randomness over L_α.*

PROOF. It is clear that weak α-ML-randomness implies randomness over L_α for any α. Suppose now α admissible and let $\bigcap_{q \in L_\alpha} \mathcal{B}_q$ be a weak α-ML test. Let $f : \omega \to L_\alpha$ be defined with $f(n)$ to be the smallest $r \in L_\alpha$, in the sense of $<_L$, such that $\mu(\bigcap_{q <_L r} \mathcal{B}_q) < 2^{-n}$. By admissibility of α there exists $\beta < \alpha$ such that $\forall n\ f(n) \in L_\beta$. We then have $\mu(\bigcap_{q \in L_\beta} \mathcal{B}_q) = 0$ and $\bigcap_{q \in L_\alpha} \mathcal{B}_q \subseteq \bigcap_{q \in L_\beta} \mathcal{B}_q$. As $\bigcap_{q \in L_\beta} \mathcal{B}_q$ has a Borel code in L_α we have that every element in $\bigcap_{q \in L_\alpha} \mathcal{B}_q$ belongs to a null set of L_α. ⊣

PROPOSITION 4.13. *Weak Σ-ML randomness is strictly stronger than randomness over L_Σ.*

PROOF. It is clear that weak Σ-ML randomness is stronger than randomness over L_Σ. Let us build $x \in 2^\omega$ that is random over L_Σ but not weakly Σ-ML random.

From Corollary 2.35, let $b : \omega \to L_\Sigma$ be a bijection which is Σ_1-definable in L_Σ with parameter ζ. One can then simply diagonalize against every measure 1 set with a Borel code in L_Σ. We define σ_0 to be the empty string and $\mathcal{F}_0$ to be 2^ω. Suppose for some n and every $i \leq n$ we have defined a string σ_i and a closed set $\mathcal{F}_i$ uniformly in i such that $\mu(\sigma_i \cap \mathcal{F}_i) > 0$, such that $\sigma_i \preceq \sigma_{i+1}$, such that $\mathcal{F}_{i+1} \subseteq \mathcal{F}_i$ and such that if $b(i)$ is the Borel code of a co-null set, then $\mathcal{F}_i \subseteq \mathcal{B}_i$. Let us define $\mathcal{F}_{n+1}$ and σ_{n+1}. If $b(n+1)$ is the Borel code of a set of measure less than 1, we define $\sigma_{n+1} = \sigma_n$ and $\mathcal{F}_{n+1} = \mathcal{F}_n$. If $b(n+1)$ is the Borel code of a set $\mathcal{B}$ of measure 1, we uniformly find a closed set $\mathcal{F} \subseteq \mathcal{B}$ with a Borel code in L_Σ and with a measure sufficiently close to 1, so that we have $\mu(\sigma_n \cap \mathcal{F}_n \cap \mathcal{F}) > 0$, using Lemma 4.1. We define $\mathcal{F}_{n+1} = \mathcal{F}_n \cap \mathcal{F}$. We then define $\sigma_{n+1} = \sigma_n i$ for $i \in \{0, 1\}$ such that $\mu(\sigma_{n+1} \cap \mathcal{F}_{n+1}) > 0$.

Let $x = \sigma_0 \preceq \sigma_1 \preceq \sigma_2 \preceq \dots$. It is clear that x is random over L_Σ. The weak Σ-ML test $\bigcap_{q \in L_\Sigma} \mathcal{B}_q$ is as follow: for $q \in L_\Sigma$, let $n = b^{-1}(q)$. Then define $\mathcal{B}_q = \sigma_n$. It is clear that $x \in \bigcap_{q \in L_\Sigma} \mathcal{B}_q$ and that $\mu(\bigcap_{q \in L_\Sigma} \mathcal{B}_q) = 0$. ⊣

We then need two lemmas. The first is the same as (1) in [6], but we believe that this paper's proof is a bit simpler.

Lemma 4.14. *Let $\beta < \alpha$ with α countable and limit and $L_\alpha \models$"everything is countable" and $L_\beta \prec_1 L_\alpha$. Let z be random over α. Then we have $L_\beta(z) \prec_1 L_\alpha(z)$.*

Proof. Let $\Phi(p, q)$ be a Δ_0 formula with $p \in P_\beta$. Suppose z is random over L_α such that $L_\alpha(z) \models \exists q\ \Phi(p[z], q)$. Consider the set $\mathcal{B}^\alpha_\Phi = \{x : L_\alpha(x) \models \exists q\ \Phi(p[x], q)\}$. We have $\mathcal{B}^\alpha_\Phi = \bigcup_{\dot{q} \in P_\alpha} \mathcal{A}_{\dot{q}}$ where:

$$\mathcal{A}_{\dot{q}} = \{x : L_\alpha(x) \models \Phi(p[x], \dot{q}[x])\}$$

Let $\mu(\mathcal{B}^\alpha_\Phi) = m$. Note that as z is random over L_α we must have $m > 0$. For every ε with $0 < \varepsilon < m$ we have $L_\alpha \models \exists \dot{r}\ \mu(\bigcup_{\dot{q} <_L \dot{r}} \mathcal{A}_{\dot{q}}) > \varepsilon$. As $L_\beta \prec_1 L_\alpha$ we then have $L_\beta \models \exists \dot{r}\ \mu(\bigcup_{\dot{q} <_L \dot{r}} \mathcal{A}_{\dot{q}}) > \varepsilon$. As this is true for every ε we then have $\mu(\bigcup_{\dot{q} \in P_\beta} \mathcal{A}_{\dot{q}}) = m$.

Suppose for a contradiction that $z \notin \bigcup_{\dot{q} \in P_\beta} \mathcal{A}_{\dot{q}}$. There exists $\dot{r} \in P_\alpha$ such that $z \in \mathcal{A}_{\dot{r}}$. Note that we have $\bigcup_{\dot{q} \in P_\beta} \mathcal{A}_{\dot{q}} \subseteq \bigcup_{\dot{q} \in P_\alpha} \mathcal{A}_{\dot{q}}$ and $\mu(\bigcup_{\dot{q} \in P_\beta} \mathcal{A}_{\dot{q}}) = \mu(\bigcup_{\dot{q} \in P_\alpha} \mathcal{A}_{\dot{q}})$. Therefore we have $\mu(\mathcal{A}_{\dot{r}} - \bigcup_{\dot{q} \in P_\beta} \mathcal{A}_{\dot{q}}) = 0$. It follows that z belongs to a set of measure 0 with a Borel code in L_α, which is a contradiction. Therefore we have $z \in \bigcup_{\dot{q} \in P_\beta} \mathcal{A}_{\dot{q}}$ which implies $L_\beta(z) \models \exists q\ \Phi(p[z], q)$. ⊣

For the following lemma, we write $=^*, \subseteq^*$ for equality and inclusion, up to a set of measure 0.

Lemma 4.15. *Let $\beta < \alpha$ with α countable and limit, such that $L_\beta \prec_2 L_\alpha$. Let $\mathcal{B}^\alpha_\Phi = \bigcup_{\dot{q}_1 < P_\alpha} \bigcap_{\dot{q}_2 < P_\alpha} \mathcal{A}_{\dot{q}_1, \dot{q}_2}$ be a Σ^α_2 set with parameters in L_β. Then we have $\mathcal{B}^\alpha_\Phi =^* \mathcal{B}^\beta_\Phi$.*

Proof. By Lemma 4.14 and Proposition 2.17 we have that if z if random over L_α, then $z \in \bigcup_{\dot{q}_1 < P_\beta} \bigcap_{\dot{q}_2 < P_\beta} \mathcal{A}_{\dot{q}_1, \dot{q}_2}$ implies that $z \in \bigcup_{\dot{q}_1 < P_\alpha} \bigcap_{\dot{q}_2 < P_\alpha} \mathcal{A}_{\dot{q}_1, \dot{q}_2}$. It follows that

$$\bigcup_{\dot{q}_1 < P_\beta} \bigcap_{\dot{q}_2 < P_\beta} \mathcal{A}_{\dot{q}_1, \dot{q}_2} \subseteq^* \bigcup_{\dot{q}_1 < P_\alpha} \bigcap_{\dot{q}_2 < P_\alpha} \mathcal{A}_{\dot{q}_1, \dot{q}_2}$$

In particular if $\mu(\bigcup_{\dot{q}_1 < P_\alpha} \bigcap_{\dot{q}_2 < P_\alpha} \mathcal{A}_{\dot{q}_1, \dot{q}_2}) = 0$ then we are done. Suppose then that we have $\mu(\bigcup_{\dot{q}_1 < P_\alpha} \bigcap_{\dot{q}_2 < P_\alpha} \mathcal{A}_{\dot{q}_1, \dot{q}_2}) = m > 0$. For every ε with $0 < \varepsilon < m$ we have:

$$L_\alpha \models \exists \langle \dot{q}_{1,0}, \dots, \dot{q}_{1,k} \rangle\ \forall \dot{r}_2\ \mu\left(\bigcup_{0 \le i \le k} \bigcap_{\dot{q}_2 <_L \dot{r}_2} \mathcal{A}_{\dot{q}_{1,i}, \dot{q}_2} \right) > \varepsilon$$

Using $L_\beta \prec_2 L_\alpha$ we deduce:

$$L_\beta \models \exists \langle \dot{q}_{1,0}, \ldots, \dot{q}_{1,k} \rangle\ \forall \dot{r}_2\ \mu\left(\bigcup_{0 \leq i \leq k} \bigcap_{\dot{q}_2 <_L \dot{r}_2} \mathcal{A}_{\dot{q}_1, \dot{q}_2}\right) > \varepsilon$$

We deduce that:

$$\mu\left(\bigcup_{\dot{q}_1 < P_\beta} \bigcap_{\dot{q}_2 < P_\beta} \mathcal{A}_{\dot{q}_1, \dot{q}_2}\right) \geq \varepsilon$$

As this is true for every ε with $0 < \varepsilon < m$, we must have the inequality $\mu(\bigcup_{\dot{q}_1 < P_\beta} \bigcap_{\dot{q}_2 < P_\beta} \mathcal{A}_{\dot{q}_1, \dot{q}_2}) \geq m$.

Together with the fact that $\bigcup_{\dot{q}_1 < P_\beta} \bigcap_{\dot{q}_2 < P_\beta} \mathcal{A}_{\dot{q}_1, \dot{q}_2} \subseteq^* \bigcup_{\dot{q}_1 < P_\alpha} \bigcap_{\dot{q}_2 < P_\alpha} \mathcal{A}_{\dot{q}_1, \dot{q}_2}$, we have the proposition. ⊣

THEOREM 4.16. *Let $\beta < \alpha$ with α limit, be such that $L_\beta \prec_2 L_\alpha$. Let z be weakly α-ML random. Then $L_\beta(x) \prec_2 L_\alpha(x)$.*

PROOF. Let $p \in P_\beta$. Let $\Phi(x_1, x_2, x_3)$ be a Δ_0 formula.

Suppose $L_\beta(z) \models \exists a\ \forall b\ \Phi(a, b, p[z])$. In particular there exists $\dot{a} \in P_\beta$ such that $L_\beta(z) \models \forall b\ \Phi(\dot{a}[z], b, p[z])$. From Lemma 4.14, $L_\alpha(z) \models \forall b\ \Phi(\dot{a}[z], b, p[z])$. Thus we have $L_\alpha(z) \models \exists a\ \forall b\ \Phi(a, b, p[z])$.

Suppose $L_\alpha(z) \models \exists a\ \forall b\ \Phi(a, b, p)$. Then, let $\mathcal{B}^\alpha_\Phi = \{x \in 2^\omega\ :\ L_\alpha(x) \models \exists a\ \forall b\ \Phi(a, b, p[x])\}$. We have $\mathcal{B}^\alpha_\Phi$ is the Σ^α_2 set given by $\bigcup_{\dot{q}_1 < P_\alpha} \bigcap_{\dot{q}_2 < P_\alpha} \mathcal{A}_{\dot{q}_1, \dot{q}_2}$, where

$$\mathcal{A}_{\dot{q}_1, \dot{q}_2} = \{x \in 2^\omega\ :\ L_\alpha(x) \models \Phi(\dot{q}_1[x], \dot{q}_2[x], p[x])\}$$

From Lemma 4.15 we have that $\mathcal{B}^\alpha_\Phi =^* \mathcal{B}^\beta_\Phi$. Let $\dot{r}$ be such that $z \in \bigcap_{\dot{q}_2 \in P_\alpha} \mathcal{A}_{\dot{r}, \dot{q}_2}$. Then we have

$$\lambda\left(\bigcap_{\dot{q}_2 \in P_\alpha} \left(\mathcal{A}_{\dot{r}, \dot{q}_2} - \mathcal{B}^\beta_\Phi\right)\right) = 0$$

It follows that $\bigcap_{\dot{q}_2 \in P_\alpha} (\mathcal{A}_{\dot{r}, \dot{q}_2} - \mathcal{B}^\beta_\Phi)$ is a weak α-ML test. As z is not weakly α-ML random it does not belong to the test and then it must belong to $\mathcal{B}^\beta_\Phi$. Thus $z \in \bigcup_{\dot{q}_1 \in P_\beta} \bigcap_{\dot{q}_2 \in P_\beta} \mathcal{A}_{\dot{q}_1, \dot{q}_2}$. It follows that $L_\beta(z) \models \exists a\ \forall b\ \Phi(a, b, p[z])$. ⊣

COROLLARY 4.17. *Let $\beta < \alpha$ such that $L_\alpha \models$ "everything is countable" and $L_\beta \prec_2 L_\alpha$. Suppose α is admissible. Let z be random over L_α. Then $L_\beta(z) \prec_2 L_\alpha(z)$.*

PROOF. If α is admissible we have that weak α-ML-randomness coincides with randomness over L_α by Proposition 4.12. Thus if z is random over L_α we must have $L_\beta(z) \prec_2 L_\alpha(z)$. ⊣

COROLLARY 4.18. *Let z be weakly Σ-ML random. Then z is ITTM-random.*

PROOF. We have $L_\zeta \prec_2 L_\Sigma$. We also have that z is ITTM-random iff z is random over L_Σ and $\Sigma^z = \Sigma$. If z is weakly Σ-ML random we have $L_\zeta(z) \prec_2 L_\Sigma(z)$. In particular (ζ^z, Σ^z) is the lexicographically smallest pair of ordinal such that $L_{\zeta^z}(z) \prec_2 L_{\Sigma^z}(z)$, which implies $\Sigma^z = \Sigma$ and $\zeta^z = \zeta$. Also if z is weakly Σ-ML random, then it is random over L_Σ. It follows that z is ITTM-random. ⊣

We now give a more combinatorial equivalent characterization the notion of ITTM-randomness: a characterization in terms of being captured by sets of measure 0 having a specific complexity. For the following proposition, by Δ_3^Σ set, we mean a set which is also Δ_3^ζ, that is, a set $\mathcal{B}_1^\Sigma = \mathcal{B}_2^\Sigma$ with $\mathcal{B}_1^\Sigma$ which is Σ_3^Σ and $\mathcal{B}_2^\Sigma$ which is Π_3^Σ, such that for the versions $\mathcal{B}_1^\zeta$ and $\mathcal{B}_2^\zeta$ we also have $\mathcal{B}_1^\zeta = \mathcal{B}_2^\zeta$.

Theorem 4.19. *The following are equivalent:*

1. *z is ITTM-random.*
2. *z belongs to no Δ_3^Σ set of measure 0, with parameters in L_ζ.*

Proof. Let us show (2) implies (1). Suppose that z is not ITTM-random. If it is not random over L_Σ then clearly (2) is false with the Σ_1^Σ set of measure 0 which is the union of all the Borel sets of L_Σ of measure 0. Otherwise z is random over L_Σ and there is a parameter $p \in L_\zeta(z)$ with a Δ_0 formula $\Phi(x_1, x_2, x_3)$ such that $L_\Sigma(z) \models \exists a\ \forall b\ \Phi(a, b, p)$ but $L_\zeta(z) \models \forall a\ \exists b\ \neg\Phi(a, b, p)$. Let $\Psi(p) \equiv \exists a\ \forall b\ \Phi(a, b, p)$.

We have $z \in \mathcal{B}_\Psi^\Sigma \cap \mathcal{B}_{\neg\Psi}^\zeta$. Also from Lemma 4.15 we have $\mathcal{B}_\Psi^\Sigma =^* \mathcal{B}_\Psi^\zeta$ and thus $\mu(\mathcal{B}_\Psi^\Sigma \cap \mathcal{B}_{\neg\Psi}^\zeta) = 0$. We now have to transform the Π_2^ζ set $\mathcal{B}_{\neg\Psi}^\zeta$ into a Π_2^Σ set $\mathcal{B}_\phi^\Sigma$ such that $\mathcal{B}_\phi^\Sigma =^* \mathcal{B}_{\neg\Psi}^\zeta$ and $\mathcal{B}_\phi^\Sigma$ still contains z. Let $\beta < \zeta$ be such that $p \in L_\beta(z)$.

We define $\mathcal{B}_\phi^\Sigma$ to be $\bigcap_{\beta \leq \alpha \prec_1 \Sigma} \mathcal{B}_{\neg\Psi}^\alpha$. Formally, the corresponding formula ϕ is given by $\phi(\beta, p) \equiv \forall \alpha \geq \beta\, [L_\alpha \text{ is not } \Sigma_1 \text{ stable or } L_\alpha(x) \models \neg\Psi(p)]$.

Using Proposition 2.18 it is clear that $\mathcal{B}_\phi^\Sigma$ is Π_2^Σ. We shall now show that as long as z is random over L_ζ we have $z \in \mathcal{B}_\phi^\Sigma$ iff $z \in \mathcal{B}_{\neg\Psi}^\zeta$. As $L_\zeta \prec_1 L_\Sigma$ we have $\mathcal{B}_\phi^\Sigma \subseteq \mathcal{B}_{\neg\Psi}^\zeta$. Let us show that if z is random over L_ζ and $z \in \mathcal{B}_{\neg\Psi}^\zeta$, then $z \in \mathcal{B}_\phi^\Sigma$.

To do so let us first show that for every α with $\zeta < \alpha < \Sigma$ we have that $\neg L_\alpha \prec_1 L_\Sigma$. Fix such an ordinal α. By Theorem 2.29, every accidentally writable ordinal becomes writable with parameter ζ. In particular $\{\alpha\}$ is Σ_1-definable in L_Σ with some Σ_1 formula $\Phi(\zeta, \alpha)$ (intuitively the program that writes α and halts). It follows that $L_\Sigma \models \exists \alpha\ \Phi(\zeta, \alpha)$ but $\neg L_\alpha \models \exists \alpha\ \Phi(\zeta, \alpha)$. Thus we do not have $L_\alpha \prec_1 L_\Sigma$.

Suppose now z is random over L_ζ and $z \in \mathcal{B}_{\neg\Psi}^\zeta$. Let $\alpha \geq \beta$ be such that $L_\alpha \prec_1 L_\Sigma$. Then we must have $\alpha \leq \zeta$. Also if $L_\Sigma \models \Phi(p)$ for some Σ_1 formula Φ with parameter $p \in L_\alpha$, we must have $L_\alpha \models \Phi(p)$ and then $L_\zeta \models \Phi(p)$. Therefore $L_\alpha \prec_1 L_\zeta$. Now as z is random over L_ζ and $z \in \mathcal{B}_{\neg\Psi}^\zeta$, we must have by Lemma 4.14 and Proposition 2.17 that $z \in \mathcal{B}_{\neg\Psi}^\alpha$. It follows that $z \in \bigcap_{\beta \leq \alpha \prec_1 \Sigma} \mathcal{B}_{\neg\Psi}^\alpha$.

We then have that $z \in \mathcal{B}_\Psi^\Sigma \cap \mathcal{B}_\phi^\Sigma$, with $\mu(\mathcal{B}_\Psi^\Sigma \cap \mathcal{B}_\phi^\Sigma) = 0$, and with $\mathcal{B}_\Psi^\Sigma \cap \mathcal{B}_\phi^\Sigma$ a Δ_3^Σ set with parameters in L_ζ. Note that $\mathcal{B}_\Psi^\zeta \cap \mathcal{B}_\phi^\zeta$ is also a Δ_3^ζ set.

Let us show (1) implies (2). Suppose now that there is a Π_3^Σ set $\mathcal{B}_\Phi^\Sigma$ and a Σ_3^Σ set $\mathcal{B}_\Psi^\Sigma$, with parameters in L_ζ, such that $z \in \mathcal{B}_\Phi^\Sigma = \mathcal{B}_\Psi^\Sigma$ and $\mu(\mathcal{B}_\Phi^\Sigma) = \mu(\mathcal{B}_\Psi^\Sigma) = 0$, with also $\mathcal{B}_\Phi^\zeta = \mathcal{B}_\Psi^\zeta$.

If $z \notin \mathcal{B}_\Phi^\zeta$ then $L_\Sigma(z) \models \Phi$ and $\neg L_\zeta(z) \models \Phi$ for the Π_3-formula Φ. By Proposition 2.17 we then have $\neg L_\zeta(x) \prec_2 L_\Sigma(x)$ and thus z is not ITTM-random.

Otherwise $z \in \mathcal{B}_\Phi^\zeta$ and thus $z \in \mathcal{B}_\Psi^\zeta$. We also have that $\mu(\mathcal{B}_\Psi^\Sigma) = 0$. Also $\mathcal{B}_\Psi^\Sigma = \bigcup_{\dot{q}_1 \in P_\Sigma} \bigcap_{\dot{q}_2 \in P_\Sigma} \bigcup_{\dot{q}_3 \in P_\Sigma} \mathcal{A}_{\dot{q}_1, \dot{q}_2, \dot{q}_3}$. For any name $\dot{q}_1 \in P_\zeta$ we have that

$\mu(\bigcap_{\dot{q}_2 \in P_\Sigma} \bigcup_{\dot{q}_3 \in P_\Sigma} \mathcal{A}_{\dot{q}_1,\dot{q}_2,\dot{q}_3}) = 0$ and from Lemma 4.15 we then must have that $\mu(\bigcap_{\dot{q}_2 \in P_\zeta} \bigcup_{\dot{q}_3 \in P_\zeta} \mathcal{A}_{\dot{q}_1,\dot{q}_2,\dot{q}_3}) = 0$. In particular there is $\dot{q}_1 \in P_\zeta$ such that $z \in \bigcap_{\dot{q}_2 \in P_\zeta} \bigcup_{\dot{q}_3 \in P_\zeta} \mathcal{A}_{\dot{q}_1,\dot{q}_2,\dot{q}_3}$. It follows that z is not random over L_Σ and thus not ITTM-random. ⊣

So ITTM-randomness is equivalent to Δ_3^Σ-randomness for sets with parameters which are at most eventually writable, but not accidentally writable. We shall now see that it is actually very close to randomness over L_Σ, which can be shown to be equivalent to a similar test notion:

THEOREM 4.20. *The following are equivalent:*

1. *z is random over L_Σ.*
2. *z is in no Σ_2^Σ set of measure 0, with parameters in L_ζ.*
3. *z is in no Π_2^Σ set of measure 0, with parameters in L_ζ.*

PROOF. It is clear that both (2) and (3) imply (1), using the Σ_1^Σ set of measure 0 which is the union of all the Borel sets of L_Σ of measure 0.

Let us show (1) implies (2). Let $\mathcal{B}_\Phi^\Sigma$ be a Σ_2^Σ set equal to $\bigcup_{\dot{q}_1 \in P_\alpha} \bigcap_{\dot{q}_2 \in P_\alpha} \mathcal{A}_{\dot{q}_1,\dot{q}_2}$ with $\mu(\mathcal{B}_\Phi^\Sigma) = 0$. The following argument is a combination of the Σ_2-stability of L_ζ in L_Σ, together with the admissibility of L_ζ.

By Lemma 4.15 we have $\mu(\mathcal{B}_\Phi^\zeta) = 0$. Then, $\forall \dot{q}_1 \in P_\zeta$ $\mu(\bigcap_{\dot{q}_2 \in P_\zeta} \mathcal{A}_{\dot{q}_1,\dot{q}_2}) = 0$. Fix $\dot{q}_1 \in P_\zeta$. By admissibility of ζ, there must exists $\dot{r} \in P_\zeta$ such that $\mu(\bigcap_{\dot{q}_2 <_L \dot{r}} \mathcal{A}_{\dot{q}_1,\dot{q}_2}) = 0$. It follows that $L_\zeta \models \forall \dot{q}_1\ \exists \dot{r}\ \mu(\bigcap_{\dot{q}_2 <_L \dot{r}} \mathcal{A}_{\dot{q}_1,\dot{q}_2}) = 0$. As $L_\zeta \prec_2 L_\Sigma$ we also have $L_\Sigma \models \forall \dot{q}_1\ \exists \dot{r}\ \mu(\bigcap_{\dot{q}_2 <_L \dot{r}} \mathcal{A}_{\dot{q}_1,\dot{q}_2}) = 0$. In particular every real in $\mathcal{B}_\Phi^\Sigma$ is in a set of measure 0 with a Borel code in L_Σ.

Let us show (1) implies (3). Let $\mathcal{B}_\Phi^\Sigma$ be the Π_2^Σ set of measure 0. By Lemma 4.15 we must have $\mathcal{B}_\Phi^\zeta =^* \mathcal{B}_\Phi^\Sigma$. Let $z \in \mathcal{B}_\Phi^\Sigma$. Suppose $z \notin \mathcal{B}_\Phi^\zeta$. Then we have $L_\Sigma(z) \models \Phi$ and $\neg L_\zeta(z) \models \Phi$ for a Π_2 formula Φ with parameters in L_ζ. By Lemma 4.14 together with Proposition 2.17 we then have that z is not random over L_Σ. Suppose now $z \in \mathcal{B}_\Phi^\zeta$. Then z is in a set of measure 0 with a Borel code in L_Σ which implies that z is not random over L_Σ. ⊣

4.3. Martin-Löf randomness in the constructibles. It was shown in [6] that randomness over L_λ is the counterpart of Δ_1^1-randomness for ITTMs, and λ-ML-randomness the counterpart of Π_1^1-ML-randomness. Carl and Schlicht asked if as in the hyperarithmetic case these two notions really differ. We give a general answer to this question by characterizing the ordinals α for which the two notions are different.

4.3.1. *Separation of randomness over L_α and α-ML-randomness.* We first give the easy relation between randomness over L_α and α-ML-randomness:

PROPOSITION 4.21. *Let α be limit. Then α-ML-randomness is stronger than randomness over L_α*

PROOF. Let $\mathcal{B}$ be a Borel set with code in L_α. By Lemma 4.1, we define an α-ML-test $\bigcap_n \mathcal{U}_n$ such that for all n, we have $\mathcal{B} \subseteq \mathcal{U}_n$, and $\mu(\mathcal{U}_n) \leq \mu(\mathcal{B}) + 2^{-n} = 2^{-n}$. Then $\mathcal{B} \subseteq \bigcap_n \mathcal{U}_n$, this proves the property. ⊣

The following theorem characterizes exactly when randomness over L_α and α-ML-randomness coincide, for α admissible or α limit and $L_\alpha \models$"everything is countable".

THEOREM 4.22. *Let α be admissible or α limit such that $L_\alpha \models$"everything is countable". The following are equivalent:*

1. *α is projectible into ω.*
2. *There is a universal α-ML-test.*
3. *α-ML-randomness is strictly stronger than randomness over L_α.*

PROOF. Note first that if α is limit, non-admissible and $L_\alpha \models$"everything is countable", then by Theorem 2.34 α is projectible into ω. Therefore for (3) implies (1) and (2) implies (1), we can suppose α admissible.

The proof that (3) implies (1) is done by contraposition and Theorem 2.21: if α is not projectible into ω, then L_α satisfies Σ_1-comprehension for subsets of ω and then every α-ML-test is in L_α, which implies that randomness over L_α is stronger than α-ML-randomness. Together with Proposition 4.21 we have that the two notions of randomness coincide.

To prove (2) implies (1), suppose we have (2) and α is not projectible into ω, in order to get a contradiction. Then by Theorem 2.21, the universal α-ML-test $\bigcap_n \mathcal{U}_n$ would be in some L_β with $\beta < \alpha$. We have that $2^\omega - \mathcal{U}_0$ is a closed set whose leftmost path is definable in L_β and then belongs to $L_{\beta+1}$. As this leftmost path is definable in L_α, it is not random over L_α, which contradicts the universality of the test.

Let us now prove (1) implies (2). Assuming that α is projectible into ω, it is then possible to α-recursively assign an integer to all the parameters in L_α, we will use this to assign an integer to every α-ML-test. We have an enumeration $\{\Phi_m(x,k,p,\sigma)\}_{m\in\omega}$ of every Δ_0 formula with four free variables and without parameters. We see any such formula as defining a uniform intersection of α-recursively enumerable open sets when given a parameter p: for some m the formula Φ_m together with a parameter p defines an intersection of open sets $\bigcap_k \mathcal{U}_k$, each $\mathcal{U}_k$ being the union of all the cylinders $[\sigma]$ such that $L_\alpha \models \exists x\ \Phi_m(x,k,p,\sigma)$.

Let π be a Σ_1-definable injection of L_α into ω. Note that if α is admissible we use the projection together with the bijection between α and L_α. Otherwise we use the bijection given by Theorem 2.34. Let p be a parameter and n an integer such that $\pi(p) = n$. If $\bigcap_k \mathcal{U}_k$ is defined in L_α by the Σ_1 formula $\Phi_m(x,k,p,\sigma)$ with parameter p, then $\bigcap_k \mathcal{U}_k$ is also defined by the following parameter-free Σ_1 formula $\Psi_{n,m}(k,\sigma) \equiv \exists p\ \exists x\ \pi(p) = n \wedge \Phi_m(x,k,p,\sigma)$. Consequently, every uniform intersection of α-recursively enumerable open set $\bigcap_k \mathcal{U}_k$ is defined by a formula in the enumeration $\{\Psi_{m,n}(k,\sigma)\}_{\langle m,n\rangle\in\omega}$.

Now for integers m, n, the formula $\Psi_{m,n}(k,\sigma)$ might not define an α-ML-test, due to the measure requirement. For any n, m let $\widetilde{\psi}_{m,n}(z,k,\sigma)$ be a Δ_0 formula such that $L_\alpha \models \exists z\ \widetilde{\psi}_{m,n}(z,k,\sigma)$ iff $L_\alpha \models \Psi_{n,m}(k,\sigma)$. We define the computable function g which to n, m associates the code $g(n,m)$ of the Δ_0 formula $\phi(z,k,\sigma)$

$$\phi(z,k,\sigma) \equiv \widetilde{\psi}_{m,n}(z,k,\sigma) \wedge \lambda\left(\bigcup\{[\tau] : \exists z' \leq_L z\ \widetilde{\psi}_{n,m}(z',k,\tau)\}\right) \leq 2^{-k}$$

The formula $\exists z \ \phi(z, k, \sigma)$ always defines a Martin-Löf test. Furthermore, if $\Psi_{n,m}(k, \sigma)$ defines an α-ML-test, then the formula $\exists z \ \phi(z, k, \sigma)$ defines the same test. It follows that $\{g(n, m)\}_{n,m\in\omega}$ is an enumeration of codes for α-ML-tests that contains all the α-ML-tests. This can then be used to define a universal α-ML test as in the lower settings: given an enumeration $\{\bigcap_k \mathcal{U}_k^n\}_{n\in\omega}$ of all the Martin-Löf tests, we define $\mathcal{V}_m = \bigcup_i \mathcal{U}^i_{i+m+1}$. We clearly have $\bigcup_n \bigcap_k \mathcal{U}_k^n \subseteq \bigcap_m \mathcal{V}_m$, and as $\mu(\mathcal{U}^i_{i+m+1}) \leq 2^{-m-i-1}$ we have $\mu(\mathcal{V}_m) \leq 2^{-m}$ which implies that $\bigcap_m \mathcal{V}_m$ is a Martin-Löf test. Thus (1) implies (2).

To prove (1) implies (3), we will build an α-ML test $\mathcal{U}$ capturing a real x which is random over L_α. Let π be a Σ_1-definable injection of L_α into ω. We proceed by stages where the stages are ordinals $s < \alpha$. The stages will approximate a set x random over L_α in a Δ^0_2 way, together with an α-ML test that capture x. To do so, for every integer n and every stage s, we will define a closed set $\mathcal{F}^n_s$ and a string σ^n_s with $|\sigma^n_s| = 2n$ and $\sigma^n_s \prec \sigma^{n+1}_s$, such that:

$$(R^s_n) \qquad \lambda\left(\bigcap_{i\leq n} \mathcal{F}^i_s \cap [\sigma^n_s]\right) > 0$$

(S^s_n) If $\pi(a) = n$ for $a \in L_s$ such that a is the code of a Borel set $\mathcal{B}_a$ of measure 1, then $\mathcal{F}^n_s \subseteq \mathcal{B}_a$

Also the definition of σ^n_s and $\mathcal{F}^n_s$ will be independent from the definition of σ^n_t and $\mathcal{F}^n_t$ for $t <_L s$. At stage s, we define $\mathcal{F}^0_s$ to be 2^ω and σ^0_s be the empty string. It is clear that R^s_0 and S^s_0 are satisfied. Suppose $\mathcal{F}^i_s$ and σ^i_s have been defined for every $i \leq n$ such that R^s_i and S^s_i are satisfied. Let us define $\mathcal{F}^{n+1}_s$ and σ^{n+1}_s. If $\pi(a) = n + 1$ for some $a \in L_s$ such that a is the Borel code of a set $\mathcal{B}_a$ of measure 1, then let $\bigcup_m \mathcal{S}_m \subseteq \mathcal{B}_a$ be a conull union of closed sets with a Borel code in L_α. Note that by Lemma 4.1 we can obtain such a union uniformly. Let then k be the smallest such that:

$$\lambda\left(\bigcup_{i\leq k} \mathcal{S}_i \cap \bigcap_{i\leq n} \mathcal{F}^i_s \cap [\sigma^n_s]\right) > 0$$

Let $F^{n+1}_s = \bigcup_{i\leq k} \mathcal{S}_i$. Let σ^s_{n+1} be the first extension of σ^s_n by two bits such that R^s_{n+1} is satisfied.

Let x_s be the sequence $\sigma^s_1 \prec \sigma^s_2 \prec \sigma^s_3 \prec \dots$. Note that for each n, the sequences $\{\sigma^n_s\}_{s<\alpha}$ and $\{\mathcal{F}^n_s\}_{s<\alpha}$ change at most once per integer $i \leq n$ such that $\pi(a) = i$ for some Borel set $\mathcal{B}_a$ of measure 1 with $a \in L_s$. Thus these sequences change at most n times. In particular the whole process converges and the sequence x_s converges to some sequence x.

This can also be used to define the α-ML-test that contains x. We define $\mathcal{U}_n = \bigcup_{s<\alpha}[\sigma^n_s]$. This is an α-ML-test as there are at most n distinct versions of σ^n_s and for each of them we have $|\sigma^n_s| = 2n$. The measure of $\mathcal{U}_n$ is then bounded by $n \times 2^{-2n} \leq 2^{-n}$. This shows that x is not α-ML-random. Also by S^s_n we have that x is in every Borel set $\mathcal{B}_s$, so it is random over L_α. We then have (1) implies (3). ⊣

COROLLARY 4.23. *We have:*

- *λ-ML-randomness is strictly stronger than randomness over L_λ.*
- *ζ-ML-randomness is equal to randomness over L_ζ.*
- *Σ-ML-randomness is strictly stronger than randomness over L_Σ.*

PROOF. By Corollary 2.31, we have that λ is projectible over ω with no parameter, and ζ is not projectible into ω. By Corollary 2.35, we have that Σ is projectible into ω with parameter ζ. ⊣

We shall now improve Corollary 4.23 for Σ-ML-randomness, by showing that it is strictly stronger than weak Σ-ML randomness and thus than ITTM-randomness.

THEOREM 4.24. *Σ-ML-randomness is strictly stronger than weak Σ-ML randomness and than ITTM-randomness.*

PROOF. We shall construct a real z such that for any Σ_1^Σ set $\bigcup_{p\in L_\Sigma}\mathcal{B}_p$ of measure 1, we have $z\in\bigcup_{p\in L_\Sigma}\mathcal{B}_p$, together with a Σ-ML test $\bigcap_{n\in\omega}\mathcal{U}_n$ containing z, and with $\mu(\mathcal{U}_n)\leq 2^{-n}$. The proof is very similar to (1) implies (3) in Theorem 4.22.

Let b be Σ_1-definable bijection of Corollary 2.35 from ω to L_Σ. Using this bijection, let $\{\bigcup_{p\in L_\Sigma}\mathcal{B}_{n,p}\}_{n\in\omega}$ be an enumeration of all the union of Borel sets of L_Σ.

We will define a computation, stage by stage, of a set z, that will be approximated in a Δ_2^0 way, together with a Σ-ML test that will capture z. To do so, for every integer n and every stage s, we will define a closed set $\mathcal{F}_s^n$ and a string σ_s^n with $|\sigma_s^n|=2n$ and $\sigma_s^n\prec\sigma_s^{n+1}$, such that for every n,s we have

$$\lambda\left(\bigcap_{i\leq n}\mathcal{F}_s^i\cap[\sigma_s^n]\right)>0$$

and for every n, if $\mu(\bigcup_{p\in L_\Sigma}\mathcal{B}_{n,p})=1$ there exists t such that for all $s\geq t$ we have $\mathcal{F}_s^n\subseteq\bigcup_{p\in L_\Sigma}\mathcal{B}_{n,p}$. Note also that the definition of $\mathcal{F}_s^n$ and σ_s^n will not depend on $\mathcal{F}_t^m$ or σ_t^m for $m\in\omega$ and $t<s$.

At stage s, we define $\mathcal{F}_s^0$ to be 2^ω and σ_s^0 to be the empty string. Suppose $\mathcal{F}_s^i$ and σ_s^i have been defined for every $i\leq n$. Let us define $\mathcal{F}_s^{n+1}$ and σ_s^{n+1}:

Suppose $\mu(\bigcup_{p\in L_s}\mathcal{B}_{n+1,p}\cap\bigcap_{i\leq n}\mathcal{F}_s^i\cap(\sigma_s^n))>0$. Then let us find some closed set $\mathcal{F}_s^{n+1}\subseteq\bigcup_{p\in L_s}\mathcal{B}_{n+1,p}$ such that $\mu(\bigcap_{i\leq n+1}\mathcal{F}_s^i\cap[\sigma_s^n])>0$. Let then σ_{n+1}^s be the first extension of σ_n^s by two bits such that $\mu(\bigcap_{i\leq n+1}\mathcal{F}_s^i\cap[\sigma_s^{n+1}])>0$.

Let z_s be the sequence $\sigma_1^s\prec\sigma_2^s\prec\sigma_3^s\prec\ldots$. Note that for each n, the sequences $\{\sigma_s^n\}_{s<\Sigma}$ and $\{\mathcal{F}_s^n\}_{s<\Sigma}$ change at most once per integer i smaller than n. Thus these sequences change at most n times. In particular the whole process converges and the sequence z_s converges to some sequence z.

This can then be used to define the α-ML-test that contains z. We define $\mathcal{U}_n=\bigcup_{s<\Sigma}[\sigma_s^n]$. This is a Σ-ML-test as there are at most n distinct versions of σ_s^n and for each of them we have $|\sigma_s^n|=2n$. The measure of $\mathcal{U}_n$ is then bounded by $n\times 2^{-2n}\leq 2^{-n}$. This shows that z is not Σ-ML-random. It is also clear that z is in every set $\bigcup_{p\in L_\Sigma}\mathcal{B}_p$ such that $\mu(\bigcup_{p\in L_\Sigma}\mathcal{B}_p)=1$. ⊣

4.3.2. *Summary.* The following picture summarizes the relations between all the randomness notions we have seen:

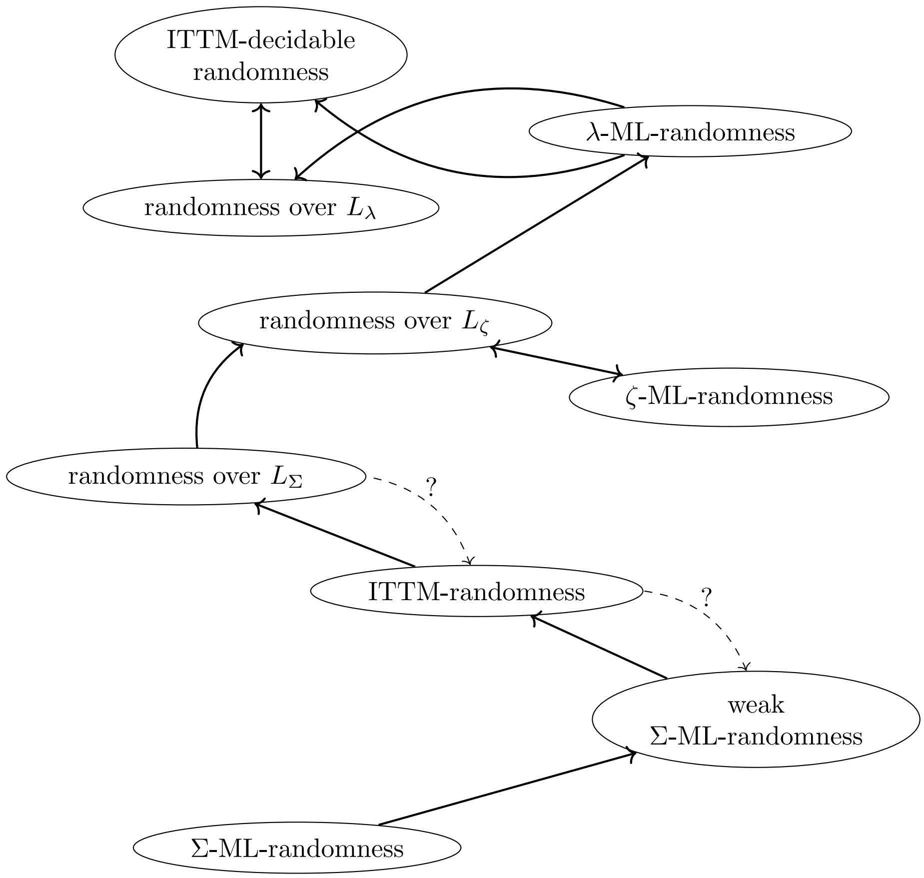

FIGURE 1. Higher randomness

We recall here the two remaining open questions:

QUESTION 4.25. *Is ITTM-randomness strictly stronger than randomness over* L_Σ *?*

QUESTION 4.26. *Is weak* Σ*-ML randomness strictly stronger than ITTM-randomness ?*

Note that by Proposition 4.13 a negative answer to one of the two questions would provide a positive answer to the other one.

4.3.3. *Mutual* λ*-ML randoms computing common reals.* When two sets are mutually random, we expect them to compute no common non-computable sets. However, depending on the randomness level we ask for, this is sometimes not the case. Carl and Schlicht asked in Question 5.5 from [6] if two mutually λ-ML-randoms could compute a common non-writable set. It is the case with Martin-Löf randomness, and sets which can be computed by two mutually Martin-Löf random must be K-trivials. We show that the same happens with ITTMs: some non-writable sets can be ITTM-computed by two mutually λ-ML-randoms. We do not study here however the notion of K-triviality for ITTMs, even though we conjecture that most of the work done about K-trivials and about higher K-trivials (K-trivials defined over $L_{\omega_1^{ck}}$) lifts to the world of computability inside L_λ, using the fact that λ is projectible into ω.

First, we need to expand, in a straightforward way, some definitions from ML-randomness to the ITTM settings. In the following, we focus on ITTMs but the proofs also work for α such that there exists a universal α-ML-test, in other word by Theorem 4.22 when α is projectible in ω and such that either α is admissible or both α is limit and $L_\alpha \models$"everything is countable".

Definition 4.27. An ITTM-Solovay test is a sequence of uniformly ITTM-semi-decidable open sets $(\mathcal{S}_s)_{s<\lambda}$ such that $\Sigma_{s<\lambda}\mu(\mathcal{S}_s) < \infty$. We say that $Z \in 2^\omega$ passes the test if Z belongs to only finitely many $\mathcal{S}_s$.

Proposition 4.28. *Let $z \in 2^\omega$. The following are equivalent:*

1. *z passes every ITTM-Solovay tests.*
2. *z is λ-ML-random.*

The proof of this characterization of λ-ML test via ITTM-Solovay tests is exactly the same as the one from the lower case, that can be found in [11]. Our witness for answering the question will be the even and odd parts of a specific λ-ML-random, an approximable one.

Definition 4.29 (Chaitin's Ω for ITTMs). Let $\bigcap_n \mathcal{U}_n$ be a universal λ-ML-test. We define Ω as being the leftmost path of $2^\omega - \mathcal{U}_0$. In particular Ω is λ-ML-random and has a left-c.e. approximation in L_λ.

In [6] Carl and Schlicht discuss the van Lambalgen theorem for λ-ML randomness. It holds using the fact that λ is projectible into ω. The proof is the same as the one for ω_1^{ck}-ML randomness (called Π^1_1-ML randomness in the literature) and works for any α limit such that α is projectible into ω. In particular for $\Omega = \Omega_1 \oplus \Omega_2$ we have that Ω_1 and Ω_2 are mutually λ-ML random.

Theorem 4.30. *There exists a non ITTM-writable set A which is ITTM-writable from both Ω_0 and Ω_1, the two halves of Chaitin's Ω for ITTMs.*

Proof. Let us first show the following version of the Hirschfeldt and Miller theorem for ITTMs (see for example [22, Theorem 5.3.15]): let $\bigcap_n \mathcal{U}_n$ be a uniform intersection of λ-recursively enumerable open sets, with $\mu(\bigcap_n \mathcal{U}_n) = 0$. Then there exists a non-writable set A such that A is x-writable in every λ-ML random $x \in \bigcap_n \mathcal{U}_n$. The set A will be a λ-recursively enumerable simple set, that is, it will be co-infinite and intersect any infinite λ-recursively enumerable set of integers. Let $\bigcap_n \mathcal{U}_n$ be a uniform intersection of λ-recursively enumerable open sets of measure 0. Note that we can suppose without loss of generality that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$. Let $\{W_e\}_{e\in\omega}$ be an enumeration of the λ-recursively enumerable sets.

The enumeration of A is defined by stages. At ordinal stage $s = \omega \times \alpha + \langle n, e\rangle$, if we have:

1. $n > 2e$,
2. $A[< s] \cap W_e[s] = \emptyset$,
3. $n \in W_e[\alpha]$
4. $\mu(\mathcal{U}_n[\alpha]) \leq 2^{-e}$

Then we add n to A at stage s.

First, let's show that A is simple. It is obviously co-infinite, as $|A \cap [0, 2n]| \leq n$ by requirements (1) and (2). Let e be such that W_e is infinite, and towards a contradiction, suppose that $W_e \cap A = \emptyset$. Then, let $m > 2e$ such that $\mu(\mathcal{U}_m) < 2^{-e}$, together with $n \geq m$ and α such that $n \in W_e[\alpha]$. Note that we have $\mu(\mathcal{U}_n) \leq \mu(\mathcal{U}_m) < 2^{-e}$. At stage $s = \omega \times \alpha + \langle n, e\rangle$ if $A[s] \cap W_e[s] = \emptyset$ then (1) (2) (3) and (4) will be met and $n \in W_e$ will be added to A at stage s.

Now, let's show that A is x-writable from every λ-ML-random element of $\bigcap_n \mathcal{U}_n$. We build the following ITTM-Solovay test: each time we add n into A at stage $s = \omega \times \alpha + \langle n, e\rangle$, we put $\mathcal{U}_n[\alpha]$ in the Solovay test. Note that by (4) we have $\mu(\mathcal{U}_n[\alpha]) < 2^{-e}$, in particular the measure requirement of the Solovay test is satisfied. Now if $x \in \bigcap_n \mathcal{U}_n$ is λ-ML-random it belongs to only finitely many such sets $\mathcal{U}_n[\alpha]$. In particular, there exists k such that for every $m \geq k$, if $m \in A$, then $m \in A[s]$ for $s = \omega \times (\alpha + 1)$ where α is the smallest such that $x \in \mathcal{U}_m[\alpha]$. We can then use x to write A.

Finally, it remains only to prove that Ω_0 and Ω_1 are both in a common uniform intersection $\bigcap_n \mathcal{U}_n$ of λ-open sets, with $\mu(\bigcap_n \mathcal{U}_n) = 0$. Each set $\mathcal{U}_n$ is given by

$$\mathcal{U}_n = \bigcup_{\alpha<\lambda} [\Omega_0[\alpha] \upharpoonright n] \cup \bigcup_{\alpha<\lambda} [\Omega_1[\alpha] \upharpoonright n]$$

It is clear that each set $\mathcal{U}_n$ is a λ-recursively enumerable open set which contains both Ω_0 and Ω_1. Let $S_0 = \{\Omega_0[\alpha] : \alpha < \lambda\} \cup \{\Omega_0\}$ and $S_1 = \{\Omega_1[\alpha] : \alpha < \lambda\} \cup \{\Omega_1\}$. To show that $\bigcap_n \mathcal{U}_n$ has measure 0, we use the following argument from [3, Proposition 5.1]: if $x \in \bigcap_n \mathcal{U}_n$, then x is at a distance of 0 from the set $S_0 \cup S_1$. Also it is clear that both S_0 and S_1 are closed sets, and thus that $S_0 \cup S_1$ is a closed set (in particular because for every i the sequences $\{\Omega_0(i)[\alpha]\}_{\alpha<\lambda}$ and $\{\Omega_1(i)[\alpha]\}_{\alpha<\lambda}$ change only finitely often). As x is at a distance 0 from a closed set, it is a member of the closed set. As the closed set is countable it has measure 0. It follows that $\mu(\bigcap_n \mathcal{U}_n) = 0$. ⊣

§5. Genericity. Just like we define as random the sequences which are in every measure 1 set, among countably many sets, we define as *generic* the sequences which are in every co-meager set, among countably many sets. Both notions are obtained by considering a notion of largeness (measure 1 sets for randomness and co-meager sets for genericity), together with a countable class of large sets. For this reason both notions present many similar properties, and of course also many differences, as they are somehow opposite notions: whereas the random sets have no atypical property, the generic sets have them all.

The notion of genericity was designed by Cohen, as a canonical forcing notion. He considered as generic, the sets that belongs to no meager set, with a Borel code, in a countable model of ZFC. Various weakenings of this notion have then been considered in the literature. This has been done in computability by Jockush and Kurtz [18] [19], in higher computability by Greenberg and Monin [13], and for ITTMs by Carl and Schlicht [5]. In the later paper, the authors mostly focus on sets that are computable from every oracle in a large set, for various notions of largeness, including co-meagerness. We focus here on various genericity notions, defined from ITTM. We define in particular the categorical analogue of ITTM-randomness, and we show that it is equivalent to ITTM-genericity over L_Σ, whereas the equivalent question remains open for the randomness case.

5.1. Genericity over the constructibles. Again, we do not work with the forcing relation, traditionally defined to deal with generic sets, but we instead directly deal with Borel sets. The following proposition is the constructible version of the fact that every Borel set has the Baire property, and is the core tool behind Cohen forcing:

Theorem 5.1 (Effective Baire property theorem). *There is a function $b \mapsto (o, m)$, which to any ∞-Borel code b, associates the ∞-Borel code o of an open set, and the ∞-Borel code m of a union of closed meager sets, such that for any $x \notin \iota(m)$ we have $x \in \iota(b)$ iff $x \in \iota(o)$. Moreover this function is uniformly $\Delta_1^{L_\alpha}$ for α limit.*

Proof. The function is defined by Δ_0 induction on the rank of sets of L_α with the bounded rank replacement of Proposition 2.10. If b is the ∞-code of an open set then $o = b$ and m is the ∞-code of the empty set. If b is the ∞-code of a closed set then o is the interior of b and m is the boundary of b. We leave to the reader the proof that the function which to an ∞-Borel code of a closed set associates the ∞-Borel code of its interior and boundary, is uniformly $\Delta_1^{L_\alpha}$ for α limit.

Consider now an ∞-Borel code $b = \bigvee_{i\in I} c_i$. Note that the rank of each c_i in L_α is smaller than the rank of b. By induction we uniformly find ∞-Borel codes o_i and m_i such that for any i and any $x \notin \iota(m_i)$ we have $x \in \iota(b_i)$ iff $x \in \iota(o_i)$. We have that o is given by a code of $\bigcup_{i\in I} \iota(o_i)$ and m is given by a code of $\bigcup_{i\in I} \iota(m_i)$. It is clear that for any $x \notin \iota(m)$ we have $x \in \iota(b)$ iff $x \in \iota(o)$.

Consider now an ∞-Borel code $b = \bigwedge_{i\in I} c_i$. Note that the rank of each c_i in L_α is smaller than the rank of b. By induction we uniformly find ∞-Borel codes o_i and m_i such that for any n and any $x \notin \iota(m_i)$ we have $x \in \iota(b_i)$ iff $x \in \iota(o_i)$. We have that o is given by a code of the open set generated by all the strings σ such that each open set $\iota(o_i)$ is dense in $[\sigma]$. For each such string σ we find $m_{\sigma,i}$, the ∞-Borel code of the closed set of empty interior $[\sigma] - \iota(o_i)$. Let m_s be a code of the meager set given by the union of each such $\iota(m_{\sigma,i})$. The meager set $\iota(m_s) \cup \bigcup_{i\in I} \iota(m_i)$ ensures that if $x \in \iota(o)$, then $x \in \iota(\bigwedge_{i\in I} c_i)$. We now need to ensure that if $x \in \iota(\bigwedge_{i\in I} c_i)$ then $x \in \iota(o)$. For that we add the following meager set: for each o_i we consider an ∞-Borel code u_i of $2^\omega - \iota(o_i)$. We then let m_t be the boundary of the closure of $\bigcup_i \iota(u_i)$. A code m of our full meager set is then given by a code of $\iota(m_t) \cup \iota(m_s) \cup \bigcup_{i\in\omega} \iota(m_i)$. Suppose now that for $x \notin \iota(m)$ we have $x \in \bigwedge_{i\in I} c_i$, and suppose that for no prefix $\sigma \prec x$ we have $[\sigma] \subseteq \iota(o)$. In particular for every prefix $\sigma \prec x$, there is an extension $\tau \succ \sigma$ and some i such that $[\tau] \subseteq \iota(u_i)$. Also because $x \notin \bigcup_{i\in I} \iota(m_i)$ we must have $x \in \iota(o_i)$ for every i

and then $\tau \not\prec x$. It follows that x is in the boundary of the closure of $\bigcup_i \iota(u_i)$, which contradicts that $x \notin \iota(m)$. ⊣

We now use the previous proposition to define the forcing relation in L_α for α limit, as follows:

Definition 5.2. Let α be limit. Let $\Phi(p)$ be a formula and $p \in L_\alpha$ a parameter. Let $\mathcal{B}^\alpha(p) = \{x \ : \ L_\alpha(x) \models \Phi(p)\}$. Let o and m be the Borel codes of Theorem 5.1, such that for $x \notin \iota(m)$ we have $x \in \iota(u_n)$ iff $x \in \mathcal{B}^\alpha(p)$. Then we define $\sigma \Vdash_\alpha \Phi(\dot{p})$ if $[\sigma] \subseteq \iota(o)$.

It is clear that for $z \succ \sigma$ generic enough, that is, which does not belong to sufficiently many meager sets, we have $L_\alpha(z) \models \Phi(p)$ iff $\sigma \Vdash_\alpha \Phi(\dot{p})$.

Proposition 5.3. *Let α be countable and limit. Let $\Phi(p)$ be a formula with parameter $p \in L_\alpha$. For any σ, there exists $\tau \succeq \sigma$ such that $\tau \Vdash_\alpha \Phi(\dot{p})$ or $\tau \Vdash_\alpha \neg\Phi(\dot{p})$.*

Proof. Let o_1 be the open set which equals $\{x \in 2^\omega \ : \ L_\alpha(x) \models \Phi(\dot{p})\}$ and o_2 be the open set which equals $\{x \in 2^\omega \ : \ L_\alpha(x) \models \neg\Phi(\dot{p})\}$, both up to a union of closed meager sets of Borel code m. Suppose we have $[\sigma] \cap (\iota(o_1) \cup \iota(o_2)) = \emptyset$ for some σ. In particular there is $z \succ \sigma$ with $z \notin \iota(m)$ (because a countable union of meager closed set is nowhere dense, here we use that α is countable). Either $L_\alpha(z) \models \Phi(p)$ or $L_\alpha(z) \models \neg\Phi(p)$. In the first case we must have $z \in \iota(o_1)$ and in the second case we must have $z \in \iota(o_2)$, which contradicts $[\sigma] \cap (\iota(o_1) \cup \iota(o_2)) = \emptyset$. ⊣

We now see that the predicate $\sigma \Vdash_\alpha \Phi(\dot{p})$ for Δ_0 formulas with parameters $\dot{p}$ is uniformly Δ_1^α. We in fact need a bit more, in order to show that the forcing relation for more complex formulas is still not too complex, even when α is not admissible (see Corollary 5.6):

Proposition 5.4. *The function which to a string σ and a Δ_0 formula $\Phi(\dot{p})$ returns 1 iff $\sigma \Vdash_\alpha \Phi(\dot{p})$ (and 0 otherwise) is $\Delta_1^{L_\alpha}$ uniformly in α limit, and more so, the function which on a Δ_0 formula $\Phi(\dot{p})$ returns the function $f : 2^{<\omega} \to \{0,1\}$ such that $f(\sigma) = 1$ iff $\sigma \Vdash_\alpha \Phi(\dot{p})$, is $\Delta_1^{L_\alpha}$ uniformly in α limit.*

Proof. By Theorem 3.6 one can uniformly find the Borel code of $\mathcal{B}^\alpha(p) = \{x \ : \ L_\alpha(x) \models \Phi(p)\}$. Then by Theorem 5.1 one can uniformly find the Borel code o of the open set such that $\mathcal{B}^\alpha(p)$ equals $\iota(o)$ up to a meager set, and let $f(\ulcorner\Phi(\dot{p})\urcorner) = o$. The function f is simply given by $f(\sigma) = 1$ iff $[\sigma] \subseteq \iota(o)$. ⊣

In the previous proposition, note that the forcing relation is uniform in α: for $\alpha_1 < \alpha_2$ both limit, the same formula defines the forcing relation, interpreted as $\Vdash_{\alpha_1}$ when working in L_{α_1} and interpreted as $\Vdash_{\alpha_2}$ when working in L_{α_2}.

Proposition 5.5. *Let α be limit. Let $\Phi(a, p)$ be some formula with parameter $p \in L_\alpha$. We have:*

$$\begin{aligned} \sigma \ &\Vdash_\alpha \ \exists a\ \Phi(a, \dot{p}) \text{ iff } \exists \dot{a}\ \sigma \Vdash_\alpha \Phi(\dot{a}, \dot{p}) \\ \sigma \ &\Vdash_\alpha \ \forall a\ \Phi(a, \dot{p}) \text{ iff } \forall \dot{a}\ \forall \tau \succ \sigma\ \exists \rho \succeq \tau\ \rho \Vdash_\alpha \Phi(\dot{a}, \dot{p}) \end{aligned}$$

Proof. This follows from the construction of the Borel code o of Theorem 5.1 together with the definition of the forcing relation: for each $a \in L_\alpha$, let $\mathcal{A}_{\dot{a}} =$

$\{x \in 2^\omega \ : \ L_\alpha \models \Phi(\dot{a}, \dot{p})\}$ and let $o_{\dot{a}}$ be Borel codes of open sets such that $\iota(o_{\dot{a}})$ equals $\mathcal{A}_{\dot{a}}$ up to a union of closed meager set.

Then we have that the Borel code of the open set o of Theorem 5.1 corresponding to $\bigcup_{\dot{a} \in P_\alpha} \mathcal{A}_{\dot{a}}$ is given by $\bigcup_{\dot{a} \in P_\alpha} \iota(o_i)$. This gives us exactly $\sigma \Vdash_\alpha \exists a \ \Phi(a, \dot{p})$ iff $\exists \dot{a} \ \sigma \Vdash_\alpha \Phi(\dot{a}, \dot{p})$.

Now the Borel code of the open set o of Theorem 5.1 corresponding to the set $\bigcap_{\dot{a} \in P_\alpha} \mathcal{A}_{\dot{a}}$ is given by Borel code of the open set generated by all the strings σ such that each $\iota(o_{\dot{a}})$ is dense in $[\sigma]$. This gives us exactly $\sigma \Vdash_\alpha \forall a \ \Phi(a, \dot{p})$ iff $\forall \dot{a} \ \forall \tau \succ \sigma \ \exists \rho \succeq \tau \ \rho \Vdash_\alpha \Phi(\dot{a}, \dot{p})$. ⊣

Corollary 5.6. *Let α be limit and $n \geq 1$. The function which to a string σ and a Σ_n formula $\Phi(\dot{p})$ returns 1 iff $\sigma \Vdash_\alpha \Phi(\dot{p})$ (and 0 otherwise) is $\Sigma_n^{L_\alpha}$ uniformly in α.*

Proof. By induction on the complexity of formula, starting with the function f of Proposition 5.4. For the induction, note the the quantifiers $\forall \tau \succ \sigma$ and $\exists \tau \succ \sigma$ are bounded, and that for the Π case, we have to use each time the function $f : 2^{<\omega} \to \{0, 1\}$ given by Proposition 5.4. ⊣

5.2. Main definitions. We now formally define the notions of genericity that will be used in this paper.

Definition 5.7. If α is an ordinal, a sequence z is *generic over* L_α if z is in every dense open set $\mathcal{U}$ with a Borel code in L_α.

This previous definition applied to ITTM give that z is generic over λ (resp. generic over ζ, resp. generic over Σ) if z is in every dense open set with a writable Borel code (resp. an eventually writable Borel code, resp. an accidentally writable Borel code). These notions are somehow analogues of Δ^1_1-genericity, in the sense that Δ^1_1-genericity corresponds to genericity over $L_{\omega_1^{ck}}$ as defined above.

Proposition 5.8. *Let α be limit. Let $\Phi(\dot{p})$ be a Δ_0 formula. Let z be generic over L_α. Then $L_\alpha(z) \models \Phi(\dot{p}[z])$ iff $\exists \sigma \prec z \ \sigma \Vdash_\alpha \Phi(\dot{p})$.*

Proof. By Theorem 3.6 one can uniformly find the Borel code of $\mathcal{B}^\alpha(p) = \{x \ : \ L_\alpha(x) \models \Phi(p)\}$. Then by Theorem 5.1 one uniformly find the Borel code m of the union of meager closed sets such that for any $x \notin \iota(m)$ we have $x \in \mathcal{B}^\alpha(p)$ iff $\exists \sigma \prec x \ \sigma \Vdash_\alpha \Phi(\dot{p})$. As z is generic over L_α it does not belong to $\iota(m)$ and the result follows. ⊣

We now define the categorical analogues of ITTM-randomness and ITTM-decidable randomness. A first idea would be to define as ITTM-generic reals those which are in every ITTM-semi-decidable open sets (open sets generated by semi-decidable set of strings). However it is clear that such open sets cannot be enumerated beyond stage λ, and the notion we get is not so interesting (it is in fact equivalent to genericity over L_λ). Instead we need to use reals as oracle and the following definition seems to be the correct one:

Definition 5.9. Let $z \in 2^\omega$. We say that z is:

- ITTM-generic if it is in no meager ITTM-semi-decidable set.
- coITTM-generic if it is no meager ITTM-co-semi-decidable set.

- ITTM-decidable generic if it is in no meager ITTM-decidable set.

The counterparts of these notions for Infinite Time Register Machines have already been studied in [4].

5.3. ITTM-genericity and ITTM-decidable genericity. In this section, we will fully characterize genericity over ITTM-decidable, semidecidable and cosemidecidable sets in terms of genericity over a level of the L-hierarchy. We will see in particular that ITTM-genericity coincides with genericity over L_Σ, whereas the analogue question remains open for randomness.

5.3.1. *ITTM-genericity.* We first see why ITTM-genericity is the categorical analogue of ITTM-randomness.

THEOREM 5.10. *Let $\alpha < \beta$ limit with $L_\alpha \prec_1 L_\beta$. Suppose $z \in 2^\omega$ is generic over L_β. Then $L_\alpha(z) \prec_1 L_\beta(z)$.*

PROOF. Suppose $L_\beta(z) \models \exists q\ \Phi(q,p)$ for a Δ_0 formula Φ and $p \in L_\alpha$. Let q be such that $L_\beta(z) \models \Phi(q,p)$. As z is generic over L_β and as Φ is Δ_0, there must exist by Proposition 5.8 a string $\sigma \prec z$ such that $\sigma \Vdash_\beta \Phi(\dot{q},\dot{p})$. In particular as $\exists \dot{q}\ \sigma \Vdash_\beta \Phi(\dot{q},\dot{p})$ we have $\sigma \Vdash_\beta \exists q\ \Phi(q,\dot{p})$. By Σ_1-stability of L_α in L_β we have $\sigma \Vdash_\alpha \exists q\ \Phi(q,\dot{p})$ and then we have $L_\alpha(z) \models \exists q\ \Phi(q,p)$. ⊣

THEOREM 5.11. *Let $z \in 2^\omega$. Then the following are equivalent*

1. *z is ITTM-generic*
2. *z is generic over L_Σ and $\Sigma^z = \Sigma$.*
3. *z is generic over L_ζ and $\zeta^z = \zeta$.*

PROOF. We first prove (1) implies (2). Suppose z is ITTM-generic. Note first that the set $\mathcal{A} = \{x \in 2^\omega\ :\ \Sigma^x > \Sigma\}$ is ITTM-semi-decidable: given z, one simply has to look for two z-accidentally writable ordinals $\alpha < \beta$ such that $L_\alpha \prec_2 L_\beta$ and then halt. Such a machine halts exactly on oracles x such that $\Sigma^x > \Sigma$. Carl and Schlicht showed [5] that if x is generic over $L_{\Sigma+1}$, then $\Sigma^x = \Sigma$ (we will improve this result with Corollary 5.14). Thus the set $\mathcal{A}$ is a meager semi-decidable set, which implies that $\Sigma^z = \Sigma$. We now have to show that z is generic over L_Σ. Suppose not for contradiction. We can then design the machine which given x on its input tape, look for all the accidentally writable Borel codes of unions of closed set of empty interior, and halt whenever it finds one such that x is in it. It is clear that such a machine semi-decides a meager set, and in particular halts on z, which contradicts that z is ITTM-generic.

Let us now show that (2) implies (1). Suppose z is generic over L_Σ and $\Sigma^z = \Sigma$. Let M be an ITTM that semi-decides a meager set M. Suppose for contradiction that $M(z)\downarrow$. As we have $\Sigma^z = \Sigma$ we must also have $\zeta^z = \zeta$, by Theorem 2.29. By Theorem 5.10 we have $L_\lambda(z) \prec_1 L_\zeta(z) = L_{\zeta^z}(z)$. As λ^z is the smallest ordinal α such that $L_\alpha(z) \prec_1 L_{\zeta^z}(z)$ and as $\lambda \leq \lambda^z$ we then have $\lambda = \lambda^z$. It follows that $M(z)\downarrow[\alpha]$ for some $\alpha < \lambda$. Thus the set $\mathcal{B} = \{x \in 2^\omega\ :\ L_\lambda(x) \models M(x)\downarrow[\alpha]\}$ is a Borel set with a code in L_λ. As M halts on a meager set, the set $\mathcal{B}$ must be meager. As $z \in \mathcal{B}$ it is not generic over L_λ, which is a contradiction.

It is clear that (2) implies (3). Let us now show (3) implies (2). Suppose z is generic over L_ζ and $\zeta^z = \zeta$. By Theorem 2.29 we have that $\Sigma^z = \Sigma$. Suppose for contradiction that z is not generic over L_Σ. Then we can design the machine M

that looks for the smallest accidentally writable ordinal α such that L_α contains the Borel code of a meager set containing z, and when it finds it, writes α and halts. As z is not generic over L_Σ the machine M with input z will write some accidentally writable ordinal α and halt. As z is generic over L_ζ it must be the case that $\alpha > \zeta$. It follows that $\lambda^z > \zeta$ and thus $\zeta^z > \zeta$, a contradiction. ⊣

COROLLARY 5.12. *There is a largest ITTM semi-decidable meager set.*

PROOF. Such a set is given in the proof of (1) implies (2), in the previous theorem: let M be the ITTM which halt on x such that $\Sigma^x > \Sigma$, or on x such that x belongs to a meager set with an accidentally writable Borel code. It is clear that M semi-decides a meager set. Also this meager set contains all the elements x which are not generic over L_Σ, or such that $\Sigma^x > \Sigma$. ⊣

We now show our main theorem for this section, that is, genericity over L_Σ coincides with ITTM-genericity.

THEOREM 5.13. *Let $\alpha < \beta$ with β limit, such that $L_\alpha \prec_2 L_\beta$. Let z be generic over L_β. Then $L_\alpha(z) \prec_2 L_\beta(z)$.*

PROOF. Let $\Phi(a, b, p)$ be a Δ_0 formula with parameter $p \in L_\alpha$. By Theorem 5.10 and Proposition 2.17 we have that if $L_\alpha(z) \models \exists a\ \forall b\ \Phi(a, b, p)$, then $L_\beta(z) \models \exists a\ \forall b\ \Phi(a, b, p)$. Suppose now that $L_\beta(z) \models \exists a\ \forall b\ \Phi(a, b, p)$. Let us show that $L_\alpha(z) \models \exists a\ \forall b\ \Phi(a, b, p)$. We shall prove that $\exists \sigma \prec z\ \sigma \Vdash_\beta \exists a\ \forall b\ \Phi(a, b, \dot{p})$. Note that this is not obvious because z is only generic over L_β and the equivalence of Proposition 5.8 works only for Δ_0 formulas.

For any γ limit such that $p \in L_\gamma$, let us define

$$\begin{aligned} A_1^\gamma &= \{\sigma \in 2^{<\omega} : \sigma \Vdash_\gamma \exists a\ \forall b\ \Phi(a, b, \dot{p})\} \\ A_2^\gamma &= \{\sigma \in 2^{<\omega} : \sigma \Vdash_\gamma \forall a\ \exists b\ \neg\Phi(a, b, \dot{p})\} \end{aligned}$$

Suppose for a contradiction that for no prefix $\sigma \prec z$ we have $\sigma \in A_1^\beta$. Suppose first that also for no prefix $\sigma \prec z$ we have $\sigma \in A_2^\beta$. By Proposition 5.3 it must be the case that either A_1^β is dense along z, or that A_2^β is dense along z (without containing z). Also by the fact that $L_\alpha \prec_2 L_\beta$ and by Corollary 5.6, we must have $A_1^\alpha = A_1^\beta$ and $A_2^\alpha = A_2^\beta$. By considering the boundary of the closure of the open set generated by whichever set among A_1^α of A_2^α is dense along z, we obtain a meager closed set containing z, with a Borel code in L_α, which contradicts that z is generic over L_β.

Thus if for no $\sigma \prec z$ we have $\sigma \notin A_1^\beta$, it must be the case that $\sigma \in A_2^\beta$ for some $\sigma \prec z$. Let us fix such a string σ. In particular we must have $\sigma \Vdash_\beta \forall a\ \exists b\ \neg\Phi(a, b, \dot{p})$. By the fact that $L_\alpha \prec_2 L_\beta$ and by Corollary 5.6 we must have $\sigma \Vdash_\alpha \forall a\ \exists b\ \neg\Phi(a, b, \dot{p})$. By Proposition 5.5 we have:

$$L_\alpha \models \forall \dot{a}\ \forall \tau \succ \sigma\ \exists \rho \succ \tau\ \exists \gamma\ \exists \dot{b} \in P_\gamma\ \rho \Vdash_\alpha \neg\Phi(\dot{a}, \dot{b}, \dot{p})$$

By Theorem 2.19 we must have that L_α is admissible. Using admissibility of L_α we must have:

$$L_\alpha \models \forall \dot{a}\ \exists \gamma\ \forall \tau \succ \sigma\ \exists \rho \succ \tau\ \exists \dot{b} \in P_\gamma\ \rho \Vdash_\alpha \neg\Phi(\dot{a}, \dot{b}, \dot{p})$$

Now coming back to the definition of forcing we easily see that we have:

$$L_\alpha \models \forall \dot{a}\ \exists \gamma\ \forall \tau \succ \sigma\ \exists \rho \succ \tau\ \rho \Vdash_\alpha \exists b \in L_\gamma\ \neg\Phi(\dot{a}, b, \dot{p})$$

Which by the fact that $L_\alpha \prec_2 L_\beta$ gives us:

$$L_\beta \models \forall \dot{a}\ \exists \gamma\ \forall \tau \succ \sigma\ \exists \rho \succ \tau\ \rho \Vdash_\beta \exists b \in L_\gamma\ \neg\Phi(\dot{a}, b, \dot{p})$$

It follows that for every $\dot{a} \in P_\beta$, there exists $\gamma < \beta$ such that the open set generated by the strings ρ for which $\rho \Vdash_\beta \exists b \in L_\gamma\ \neg\Phi(\dot{a}, b, \dot{p})$, is dense in $[\sigma]$. Also this open set is clearly a set of L_β, and its complement in $[\sigma]$ is a meager closet set of L_β. It follows that we must have a prefix $\rho \prec z$ such that $\rho \Vdash_\beta \exists b \in L_\gamma\ \neg\Phi(\dot{a}, b, \dot{p})$, which implies $L_\beta(z) \models \exists b \in L_\gamma\ \neg\Phi(a, b, p)$. As this is true for every $\dot{a} \in P_\beta$, we must have that $L_\beta(z) \models \forall a\ \exists b\ \neg\Phi(a, b, p)$, which contradicts that $L_\beta(z) \models \exists a\ \forall b\ \Phi(a, b, p)$.

Thus it must be in the first place that $\sigma \Vdash_\beta \exists a\ \forall b\ \Phi(a, b, \dot{p})$ for some prefix $\sigma \prec z$. Then we also must have $\sigma \Vdash_\alpha \exists a\ \forall b\ \Phi(a, b, \dot{p})$ which implies $L_\alpha(z) \models \exists a\ \forall b\ \Phi(a, b, \dot{p})$. This concludes the proof. ⊣

Corollary 5.14. *If z is generic over L_Σ then $\Sigma^z = \Sigma$. In particular the set*

$$\{z \in 2^\omega\ :\ \Sigma^z > \Sigma\}$$

is meager.

Proof. This is because $L_\zeta \prec_2 L_\Sigma$, and because Σ^z is the smallest ordinal such that $L_\alpha(z) \prec_2 L_{\Sigma^z}(z)$ for some α. By the previous theorem we must have $\Sigma^z = \Sigma$. ⊣

Corollary 5.15. *Let $z \in 2^\omega$. The following are equivalent:*

1. *z is generic over L_Σ.*
2. *z is ITTM-generic.*

Proof. The equivalence is given by the conjunction of Theorem 5.13 and 5.11. ⊣

5.3.2. *ITTM-decidable genericity.*

Theorem 5.16. *Let $z \in 2^\omega$. The following are equivalent:*

1. *z is generic over L_λ,*
2. *z is ITTM-decidable generic,*
3. *z is co-ITTM generic.*

Proof. The implications (3) ⇒ (2) and (2) ⇒ (1) are trivial. Thus, it remains only to prove (1) ⇒ (3). Let z be a real generic over L_λ. Let M be a machine that halts on a co-meager set. By Corollary 5.14 we have that the set $\{x \in 2^\omega\ :\ \Sigma^x > \Sigma\}$ is meager. Note also that if z is generic over L_Σ we have $L_\lambda(x) \prec_1 L_\zeta(x)$ together with $L_\zeta(x) \prec_2 L_\Sigma(x)$. Thus the set $\{x \in 2^\omega\ :\ \lambda^x > \lambda\}$ is actually also meager. It follows that the set $\{x \in 2^\omega\ :\ \exists \alpha < \lambda\ M(x) \downarrow [\alpha]\}$ is already co-meager.

In particular the set $\{\sigma\ :\ \sigma \Vdash_\lambda \exists \alpha\ M(x) \downarrow [\alpha]\}$ must be a dense set of strings. By admissibility of λ, there must exists $\beta < \lambda$ such that the set $\{\sigma\ :\ \sigma \Vdash_\lambda \exists \alpha < \beta\ M(x) \downarrow [\alpha]\}$ is already a dense set of strings.

It follows that $\{x \in 2^\omega\ :\ M(x) \downarrow [\beta]\}$ is co-meager in a dense open set and thus comeager. Furthermore its complement is a union of nowhere dense closed sets with Borel code in L_λ. In particular as z is generic over L_λ, it must be that $M(z) \downarrow [\beta]$. Thus z also is co-ITTM generic. ⊣

REFERENCES

[1] PAUL-ELLIOT ANGLÈS D'AURIAC and BENOIT MONIN, *Another characterization of the higher k-trivials*, ***Lipics-leibniz international proceedings in informatics***, vol. 83, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.

[2] JON BARWISE, ***Admissible sets and structures***, vol. 7, Cambridge University Press, 2017.

[3] LAURENT BIENVENU, NOAM GREENBERG, and BENOIT MONIN, *Continuous higher randomness*, ***Journal of Mathematical Logic***, vol. 17 (2017), no. 01, p. 1750004.

[4] MERLIN CARL, *Randomness and degree theory for infinite time register machines*, ***Computability***, vol. 5 (2016), no. 2, pp. 181–196.

[5] MERLIN CARL and PHILIPP SCHLICHT, *Infinite computations with random oracles*, ***Notre Dame Journal of Formal Logic***, vol. 58 (2013), no. 2, pp. 249–270.

[6] MERLIN CARL and PHILIPP SCHLICHT, *Randomness via infinite computation and effective descriptive set theory*, ***The Journal of Symbolic Logic***, vol. 83 (2018), no. 2, pp. 766–789.

[7] CHI TAT CHONG, ANDRÉ NIES, and LIANG YU, *Lowness of higher randomness notions*, ***Israel J. Math.***, vol. 166 (2008), no. 1, pp. 39–60.

[8] CHI TAT CHONG and LIANG YU, *Randomness in the higher setting*, ***The Journal of Symbolic Logic***, vol. 80 (2015), no. 4, pp. 1131–1148.

[9] PAUL J COHEN, ***Set theory and the continuum hypothesis***, Courier Corporation, 2008.

[10] KEITH J DEVLIN, ***Constructibility***, vol. 6, Cambridge University Press, 2017.

[11] RODNEY G. DOWNEY and DENIS R. HIRSCHFELDT, ***Algorithmic randomness and complexity***, Theory and Applications of Computability, Springer, 2010.

[12] SY-DAVID FRIEDMAN and PHILIP D WELCH, *Two observations concerning infinite time turing machines*, ***BIWOC***, (2007), pp. 44–47.

[13] NOAM GREENBERG and BENOIT MONIN, *Higher randomness and genericity*, ***Forum of Mathematics, Sigma***, vol. 5 (2017), p. e31.

[14] JOEL DAVID HAMKINS and ANDY LEWIS, *Infinite time turing machines*, ***The Journal of Symbolic Logic***, vol. 65 (2000), no. 02, pp. 567–604.

[15] JOEL DAVID HAMKINS and DANIEL EVAN SEABOLD, *Infinite time turing machines with only one tape*, ***Mathematical Logic Quarterly***, vol. 47 (2001), no. 2, pp. 271–287.

[16] GREG HJORTH and ANDRÉ NIES, *Randomness via effective descriptive set theory*, ***Journal of the London Mathematical Society***, vol. 75 (2007), no. 2, pp. 495–508.

[17] PETER KOEPKE and BENJAMIN SEYFFERTH, *Ordinal machines and admissible recursion theory*, ***Annals of Pure and Applied Logic***, vol. 160 (2009), no. 3, pp. 310 – 318, Computation and Logic in the Real World: CiE 2007.

[18] STUART KURTZ, *Randomness and genericity in the degrees of unsolvability.*, ***Dissertation Abstracts International Part B: Science and Engineering***, vol. 42 (1982), no. 9, p. 1982.

[19] ———, *Notions of weak genericity*, ***The Journal of symbolic logic***, vol. 48 (1983), no. 03, pp. 764–770.

[20] PER MARTIN-LÖF, *The definition of random sequences*, ***Information and Control***, vol. 9 (1966), pp. 602–619.

[21] BENOIT MONIN, *Higher randomness and forcing with closed sets*, ***Lipics-leibniz international proceedings in informatics***, vol. 25, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2014.

[22] ANDRÉ NIES, ***Computability and randomness***, Oxford Logic Guides, Oxford University Press, 2009.

[23] PHILIP D WELCH, *Eventually infinite time turing machine degrees: Infinite time decidable reals*, ***The Journal of Symbolic Logic***, vol. 65 (2000), no. 3, pp. 1193–1203.

[24] ———, *The length of infinite time turing machine computations*, ***Bulletin of the London Mathematical Society***, vol. 32 (2000), no. 2, p. 129–136.

[25] ———, *Characteristics of discrete transfinite time turing machine models: Halting times, stabilization times, and normal form theorems*, ***Theoretical Computer Science***, vol. 410 (2009), p. 426–442.

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