ON THE BORELNESS OF UPPER CONES OF HYPERDEGREES

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ABSTRACT. We show that for any x, $\mathcal{U}_h(x) = \{y \mid y \geq_h x\}$ is Borel if and only if $x \in L$. Moreover, the Borel rank of $\mathcal{U}_h(x)$ when it exists is less than $(\omega_1)^L$. We also show using Steel forcing with tagged trees that Borel ranks of these sets range unboundedly below $(\omega_1)^L$.

1. INTRODUCTION

We assume the reader is familiar with descriptive set theory, Gödel's constructible hierarchy and higher recursion theory. The reader can refer to [2] and [6] for more details about these topics.

We use $\leq_h$ to denote hyperarithmetic reduction partial order.

Definition 1.1. *For any real x, let*

$$\mathcal{U}_h(x) = \{y \mid y \geq_h x\}.$$

The goal of this paper is to show the two following theorems:

Theorem 1.2. *Let $x \in 2^\omega$. Then $\mathcal{U}_h(x)$ is Borel iff $x \in L$. Moreover when $x \in L$ the Borel rank of $\mathcal{U}_h(x)$ is less than $(\omega_1)^L$.*

Theorem 1.3. *For any $\gamma < (\omega_1)^L$ there is $x \in L$ such that the Borel rank of $\mathcal{U}_h(x)$ is greater than γ.*

The heart of Theorem 1.3 lies in the following result from Steel:

Theorem 1.4 (Steel [8])**.** *For any countable admissible ordinal α, the set*

$$\{x \in 2^\omega \mid \omega_1^x = \alpha\}$$

is properly $\mathbf{\Pi}^0_{\alpha+2}$.

Steel has developped in [8] a new forcing notion - forcing with tagged trees - in order to study countable Δ^1_1 subsets of ω^ω, as well as independence results for subsystems of analysis. Steel mentions in his paper that his forcing can be used to show Theorem 1.4, without formally writting the proof, which is why we think it is of interest to have it fully exposed here. It is also an opportunity for the reader to read the details about Steel forcing itself, which we think is a very nice and intriging notion, the heart of its possibilities lying in the rettaging tool (Lemma 2.5) and the rettaging lemma (Lemma 2.6).

2010 *Mathematics Subject Classification.* 03D30,03D65, 03E15, 03E35.

Yu was partially supported by NSF of China No. 11671196, Humboldt foundation for experienced fellows and the Fundamental Research Funds for the Central Universities No. 14380028.

In the next section we give a detailed explanation of the forcing. In the section after that we show Theorem 1.4 and Theorem 1.3. Finally we show Theorem 1.2 in the last section.

2. Steel forcing

2.1. **The trees.** Let $\mathcal{T}$ be the set of trees of the Baire space, both finite and infinite. Let us fix a computable bijection $b : \omega \to \omega^{<\omega} - \{\epsilon\}$ (where ϵ is the empty word). We say that an element $x \in 2^\omega$ represents a tree T if $n \in x$ iff $b(n) \in T$. We easily observes that the set of sequences representing elements of $\mathcal{T}$ is a closed subset of 2^ω. Indeed, the condition for x to represent a tree is Π^0_1: "For every n, if $x(n) = 1$ then for every prefix τ of $b(n)$ we should have $x(b^{-1}(\tau)) = 1$".

It is clear that any tree is uniquely represented by a sequence this way. Also sometimes we will blur the distinction between an element of $\mathcal{T}$ and its representation in the Cantor space. We use on $\mathcal{T}$ the topology of the Cantor space induced on the set of representations of elements of $\mathcal{T}$. We easily verify that the set of representations of elements in $\mathcal{T}$ has no isolated point, therefore its elements are the paths of a perfect subtree of $2^{<\omega}$. It follows that $\mathcal{T}$ is topologically the same space as the Cantor space[1].

We will denote by F the set of 'finite trees' corresponding to a cylinder in the set of representation of elements of $\mathcal{T}$. In particular an element $p \in F$ specifies a set of nodes that are in the tree, and also a set of nodes that are not in the tree. Given an element $p \in F$, we denote by $[p]$ the set of all trees of $\mathcal{T}$ that extend p. If $T \in \mathcal{T}$ extends p we write $p \prec T$, and if another finite tree q extends p we write $p \preceq q$. It is clear that for any cylinder $[p]$, there are two finite sets of strings $\{\sigma_1, \dots, \sigma_n, \tau_1 \dots, \tau_m\}$ such that any tree T is in $[p]$ iff for $i \leq n$ we have $\sigma_i \in T$ and for $i \leq m$ we have $\tau_i \notin T$.

For a well-founded tree T, we write $|T|$ to denote the ordinal coded by T : for every node $\sigma \in T$ we inductively define $|\sigma| = \sup_i(|\sigma^\frown n_i| + 1))$ where $\langle \sigma^\frown n_i \rangle_{i\in\omega}$ are all the children of σ. We then have $|T| = |\sigma|$ where σ is the root of T. For an ill-founded tree T we write $|T| = \infty$. If $\sigma \in T$ is such that the tree of nodes compatible with σ is ill-founded we write $|\sigma| = \infty$. For every countable ordinal α, we denote by $\mathcal{T}_\alpha$ the set of trees T in $\mathcal{T}$ such that for every node $\sigma \in T$ of length 1 (direct child of the root), either $|\sigma| < \alpha$ or $|\sigma| = \infty$. In particular for any tree $T \in \mathcal{T}_\alpha$ we have either $|T| = \infty$ or $|T| \leq \alpha$.

2.2. **The tagging.** We now define P to be the set of elements $p \in F$, paired with a valid tagging function h which assigns to each node of p a countable ordinal, or the value ∞. A tagging is said to be *valid* if for any $\sigma_1 \prec \sigma_2 \in p$, we have $h(\sigma_1) > h(\sigma_2)$. By convention, ∞ is considered greater than any countable ordinal, and also greater than itself.

So an element of P is given by a pair (p, h) where $p \in F$ and where h is a valid tagging of p. Given $(p, h) \in P$ we write $[(p, h)]$ to denote the set of trees $T \in [p]$ such that for every node $\sigma \in p$, we have $|\sigma| = h(\sigma)$ (where $|\sigma|$ is performed within T). For

[1]One can easily prove it directly by constructing the homeomorphism, or use Brouwer's theorem, saying that any compact, metrisable, perfect, 0-dimensional space is homeomorphic to the Cantor space, see [4] for details.

$(p,h) \in P$ and $(q,g) \in P$, we say that $(p,h) \preceq (q,g)$ if $p \preceq q$ and $h \preceq g$ (the taggings g and h coincide on elements of p).

For any countable α, we then let P_α denotes the set of elements $(p,h) \in P$ such that for nodes $\sigma \in p$ of length 1 we have $h(\sigma) < \alpha$ or $h(\sigma) = \infty$. Given $(p,h) \in P_\alpha$, we write $[(p,h)]^\alpha$ to denote the set

$$\{T \in [(p,h)] \cap \mathcal{T}_\alpha : \ \forall \sigma \in T \cap p \ |\sigma| = h(\sigma)\}$$

In particular if $T \in [(p,h)]^\alpha$ then for every node $\sigma \in T$ distinct from the root, we have either $|\sigma| < \alpha$ or $|\sigma| = \infty$.

2.3. **The forcing relation.** For any countable α, β, we now define the forcing relation between $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ or $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ subsets of $\mathcal{T}$ and elements of P_β.

For $(p,h) \in P_\beta$ and $(q,g) \in P$, we say that $(p,h) \preceq_\beta (q,g)$ if $(p,h) \preceq (q,g)$ and if in addition we have $(q,g) \in P_\beta$. Let $(p,h) \in P_\beta$ and let us define the relation $\Vdash_\beta$ by induction on the Borel complexity of sets.

- If $\mathcal{A}$ is $\mathbf{\Delta}^0_{\mathbf{1}}$ (a finite union of cylinders) we say that $(p,h) \Vdash_\beta \mathcal{A}$ iff $[p] \subseteq \mathcal{A}$.
- If $\mathcal{A}$ is $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ with $\mathcal{A} = \bigcup_n \mathcal{A}_n$, we say that $(p,h) \Vdash_\beta \mathcal{A}$ iff $\exists n \ (p,h) \Vdash_\beta \mathcal{A}_n$.
- If $\mathcal{A}$ is $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$, we say that $(p,h) \Vdash_\beta \mathcal{A}$ iff $\forall (q,g) \succeq_\beta (p,h) \ (q,g) \nVdash_\beta \mathcal{A}^c$.

Note that the forcing relation that we gave might depend on the presentation of a given Borel set. Also for two different ways to write $\mathcal{A} = \bigcup_n \mathcal{A}_n$ or $\mathcal{A} = \bigcup_n \mathcal{A}'_n$, we might have that some $(p,h) \Vdash_\beta \bigcup_n \mathcal{A}_n$ but $(p,h) \nVdash_\beta \bigcup_n \mathcal{A}'_n$. We can show however that if $(p,h) \Vdash_\beta \bigcup_n \mathcal{A}_n$ then there must be some $(q,g) \succeq_\beta (p,h)$ such that $(q,g) \Vdash_\beta \bigcup_n \mathcal{A}'_n$, which will be good enough for us. Furthermore one can prove by induction that as long as our unions are increasing, the forcing relation then does not depend anymore on the presentation of a given Borel set.

To simplify the reading, instead of writing (p,h) for elements of P, we sometimes simply write p, the tagging function being implicit. When we do so, we will always precise it, so that there is no ambiguity. This slight abuse of notation starts with the next lemma, for which the tagging function is implicit:

Lemma 2.1. *For a $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ set $\mathcal{A} = \bigcap_n \mathcal{A}_n$, any countable β and any $(p,h) \in P_\beta$, we have*

$$p \Vdash_\beta \mathcal{A} \textit{ iff } \forall n \ \forall q \succeq_\beta p \ \exists r \succeq_\beta q \ r \Vdash_\beta \mathcal{A}_n$$

Proof. Suppose $p \Vdash_\beta \mathcal{A}$, then by definition, $\forall q \succeq_\beta p \ q \nVdash_\beta \bigcup_n \mathcal{A}^c_n$. Still following the definition of forcing we then have $\forall q \succeq_\beta p \ \forall n \ q \nVdash_\beta \mathcal{A}^c_n$ with $\mathcal{A}^c_n$ a $\mathbf{\Pi}^0_{\boldsymbol{\gamma}}$ set for some $\gamma < \alpha$, and then $\forall n \ \forall q \succeq_\beta p \ \exists r \succeq_\beta q \ r \Vdash_\beta \mathcal{A}_n$.

Suppose $p \nVdash_\beta \mathcal{A}$, then by definition, $\exists q \succeq_\beta p \ q \Vdash_\beta \bigcup_n \mathcal{A}^c_n$. Still following the definition of forcing we have $\exists q \succeq_\beta p \ \exists n \ q \Vdash_\beta \mathcal{A}^c_n$ with $\mathcal{A}^c_n$ a $\mathbf{\Pi}^0_{\boldsymbol{\gamma}}$ set for some $\gamma < \alpha$, and then $\exists n \ \exists q \succeq_\beta p \ \forall r \succeq_\beta q \ r \nVdash_\beta \mathcal{A}_n$. □

2.4. **The β-topology.** For any ordinal β, we call β-topology, the topology on $\mathcal{T}_\beta$ generated by the basis $[(p,h)]^\beta$ for any $(p,h) \in P_\beta$. Recall that the topology on $\mathcal{T}$ is such that for any $(p_1,h_1),(p_2,h_2) \in P_\beta$ we must have $[(p_1,h_1)] \cap [(p_2,h_2)] = \emptyset$ or $(p_1,h_1) \preceq (p_2,h_2)$ or $(p_2,h_2) \preceq (p_1,h_1)$. We would like to study genericity with respect to the β-topology, that is, elements of $\mathcal{T}_\beta$ which are in 'sufficiently many' dense open sets of this topological space.

This study can make sense only after we prove that generic elements actually exist, that is, we should make sure that $\mathcal{T}_\beta$ endowed with the β-topology is a Baire space:

Proposition 2.2. *For any β, the set $\mathcal{T}_\beta$, together with the β-topology is a Baire space.*

Proof. Recall that $\mathcal{T}$ is homeomorphic to the Cantor space.

Suppose that we have a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of subsets of $\mathcal{T}_\beta$ which are open in the β-topology. Each of them is a union of cylinders, so that for any n and any $(p,h) \in P_\beta$, there is some cylinder $[(q,g)]^\beta \subseteq \mathcal{U}_n$ so that $[(q,g)]^\beta \subseteq [(p,h)]^\beta$. Consider any condition $(p,h) \in P_\beta$. Let us show that $\bigcap_n \mathcal{U}_n \cap [(p,h)]^\beta$ is not empty by building an element inside it.

There must exist some $[(p_0,h_0)]^\beta \subseteq \mathcal{U}_0$ which is such that $[(p_0,h_0)]^\beta \subseteq [(p,h)]^\beta$. Then inductively for any n, assuming (p_n,q_n) is defined, we define (p_{n+1},q_{n+1}). We first define a pair (q,g) extending (p_n,q_n) the following way: we start by putting in (q,g) all tagged nodes of (p_n,q_n). Then for any leaf σ in p_n with tagging $\alpha+1$, we add $(\sigma^\frown k, \alpha)$ in (q,g) for some $\sigma^\frown k$ so that no string $\tau \succeq \sigma^\frown k$ is mentioned in p_n or in q so far.

For any node σ in p_n with tagging α limit, if no sequence $\{\alpha_m\}_{m\in\omega}$ is assigned to σ yet, we assign one so that $\alpha = \sup_m \alpha_m$. If a sequence $\{\alpha_m\}_{m\in\omega}$ is assigned to σ, we put $(\sigma^\frown k, \alpha_n)$ in (q,g) for some $\sigma^\frown k$ so that no string $\tau \succeq \sigma^\frown k$ is mentioned in p_n or in q so far.

Finally for every node σ in p_n with tagging ∞, we add $(\sigma^\frown k, \infty)$ in (q,g) for some $\sigma^\frown k$ so that no string $\tau \succeq \sigma^\frown k$ is mentioned in p_n or in q so far. Then as q should correspond to a cylinder in the set of representations of trees, we might need to actively specify that some nodes are not in q (and then not in any extension of q). If needed we do so.

Now (q,g) is a valid extension of (p_n,h_n) and then there must exists a cylinder $[(p_{n+1},h_{n+1})]^\beta \subseteq \mathcal{U}_{n+1}$ such that $[(p_{n+1},h_{n+1})]^\beta \subseteq [(q,g)]^\beta$.

It is clear by construction that $\bigcap_n [(p_n,q_n)]^\beta \subseteq \bigcap_n \mathcal{U}_n$. We should now prove that $\bigcap_n [(p_n,q_n)]^\beta$ is not empty. Because $p_0 \preceq p_1 \preceq p_2 \preceq \dots$ and $h_0 \preceq h_1 \preceq h_2 \preceq \dots$ we have that $\bigcap_n [p_n]$ contains a unique element T and $\bigcap_n [h_n]$ contains a unique element H tagging every node in T (and saying nothing on nodes which are not in T). One easily show by induction on α that for any node $\sigma \in T$ such that $H(\sigma) = \alpha$ we have $H(\sigma) = |\sigma| = \alpha$. One also easily show that $H(\sigma) = \infty$ iff $|\sigma| = \infty$. ☐

We shall now prove that if $(p,h) \Vdash_\beta \mathcal{A} \subseteq \mathcal{T}$, then $\mathcal{A} \cap \mathcal{T}_\beta$ is co-meager in $[(p,h)]^\beta$ for the β-topology (we will simply say that $\mathcal{A}$ is co-meager in $[(p,h)]^\beta$). In particular, whenhever $(p,h) \Vdash_\beta \mathcal{A}$, any generic enough element of $[p,h]^\beta$ belongs to $\mathcal{A}$.

Lemma 2.3. *Let $\mathcal{A}$ be any $\boldsymbol{\Sigma}^0_\alpha$ or $\boldsymbol{\Pi}^0_\alpha$ set and let $(p,h) \in P_\beta$. If $(p,h) \Vdash_\beta \mathcal{A}$ then $\mathcal{A} \cap \mathcal{T}_\beta$ is co-meager in $[(p,h)]^\beta$ for the β-topology.*

Proof. Consider $\mathcal{A}$ a $\boldsymbol{\Delta}^0_1$ set and suppose that for some β and $(p,h) \in P_\beta$ we have $(p,h) \Vdash_\beta \mathcal{A}$. Then $[p] \subseteq \mathcal{A}$ and then also $[(p,h)]^\beta \subseteq \mathcal{A}$, so clearly $\mathcal{A}$ is co-meager in $[(p,h)]^\beta$.

The tagging function is now implicit. Consider $\mathcal{A} = \bigcap_n \mathcal{A}_n$ a $\boldsymbol{\Pi}^0_\alpha$ set and suppose that for some β and $p \in P_\beta$ we have $p \Vdash_\beta \mathcal{A} = \bigcap_n \mathcal{A}_n$. Then $\forall n\ \forall q \geq_\beta p\ \exists r \geq_\beta$

$q\ r \Vdash_\beta \mathcal{A}_n$. Therefore, for all n, by induction hypothesis, the set $\mathcal{A}_n$ is co-meager in a dense open subset of $[p]^\beta$. Therefore it is also co-meager in $[p]^\beta$. Also as every $\mathcal{A}_n$ is co-meager in $[p]^\beta$, then $\bigcap_n \mathcal{A}_n$ is co-meager in $[p]^\beta$.

Consider $\mathcal{A} = \bigcup_n \mathcal{A}_n$ a $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ set and suppose that for some β and $p \in P_\beta$ we have $p \Vdash_\beta \mathcal{A}$. Then $p \Vdash_\beta \mathcal{A}_n$ for some n. By induction hypothesis we have that $\mathcal{A}_n$ is co-meager in $[p]^\beta$ and then that $\bigcup_n \mathcal{A}_n$ is co-meager in $[p]^\beta$. □

Just a small step now remains to prove the Baire property of any Borel $\mathcal{A}$, for the β-topology, that is, any Borel set $\mathcal{A}$ is equal to an open set, up to a meager set. The tagging function is implicit in the following lemma.

Lemma 2.4. *For any $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ or $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ set $\mathcal{A}$ and any β, the set $\{[p]^\beta\ :\ p \in P_\beta\ \wedge\ (p \Vdash_\beta \mathcal{A} \vee p \Vdash_\beta \mathcal{A}^c)\}$ is dense in $\mathcal{T}_\beta$, for the β-topology.*

Proof. Let $\mathcal{A}$ be $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$. Consider any $p \in P_\beta$. Then either $p \Vdash_\beta \mathcal{A}^c$ or $p \nVdash_\beta \mathcal{A}^c$, in which case by definition $\exists q \succeq_\beta p\ q \Vdash_\beta \mathcal{A}$. □

For a fixed β, the more dense open sets (for the β-topology) T belongs to, the more generic it is. We argue that for any β and any countably many Borel sets $\{\mathcal{A}_n\}_{n\in\omega}$, if a tree $T \in \mathcal{T}_\beta$ is generic enough, we have for any n that $T \in \mathcal{A}_n$ iff there is a prefix p of T such that $(p, |T|\restriction_p) \Vdash_\beta \mathcal{A}_n$. In what follows, the tagging function $|T|\restriction_p$ is implicit.

Pick some n and suppose that for some prefix p of T we have $p \Vdash_\beta \mathcal{A}_n$. Then using Lemma 2.3 we have that $\mathcal{A}_n$ is co-meager in $[p]^\beta$ and then if T is generic enough it belongs to $\mathcal{A}_n$. Suppose now that $T \in \mathcal{A}_n$. In particular if T is generic enough, it is in the dense open set $\{[p]^\beta\ :\ p \in P_\beta\ \wedge p \Vdash_\beta \mathcal{A}_n \vee p \Vdash_\beta \mathcal{A}^c_n\}$. Also we cannot have that $p \Vdash_\beta \mathcal{A}^c$ for some $p \prec T$, as we just proved that in this case $T \in \mathcal{A}^c$ for T generic enough. Therefore, for some prefix p of T we have $p \Vdash_\beta \mathcal{A}_n$.

2.5. **The retagging lemma.** We now prove the main lemma of Steel forcing. For any ordinal α, any two ordinals $\beta_1, \beta_2 \geq \omega\alpha$, and $(p, h_1) \in P_{\beta_1}$, $(p, h_2) \in P_{\beta_2}$, we write $(p, h_1) \sim_{\omega\alpha} (p, h_2)$ if for every node σ in p we have $h_1(\sigma) < \omega\alpha$ iff $h_2(\sigma) < \omega\alpha$ iff $h_1(\sigma) = h_2(\sigma)$.

Lemma 2.5 (The retagging tool)**.** *Let β, α be countable ordinals with $\beta < \alpha$. Let $\beta_1, \beta_2 \geq \omega\alpha$ and $p \in F$ with $(p, h_1) \in P_{\beta_1}$, $(p, h_2) \in P_{\beta_2}$ and suppose $(p, h_1) \sim_{\omega\alpha} (p, h_2)$. Then for any $(q, g_1) \succeq_{\beta_1} (p, h_1)$, there exists a retagging g_2 of q such that $(q, g_2) \succeq_{\beta_2} (p, h_2)$ and with $(q, g_1) \sim_{\omega\beta} (q, g_2)$.*

Proof. We simply build g_2. On nodes σ of p we set $g_2(\sigma) = h_2(\sigma)$, so the tagging g_2 will extend the tagging h_2. As $(p, h_1) \sim_{\omega\alpha} (p, h_2)$ then also $(p, g_1\restriction_p) \sim_{\omega\alpha} (p, g_2\restriction_p)$.

Also because $(p, h_1) \sim_{\omega\alpha} (p, h_2)$ and because $\omega\beta + \omega \leq \omega\alpha$, for every other node σ of q that is not in p and such that $g_1(\sigma) < \omega\beta + \omega$, we can set $g_2(\sigma) = g_1(\sigma)$ and have that g_2 is still a valid tagging so far.

Let M be the largest integer such that every node $\sigma \in q$ tagged by something strictly smaller than $\omega\beta + \omega$ is tagged by something strictly smaller than $\omega\beta + M$. So every node tagged in q so far by g_2 is such that $g_2(\sigma) < \omega\beta + M$ or $g_2(\sigma) \geq \omega\beta + \omega$. We have infinitely many values between $\omega\beta + M$ and $\omega\beta + \omega$ that we can use to extend g_2 in a valid tagging. It is then easy to check that $(q, g_1) \sim_{\omega\beta} (q, g_2)$ and that $(q, g_2) \succeq_{\beta_2} (p, h_2)$. □

Lemma 2.6 (The retagging lemma). *For any $\mathbf{\Pi^0_\alpha}$ or $\mathbf{\Sigma^0_\alpha}$ set $\mathcal{A}$, any countable ordinal $\beta_1, \beta_2 \geq \omega\alpha$ and any $p \in F$ with $(p, h_1) \in P_{\beta_1}$ and $(p, h_2) \in P_{\beta_2}$, if $(p, h_1) \sim_{\omega\alpha} (p, h_2)$, then $(p, h_1) \Vdash_{\beta_1} \mathcal{A}$ iff $(p, h_2) \Vdash_{\beta_2} \mathcal{A}$.*

Proof. Let $\alpha = 1$. Let $\beta_1, \beta_2 \geq \omega$. Let $p \in F$ with $(p, h_1) \in P_{\beta_1}$ and $(p, h_2) \in P_{\beta_2}$. Suppose $\mathcal{A}$ is a $\mathbf{\Pi^0_1}$. Let us suppose that $(p, h_1) \nVdash_{\beta_1} \mathcal{A}$. Then $\exists (q, g_1) \succeq_{\beta_1} (p, h_1)$ $(q, g_1) \Vdash_{\beta_1} \mathcal{A}^c$. Also $\mathcal{A}^c$ is given by a union of clopen set $\bigcup_n \mathcal{A}_n$ and we have by definition that $[q] \subseteq \mathcal{A}_n$ for some n. By the retagging tool there is a tagging g_2 of q such that $(q, g_2) \succeq_{\beta_2} (p, h_2)$. We also have $(q, g_2) \Vdash_{\beta_2} \mathcal{A}^c$ and then $(p, h_2) \nVdash_{\beta_2} \mathcal{A}$. Similarly one shows that $(p, h_2) \nVdash_{\beta_2} \mathcal{A}$ implies $(p, h_1) \nVdash_{\beta_1} \mathcal{A}$.

Suppose now that $\mathcal{A}$ is a $\mathbf{\Pi^0_\alpha}$ set. Let us suppose that $(p, h_1) \nVdash_{\beta_1} \mathcal{A}$. Then we have an extension $(q, g_1) \succeq_{\beta_1} (p, h_1)$ such that $(q, g_1) \Vdash_{\beta_1} \mathcal{A}^c$. Let $\bigcup_n \mathcal{A}_n$ be the complement of $\mathcal{A}$. Then for some n we have $(q, g_1) \Vdash_{\beta_1} \mathcal{A}_n$. Let $\beta < \alpha$ be such that $\mathcal{A}_n$ is Π^0_β. By the retagging tool we have a tagging g_2 with $(q, g_2) \sim_{\omega\beta} (q, g_1)$ and such that $(q, g_2) \succeq_{\beta_2} (p, h_2)$. By induction hypothesis we have $(q, g_2) \Vdash_{\beta_2} \mathcal{A}_n$. As $(q, g_2) \succeq_{\beta_2} (p, h_2)$, it follows that $(p, h_2) \nVdash_{\beta_2} \mathcal{A}$. Similarly one shows that $(p, h_1) \nVdash_{\beta_1} \mathcal{A}$ implies $(p, h_2) \nVdash_{\beta_2} \mathcal{A}$.

Suppose now that the lemma is true for any $\mathbf{\Pi^0_\alpha}$ set (for α countable). For any $\mathbf{\Sigma^0_\alpha}$ set $\mathcal{A} = \bigcup_n \mathcal{A}_n$ where each $\mathcal{A}_n$ is $\mathbf{\Pi^0_\beta}$ for some $\beta < \alpha$. Let $\beta_1, \beta_2 \geq \omega\alpha$. Let $(p, h_1) \in P_{\beta_1}$ and $(p, h_2) \in P_{\beta_2}$. We have $(p, h_1) \Vdash_{\beta_1} \mathcal{A}$ iff $(p, h_1) \Vdash_{\beta_1} \mathcal{A}_n$ for some n iff $(p, h_2) \Vdash_{\beta_2} \mathcal{A}_n$ iff $(p, h_2) \Vdash_{\beta_2} \mathcal{A}$. □

3. On the Borel rank of $\mathcal{U}_h(x)$

The goal of this section is to show Theorem 1.3:

Theorem (1.3). *For any $\gamma < (\omega_1)^L$ there is $x \in L$ such that the Borel rank of $\mathcal{U}_h(x)$ is greater than γ.*

To do so we show using Steel's forcing that for any countable admissible α, the set $\{x \in 2^\omega \mid \omega_1^x = \alpha\}$ is properly $\mathbf{\Pi}^0_{\alpha+2}$ and in particular not $\mathbf{\Sigma}^0_{\alpha+2}$. Let us first show that the set $\{x \in 2^\omega \mid \omega_1^x = \alpha\}$ is $\mathbf{\Pi}^0_{\alpha+2}$.

In what follows, given L_α admissible with α countable, we allow Borel codes of L_α to have their unions and/or intersetions to be indexed by ordinals $\beta < \alpha$ and not necessarily indexed by ω. This is less restrictive when in particular L_α is not a model of "everything is countable" or when α is countable but not constructibly countable.

Definition 3.1. *For $x \in 2^\omega$ let R^x_e be the c.e. subset of $\omega \times \omega$ of code e. We write $\mathcal{W}^x$ for the set of codes e such that $R^x_e \subseteq \omega \times \omega$ enumerates a well-order. For $e \in \mathcal{W}^x$ we write $|e|_x$ for the order-type of the order enumerated by R^x_e. We write $\mathcal{W}^x_{<\alpha}$ for the elements $e \in \mathcal{W}^x_{<\alpha}$ such that $|e|_x < \alpha$.*

In this document, given L_α admissible with α countable, we allow Borel codes of L_α to have their unions and/or intersetions to be indexed by ordinals $\beta < \alpha$ and not necessarily indexed by ω. This is less restrictive when in particular L_α is not a model of "everything is countable" or if α is countable but not constructibly countable.

Proposition 3.2. *Let α be admissible. Let $e \in \omega$. Let $\omega \leq \beta < \alpha$. The set $\{x \in 2^\omega \ : \ e \in \mathcal{W}^x_{<\beta}\}$ has a Σ^0_β Borel code in L_α. A function which to e, β assigns a Σ^0_β Borel code for $\{x \in 2^\omega \ : \ e \in \mathcal{W}^x_{<\beta}\}$ is $\Delta_1^{L_\alpha}$-definable uniformly in α admissible.*

Proof. We have $e \in \mathcal{W}^x_{<\omega}$ iff R^x_e describe a linear order (a $\Pi^0_2(x)$ statement), with a finite support (a $\Sigma^0_2(x)$ statement). The set $\{x \in 2^\omega \ : \ e \in \mathcal{W}^x_{<\omega}\}$ is then Σ^0_ω.

Then inductively we have $e \in \mathcal{W}^x_{<\beta+1}$ if R^x_e enumerates a linear order (a Π^0_2 statement), if there exists m such that m is maximal in R^x_e and such that $f(e,m) \in \mathcal{W}^x_{<\beta}$ where $R^x_{f(e,m)}$ is R^x_e restricted to elements strictly smaller than m. By induction $\{x \in 2^\omega \ : \ e \in \mathcal{W}^x_{<\beta+1}\}$ is Σ^0_β and then $\Sigma^0_{\beta+1}$. Finally for β limit we have $\{x \in 2^\omega \ : \ e \in \mathcal{W}^x_{<\beta}\} = \bigcup_{\gamma<\beta}\{x \ : \ e \in \mathcal{W}^x_{<\gamma}\}$. It is then a Σ^0_β set.

The construction of each Σ^0_β Borel code for the set $\{x \in 2^\omega \ : \ e \in \mathcal{W}^x_{<\beta}\}$ is clearly absolute within any admissible L_α. □

Proposition 3.3. *Let α be limit. The set $\{x \in 2^\omega \ : \ \omega^x_1 = \alpha\}$ is $\mathbf{\Pi^0_{\alpha+2}}$.*

Proof. Let us first show that the set $\{x \in 2^\omega \ : \ \omega^x_1 \leq \alpha\}$ is $\mathbf{\Pi^0_{\alpha+2}}$. For a given e and a given n, the set:

$$\mathcal{A}_{e,n} = \{x \in 2^\omega \ : \ \forall \beta < \alpha \ \Phi_e(x,n) \notin \mathcal{W}^x_{<\beta}\}$$

is $\mathbf{\Pi^0_\alpha}$. Also for a given e the set:

$$\mathcal{B}_e = \{x \in 2^\omega \ : \ \exists \beta < \alpha \ \forall n \ \Phi_e(x,n) \in \mathcal{W}^x_{<\beta}\}$$

is $\mathbf{\Sigma^0_\alpha}$. Then the set $\{x \in 2^\omega \ : \ \omega^x_1 \leq \alpha\}$ is equal to $\bigcap_e((\bigcup_n \mathcal{A}_{e,n}) \cup \mathcal{B}_e))$ which is clearly a $\mathbf{\Pi^0_{\alpha+2}}$ set. Now the set $\{x \in 2^\omega \ : \ \omega^x_1 = \alpha\}$ is the intersection of $\{x \in 2^\omega \ : \ \omega^x_1 \leq \alpha\}$ together with the set $\bigcap_{\beta<\alpha \text{ with } \beta \text{ limit}} \{x \in 2^\omega \ : \ \omega^x_1 > \beta\}$. By induction this set is $\mathbf{\Pi^0_\alpha}$ and thus the set $\{x \in 2^\omega \ : \ \omega^x_1 = \alpha\}$ is $\mathbf{\Pi^0_{\alpha+2}}$. □

We shall now show that the set $\{x \in 2^\omega \mid \omega^x_1 = \alpha\}$ is not $\mathbf{\Sigma}^0_{\alpha+2}$. To do so, we need to use Steel's proof of Sacks' result saying that α is admissible and countable iff it equals ω^y_1 for some $y \in 2^\omega$. Sacks showed something even stronger : for any countable admissible α and any $x \notin L_\alpha$ we can find $y \in 2^\omega$ such that $x \notin L_{\omega^y_1}[y]$ with $\omega^y_1 = \alpha$. This full version of Sack's theorem will be needed for the last section of this paper, which is why we start by proving it here.

3.1. Sacks's theorem.

It is clear that for α admissible and $\beta < \alpha$ we have $P_\beta \in L_\alpha$ and P_β is uniformly Δ_1-definable over L_α. Furthermore

Lemma 3.4. *The relation $p \Vdash_\beta \mathcal{A}$ is Δ_1-definable over L_α uniformly in any admissible α and in any set $\mathcal{A}$ with a Borel code in L_α.*

Proof. It is clear by definition of the forcing. □

There are many different proofs of the following theorem. The first one by Sacks uses Sacks forcing. We present here the proof of Steel, using Steel's forcing. It can also be proved using the Friedman-Jensen's application of Barwise compactness (see [3]) together with omitting type theorem [1].

Theorem 3.5. *Let α be admissible. Let $T \in \mathcal{T}_\alpha$ be generic enough, then $\omega_1^T = \alpha$. Furthermore if $x \notin L_\alpha$ then for T generic enough we have $x \notin L_{\omega_1^T}[T]$.*

Proof. We start by showing that if $T \in \mathcal{T}_\alpha$ is generic enough then $\omega_1^T = \alpha$. We will then show that if $T \in \mathcal{T}_\alpha$ is generic enough then for any $\beta < \alpha$ we have $x \notin L_\beta[T]$. Consider a functional $\Phi : \mathcal{T} \times \omega \to \omega$ and the set

$$\mathcal{A} = \{T \ : \ \forall n \ \exists \beta < \alpha \ \Phi(T, n) \in \mathcal{W}^T_{<\beta}\}$$

Let $\mathcal{A}_n = \{T \ : \ \exists \beta < \alpha \ \Phi(T, n) \in \mathcal{W}^T_{<\beta}\}$ and $\mathcal{A}_{n,\beta} = \{T \ : \ \Phi(T, n) \in \mathcal{W}^T_{<\beta}\}$. Note that from Proposition 3.2, for each $\beta < \alpha$ and each n the set $\mathcal{A}_{n,\beta}$ is $\mathbf{\Sigma}^0_{\boldsymbol{\beta}}$ uniformly in n and β, with a Borel code in L_α.

Suppose that for some $T \in \mathcal{T}_\alpha$ we have $T \in \mathcal{A}$. Suppose also that T is generic enough, so that T belongs to some $[(p, h)]^\alpha$ such that $(p, h) \Vdash_\alpha \mathcal{A}$. In particular there is a smallest $\beta_0 < \alpha$ such that $(p, h) \in P_{\beta_0}$. In what follows the tagging is implicit.

Consider the $\Sigma_1^{L_\alpha}$ function $f : \alpha \to \alpha$ which to each β associates the smallest ordinal $\gamma \geq \omega\beta$ such that:

$$\forall n \ \forall q \succeq_{\omega\beta} p \ \exists r \succeq_\gamma q \ r \Vdash_\gamma \bigcup_{\zeta<\gamma} \mathcal{A}_{n,\zeta}$$

The fact that f is Σ_1-definable over L_α follows from Lemma 3.4. Let us show that f is defined on every ordinal $\beta \geq \beta_0$. As we have $p \Vdash_\alpha \bigcap_n \bigcup_{\zeta<\alpha} \mathcal{A}_{n,\zeta}$, then also we have:

$$\forall n \ \forall q \succeq_\alpha p \ \exists r \succeq_\alpha q \ r \Vdash_\alpha \bigcup_{\zeta<\alpha} \mathcal{A}_{n,\zeta}.$$

So consider any n and any $q \succeq_{\omega\beta} p$. In particular there must exist some $r \succeq_\alpha q$ such that $r \Vdash_\alpha \bigcup_{\zeta<\alpha} \mathcal{A}_{n,\zeta}$. Therefore, by the definition of the forcing relation we must have $r \Vdash_\alpha \mathcal{A}_{n,\zeta}$ already for some $\zeta < \alpha$. Also let γ be the smallest ordinal bigger than $\max(\omega\zeta, \omega\beta)$ such that $r \in P_\gamma$. Then by the retagging lemma, as $\mathcal{A}_{n,\zeta}$ is a $\mathbf{\Sigma}^0_{\boldsymbol{\zeta}}$ set, we must have $r \Vdash_\gamma \mathcal{A}_{n,\zeta}$ and then $r \Vdash_\gamma \bigcup_{\zeta<\gamma} \mathcal{A}_{n,\zeta}$. As we can find such a γ for any n and any $q \succeq_{\omega\beta} p$, then by admissibility of L_α, the supremum of all those γ is still smaller than α. So the function is f is defined everywhere.

It is straightforward to check that the function f is continuous, that is, $f(\sup_n \beta_n) = \sup_n f(\beta_n)$. Therefore if we define $\beta_{n+1} = f(\beta_n)$ for each n, we then have that $\beta_\omega = \sup_n \beta_n$ is a fixed point of f. Note also that as $\beta_{n+1} \geq \omega\beta_n$ we have $\omega\beta_\omega = \beta_\omega$. It follows that:

$$\forall n \ \forall q \succeq_{\beta_\omega} p \ \exists r \succeq_{\beta_\omega} q \ r \Vdash_{\beta_\omega} \bigcup_{\zeta<\beta_\omega} \mathcal{A}_{n,\zeta}$$

Then we have $p \Vdash_{\beta_\omega} \bigcap_n \bigcup_{\zeta<\beta_\omega} \mathcal{A}_{n,\zeta}$. We now have to prove that $p \Vdash_\alpha \bigcap_n \bigcup_{\zeta<\beta_\omega} \mathcal{A}_{n,\zeta}$. Note that we cannot apply the tagging lemma directly because $\bigcap_n \bigcup_{\zeta<\beta_\omega} \mathcal{A}_{n,\zeta}$ is only a $\mathbf{\Pi}^0_{\boldsymbol{\beta_\omega+1}}$ set. This is here that we exploit the possibility for a tagging to be ∞. Using this, we shall argue that we actually already have:

$$(*) \qquad \forall n \ \forall q \succeq_\alpha p \ \exists r \succeq_\alpha q \ r \Vdash_\alpha \bigcup_{\zeta<\beta_\omega} \mathcal{A}_{n,\zeta}$$

Consider any n and any $q \succeq_\alpha p$, and let q^* be a retagging of q, so that every node in q that is tagged by something bigger than or equal to β_ω is retagged by ∞ in q^*. Then we have some $r^* \succeq_{\beta_\omega} q^*$ with $r^* \Vdash_{\beta_\omega} \bigcup_{\zeta<\beta_\omega} \mathcal{A}_{n,\zeta}$. In particular for some $\gamma < \beta_\omega$ we have $r^* \Vdash_{\beta_\omega} \mathcal{A}_{n,\gamma}$. Also by the retagging tool, as $q \sim_{\omega\beta_\omega} q^*$, we have some $r \succeq_\alpha q$ with $r \sim_{\omega\gamma} r^*$ and then, by the retagging lemma, we have $r \Vdash_\alpha \mathcal{A}_{n,\gamma}$, as $\mathcal{A}_{n,\gamma}$ is a $\mathbf{\Sigma^0_\gamma}$ set. It follows that $r \Vdash_\alpha \bigcup_{\zeta<\beta_\omega} \mathcal{A}_{n,\zeta}$ and then that (*) is actually true. Then we have $p \Vdash_\alpha \bigcap_n \bigcup_{\zeta<\beta_\omega} \mathcal{A}_{n,\zeta}$.

It follows that $\sup_n |\Phi(T,n)|_T \leq \beta_\omega < \alpha$. As we have this for every functional Φ for T generic enough, we then have $\omega_1^T = \alpha$.

Let us now show that if $T \in \mathcal{T}_\alpha$ is generic enough then for any $\beta < \alpha$ we have $x \notin L_\beta[T]$. Consider a name $\dot{x}$ for $x \in 2^\omega$. Then for any $n \in \omega$ and any β the sets $A_{\beta,n} = \{T : L_\beta[T] \models n \in \dot{x}\}$ have a Borel code in L_α uniformly in n and β. Let us show that the set $\{p \in P_\alpha : p \Vdash_\alpha A_{\beta,n} \text{ for } n \notin x \text{ or } p \Vdash_\alpha 2^\omega - A_{\beta,n} \text{ for } n \in x\}$ is dense. For $\beta < \alpha$ let $\beta^* < \alpha$ be such that each $A_{\beta,n}$ is $\mathbf{\Sigma^0_{\beta^*}}$. Consider a forcing condition $p \in P_\alpha$. Let p^* be a rettaging of p where all the nodes tagged by something greater than $\omega(\beta^*+1)$ is rettaged by ∞. Note that we have $p \sim_{\omega(\beta^*+1)} p^*$.

Let $A = \{n \in \omega : \exists q^* \succeq_{\omega(\beta^*+1)} p^* : q^* \Vdash_{\omega(\beta^*+1)} A_{\beta,n}\}$ Note that by Lemma 3.4 $A \in L_\alpha$. Suppose $x \nsubseteq A$. Then there is $q^* \succeq_{\omega(\beta^*+1)} p^*$ and $n \notin x$ such that $q^* \Vdash_{\omega(\beta^*+1)} A_{\beta,n}$. By the rettaging tool there is a rettaging $q \succeq_\alpha p$ such that $q \sim_{\omega\beta^*} q^*$. By the rettaging lemma we have $q \Vdash_\alpha A_{\beta,n}$. Suppose now $x \subseteq A$. As $x \notin L_\alpha$ and $A \in L_\alpha$ there must be $n \in x$ and $n \notin A$. By Lemma 2.4 there must be $q^* \succeq_{\omega(\beta^*+1)} p^*$ such that $q^* \Vdash_{\omega(\beta^*+1)} 2^\omega - A_{\beta,n}$. By the rettaging tool there is a rettaging $q \succeq_\alpha p$ such that $q \sim_{\omega\beta^*} q^*$. By the rettaging lemma we have $q \Vdash_\alpha 2^\omega - A_{\beta,n}$.

In any case the set $\{p \in P_\alpha : p \Vdash_\alpha A_{\beta,n} \text{ for } n \notin x \text{ or } p \Vdash_\alpha 2^\omega - A_{\beta,n} \text{ for } n \in x\}$ is dense. It follows that if $T \in \mathcal{T}_\alpha$ is generic enough then $x \notin L_\beta[T]$ for any $\beta < \alpha$. □

3.2. The Borel complexity of $\{X \in 2^\omega : \omega_1^X = \alpha\}$.

Theorem 3.6. *The set* $\{X \in 2^\omega : \omega_1^X = \alpha\}$ *is not* $\mathbf{\Sigma^0_{\alpha+2}}$.

Proof. We shall prove that the set of representations of elements of $\mathcal{T}$ which preserve α is not $\mathbf{\Sigma^0_{\alpha+2}}$. As this set is a closed subset of the Cantor space, it follows that also the set $\{X \in 2^\omega : \omega_1^X = \alpha\}$ is not $\mathbf{\Sigma^0_{\alpha+2}}$. In what follows, the tagging functions are implicit.

Suppose that $\{T \in \mathcal{T} : \omega_1^T = \alpha\} = \bigcup_m \bigcap_n \mathcal{A}_{n,m}$ where each $\mathcal{A}_{n,m}$ is a $\mathbf{\Sigma^0_\alpha}$ set. Then using Theorem 3.5, there must be some m such that the set $\bigcap_n \mathcal{A}_{n,m}$ contains some tree T which is generic enough for Steel forcing over P_α, so that $\omega_1^T = \alpha$. In particular we have $p \Vdash_\alpha \bigcap_n \mathcal{A}_{n,m}$ for some $p \prec T$ with $(p, |T|\restriction_p) \in P_\alpha$.

So also we have $\forall n\ \forall q \succeq_\alpha p\ \exists r \succeq_\alpha q\ r \Vdash_\alpha \mathcal{A}_{n,m}$. Let α^+ be the smallest admissible ordinal bigger than α. We should now prove that we actually have:

$$(*) \qquad \forall n\ \forall q \succeq_{\alpha^+} p\ \exists r \succeq_{\alpha^+} q\ r \Vdash_{\alpha^+} \mathcal{A}_{n,m}$$

Consider now any n and any $q \succeq_{\alpha^+} p$ and let q^* be a retagged version of q where each ordinal bigger than or equal to α in q is retagged by ∞ in q^*. Then $q^* \succeq_\alpha p$ and in particular we have some $r^* \succeq_\alpha q^*$ such that $r^* \Vdash_\alpha \mathcal{A}_{n,m}$. Let $\mathcal{A}_{n,m} = \bigcup_{k\in\omega} \mathcal{A}_{n,m,k}$

with each $\mathcal{A}_{n,m,k}$ is $\mathbf{\Pi^0_\beta}$ set for some $\beta < \alpha$. In particular for some $\beta < \alpha$ we have $r^* \Vdash_\alpha \mathcal{A}_{n,m,k}$ where $\mathcal{A}_{n,m,k}$ is a $\mathbf{\Pi^0_\beta}$ set. Also by the retagging tool, as $q \sim_{\omega\alpha} q^*$, we have some $r \succeq_{\alpha^+} q$ with $r \sim_{\omega\beta} r^*$ and then, by the retagging lemma, we have $r \Vdash_{\alpha^+} \mathcal{A}_{n,m,k}$, as $\mathcal{A}_{n,m,k}$ is a $\mathbf{\Pi^0_\beta}$ set. It follows that $r \Vdash_{\alpha^+} \mathcal{A}_{n,m}$ and then that (*) is actually true. Then we have $p \Vdash_{\alpha^+} \bigcap_n \mathcal{A}_{n,m}$.

It follows that any $q \succeq_{\alpha^+} p$ also forces $\bigcap_n \mathcal{A}_{n,m}$. Take such an extension q with a node tagged by the ordinal α. For any $T \in [q]^{\alpha^+}$ we have some node of T starting a well-founded tree coding for α. Thus α is computable in T and $\omega_1^T > \alpha$. Also as $q \Vdash_{\alpha^+} \bigcap_n \mathcal{A}_{n,m}$, the set $\bigcap_n \mathcal{A}_{n,m}$ contains some generic tree that is in $[q]^{\alpha^+}$. Then $\bigcup_m \bigcap_n \mathcal{A}_{n,m}$ contains an element making α computable, which is a contradiction. □

We are now ready to show Theorem 1.3.

Theorem (1.3)**.** *For any $\gamma < (\omega_1)^L$ there is $x \in L$ such that the Borel rank of $\mathcal{U}_h(x)$ is greater than γ.*

Proof. We show that for any $\gamma < (\omega_1)^L$ there is $x \in L$ such that the Borel rank of $\mathcal{U}_h(x)$ is not $\mathbf{\Sigma}^0_\gamma$. Fix an ordinal $\gamma < (\omega_1)^L$. Let $\nu < \alpha_0 < \alpha_1 < (\omega_1)^L$ be three contiguous admissible ordinals greater than $\gamma + \omega$ so that there is a real in $L_{\alpha_1} \setminus L_{\alpha_0}$. So we may assume that there is a real $x \in L_{\alpha_1} \setminus L_{\alpha_0}$ which is a master code. Thus $\omega_1^x = \alpha_1$. Then

$$\mathcal{U}_h(x) = \{y \mid \omega_1^y \geq \omega_1^x\} = \{y \mid \omega_1^y \geq \alpha_1\} = \{y \mid \omega_1^y > \alpha_0\} = 2^\omega \setminus \bigcup_{\beta \leq \alpha_0} \{y \mid \omega_1^y = \beta\}.$$

Suppose for contradiction that $\mathcal{U}_h(x)$ is $\mathbf{\Sigma}^0_\gamma$. Then $\{y \mid \omega_1^y = \alpha_0\} = (2^\omega \setminus \mathcal{U}_h(x)) \setminus \bigcup_{\beta \leq \nu} \{y \mid \omega_1^y = \beta\}$ would be $\mathbf{\Pi}^0_{\nu+3}$. Since $\alpha_0 > \nu + \omega$, we have a contradiction to Theorem 1.4. □

4. On the Borelness of $\mathcal{U}_h(x)$

The goal of this section is to show Theorem 1.2:

Theorem (1.2)**.** *Let $x \in 2^\omega$. Then $\mathcal{U}_h(x)$ is Borel iff $x \in L$. Moreover when $x \in L$ the Borel rank of $\mathcal{U}_h(x)$ is less than $(\omega_1)^L$.*

We start with the following lemma:

Lemma 4.1. *If $x \in L$, then the set $\mathcal{U}_h(x)$ is a $\Delta^1_1(z)$ set for some real $z \in L$. In particular the rank of z must be smaller than $(\omega_1)^L$ and so must be the rank of the Borel code of $\mathcal{U}_h(x)$.*

Proof. For any x we have $y \geq_h x$ if and only if there is an ordinal β computable in y such that $x \in L_\beta[y]$. Suppose now $x \in L$, then there is an ordinal $\alpha < (\omega_1)^L$ so that $x \in L_\alpha$. Suppose $y \geq_h x$. If $\omega_1^y \leq \alpha$ then we must have $x \in L_\beta[y]$ for some $\beta \leq \alpha$ which is computable in y. If $\omega_1^y > \alpha$, as $x \in L_\alpha$ then we must have $x \in L_\beta[y]$ for some $\beta \leq \alpha$ which is computable in y. In any case the set $\{y \in 2^\omega : y \geq_h x\}$ equals the set $\{y \in 2^\omega : x \in L_\beta[y]$ for some ordinal $\beta \leq \alpha$ computable in $y\}$.

Now fix a real $z \in L$ coding a well order of ω with order type α. Then $\mathcal{U}_h(x)$ is $\Delta^1_1(z)$. □

To show the last theorem we use the two following results :

Theorem 4.2 (Simpson [7]). *Let $r \in 2^\omega$. There is an uncountable $\Sigma^1_1(r)$ closed set A so that for any reals $y_0 \neq y_1 \in A$, $L_{\omega_1^r}[y_0 \oplus r] \cap L_{\omega_1^r}[y_1 \oplus r] = L_{\omega_1^r}[r]$.*

Theorem 4.3 (Martin [5]). *Let $r \in 2^\omega$. If A is an uncountable $\Sigma^1_1(r)$-closed set, then for any y, there is a real $x \in A$ so that $y \in L_{\omega_1^{x\oplus r}}[x \oplus r]$.*

We use $\langle *, * \rangle$ to denote the Gödel paring function.

Lemma 4.4. *Suppose that $x \notin L$, then $\mathcal{U}_h(x)$ is not Borel.*

Proof. For a contradiction, suppose that $\mathcal{U}_h(x)$ is Borel. So there is a real $z \geq_T x$ so that $\mathcal{U}_h(x)$ is $\Delta^1_1(z)$. We claim that there is a countable ordinal α so that

$$\forall y (y \geq_h x \rightarrow x \in L_\alpha[y]).$$

Since $\mathcal{U}_h(x)$ is $\Delta^1_1(z)$, by Spector-Gandy's theorem (see [2]), there is a Σ_1-formula φ so that

$$\forall y (y \not\geq_h x \leftrightarrow L_{\omega_1^{y\oplus z}}[y \oplus z] \models \varphi).$$

So there is a total $\Pi^1_1(z)$-function $f : 2^\omega \rightarrow \omega$ uniformizing the following Π^1_1-relation

$$R(y,m) = \begin{cases} m = \langle 0, n \rangle, & y \geq_h x \wedge n \in \mathcal{W}^y \wedge x \in L_{|n|_y}[y]; \\ m = \langle 1, n \rangle, & y \not\geq_h x \wedge n \in \mathcal{W}^{y\oplus z} \wedge L_{|n|_{y\oplus z}}[y \oplus z] \models \varphi. \end{cases}$$

So f is also a $\Delta^1_1(z)$-function. Thus the range of f on $\mathcal{U}_h(x)$ is a $\Sigma^1_1(z)$ set and so there must be an ordinal $\alpha < \omega_1^z$ so that for any $y \geq_h x$ and ordinal notation $n \in \mathcal{W}^y$ with $f(y) = \langle 0, n \rangle$, we have that $|n|_y \leq \alpha$. The α is exactly what we want.

We may let α be an admissible ordinal. By Lemma 3.4, there is a real r so that $\omega_1^r = \alpha$ and $x \not\leq_h r$. Now by Theorem 4.2, there is an uncountable $\Sigma^1_1(r)$ closed set A so that for any reals $y_0 \neq y_1 \in A$, $L_{\omega_1^r}[y_0 \oplus r] \cap L_{\omega_1^r}[y_1 \oplus r] = L_{\omega_1^r}[r]$. By Theorem 4.3, there must be a two reals $y_1 \neq y_2 \in A$ so that $x \in L_{\omega_1^{y_1\oplus r}}[y_1 \oplus r]$ and $x \in L_{\omega_1^{y_2\oplus r}}[y_2 \oplus r]$. By the choice of A we must have $x \notin L_{\omega_1^r}[y_i \oplus r]$ for some $i \in \{0, 1\}$. This contradicts the above claim saying that as $x \in L_{\omega_1^{y_i\oplus r}}[y_i \oplus r]$ we must have $x \in L_\alpha[y_i \oplus r]$ where $\omega_1^r = \alpha$. Thus $\mathcal{U}_h(x)$ is not Borel. □

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