UNIVERSITE PARIS-EST CRÉTEIL (Paris 13)

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Mathias Forcing and the Ramsey theorem for pairs

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Introduction

Nous présentons dans ce document plusieurs contributions sur l'étude du contenu calculatoire du théorème de Ramsey. Le résultat principal, présenté dans le dernier chapitre, est une réponse à une question ouverte depuis longtemps : la séparation des principes de mathématiques à rebours SRT^2_2 et RT^2_2 dans les ω-modèles.

Mathématiques à rebours

Les mathématiques à rebours sont un programme de logique mathématique visant à déterminer quels axiomes sont nécessaires pour prouver des théorèmes mathématiques. La méthode peut être succinctement décrite comme "Aller à rebours, des théorèmes vers les axiomes", contrastant avec la pratique ordinaire des mathématiques, consistant à dériver les théorèmes des axiomes.

La majeure partie de la recherche en mathématiques à rebours se place dans l'arithmétique du second ordre. L'ensemble de la recherche dans ce domaine a établi que des sous-systèmes faibles de l'arithmétique du second ordre sont suffisants pour formaliser presque toutes les mathématiques de Licence. En arithmétique du second ordre, les objets sont représentés par des entiers naturels ou des ensembles d'entiers naturels. Par exemple pour parler de théorèmes portant sur les nombre réels, ces derniers peuvent être représentés par des séquences de Cauchy de nombres rationnels, chacune d'entre elle pouvant être représentée comme un ensemble d'entiers naturels.

Simpson [38] décrit cinq sous-systèmes particuliers de l'arithmétique du second ordre, apparaissant fréquemment en mathématique à rebours. Par ordre de force, ces systèmes sont dénotés par les initiales suivantes : RCA_0, WKL_0, ACA_0, ATR_0, et Π^1_1-CA_0.

1. RCA_0[1] (Axiome récursif de compréhension) : Il s'agit du fragment de l'arithmétique du second ordre qui comprend les axiomes de l'arithmétique de Robinson, l'induction pour les formules Σ^0_1 et la compréhension pour les formules Δ^0_1. Ces axiomes sont nécessaires et suffisants pour montrer l'existance de tous les ensembles d'entiers calculables.

   Le sous-système RCA_0 est le système de base des mathématiques à rebours. En dépit de sa faiblesse (il ne prouve pas l'existence d'ensembles non calculables), RCA_0 est suffisant pour prouver un certain nombre de théorèmes classiques, ne requérant qu'une puissance axiomatique minimale.

2. WKL_0[2] (Le lemme faible de König) : Le sous-système WKL_0 est composé de RCA_0 plus le lemme faible de König, stipulant que tout sous-arbre infini de $2^{<\omega}$ a un chemin infini.

[1] de l'anglais : Recursive Comprehension Axiom
[2] de l'anglais : Weak König's Lemma

WKL$_0$ peut prouver un certain nombre de résultats mathématiques ne découlant pas de RCA$_0$. Par exemple le théorème de complétude de Gödel.

3. ACA$_0$[3] (Axiome de compréhension arithmétique) : ACA$_0$ consiste en RCA$_0$ plus le schéma d'axiome de compréhension pour les formules arithmétiques. ACA$_0$ permet de former les ensembles d'entiers naturels satisfaisant un nombre arbitraire de formules arithmétiques (c'est à dire avec aucune quantification sur les ensembles d'entiers, mais possiblement avec des ensembles d'entiers en paramètre).

   ACA$_0$ peut être vu comme le cadre dans lequel faire des mathématiques prédicatives, même si il y a des théorèmes prédicatifs qui ne sont pas prouvables dans ACA$_0$. La plupart des résultats fondamentaux sur les entiers naturels, et beaucoup d'autres théorèmes mathématiques, sont prouvables dans ACA$_0$.

4. ATR$_0$[4] (Récursion arithmétique transfinie) : ATR$_0$ consiste en RCA$_0$ plus la possibilité de définir des ensembles par récursion arithmétique transfinie sur les bons ordres.

5. Π^1_1-CA$_0$ [5] (Axiome de compréhension Π^1_1) : Π^1_1-CA$_0$ consiste en RCA$_0$ plus l'axiome de compréhension pour les formules Π^1_1.

La plupart des théorèmes mathématiques s'avèrent être équivalents, relativement à RCA$_0$, à l'un des cinq systèmes axiomatiques ci dessus, ce qui a conduit à leur surnom de “Big Five”, traduit en français par Ludovic Patey dans sa thèse de doctorat par “Club des Cinq”.

Les modèles des sous-systèmes de l'arithmétique du second ordre ont une *partie du premier ordre* : les entiers, et une *partie du second ordre* : les ensembles. Un modèle dans lequel la partie du premier ordre est simplement ω, est appelé un *ω-modèle*. Sauf mention contraire, tous les modèles considérés dans ce document seront des ω-modèles et nous écrirons simplement modèle pour signifier ω-modèle, considérant simplement leur partie du deuxième ordre, celle du premier ordre étant implicite.

Pour séparer deux systèmes de l'arithmétique du second ordre, par le théorème de compétude de Gödel, il est nécessaire et suffisant de construire un modèle de l'un qui n'est pas modèle de l'autre. Chacun des cinq sous-systèmes du Club des Cinq diffère déjà sur les ω-modèles. Nous seront principalement intéressé par les trois premiers, RCA$_0$, WKL$_0$ et ACA$_0$, pour lesquels nous donnons plus de détails.

1. Les modèles de RCA$_0$ sont appelés *Idéaux Turing* : Les classes d'ensembles closes par jointure Turing et réduction Turing. La classe des ensembles calculables est le plus petit de ces modèles.

2. Les modèles de WKL$_0$ sont appelés les *ensembles de Scott* : Les idéaux Turing qui sont aussi des modèles du lemme faible de König : Pour tout arbre infini $T \subseteq 2^{<\omega}$ qui est X-calculable pour X dans le modèle, il y a un ensemble $Y \in [T]$ qui est aussi dans le modèle. Il est facile de construire un arbre calculable infini ne contenant aucun chemin infini calculable, ce qui sépare WKL$_0$ de RCA$_0$.

   Friedman a aussi montré que RCA$_0$ et WKL$_0$ diffèrent uniquement par leur partie du second ordre : les deux systèmes prouvent les mêmes énoncés du premier ordre.

3. Les modèles de ACA$_0$ sont les idéaux Turing qui sont aussi clôt par saut Turing. Par le théorème “low basis”, la classe des ensembles “low” (dont le saut Turing est

[3]de l'anglais : Arithmetic Comprehension Axiom
[4]de l'anglais : Arithmetic Transfinite Recursion
[5]de l'anglais : Π^1_1 Comprehension Axiom

calculable par $\emptyset'$) est un modèle de WKL_0, et ce modèle n'est pas clôt par saut Turing et donc donne une séparation entre ACA_0 et WKL_0.

La partie du premier ordre de ACA_0 est exactement celle de l'arithmétique du premier ordre de Péano; Ont dit que ACA_0 est une *extension conservative* de l'arithmétique du premier ordre de Péano.

Le théorème de Ramsey

Parmi les théorèmes étudiés en mathématiques à rebours, le théorème de Ramsey a reçu une attention particulière de la part de la communauté, en raison du fait que sa version pour les pairs d'entiers fut historiquement le premier théorème dont on a mis en évidence qu'il échappait au phénomène du Club des Cinq. Nous en donnons ici une présentations succincte, et le lecteur peut consulter Hirschfeldt [16] pour une introduction détaillée aux mathématiques à rebours du théorème de Ramsey.

Etant donné un ensemble d'entiers X, $[X]^n$ dénote l'ensemble des sous-ensembles de X de taille n. Etant donné une couleur $f : [\omega]^n \to k$, un ensemble d'entier H est *homogène* pour cette couleur si f est constante sur $[H]^n$.

Statement (Théorème de Ramsey) : RT^n_k: "Chaque k-coloration de $[\omega]^n$ admet un ensemble homogène infini". ♢

Dans ce qui suit, le principe RT^n_k est toujours considéré relativement à RCA_0 — comme pour n'importe quel principe de mathématiques à rebours. Par exemple quand on écrit "RT^3_2 implique ACA_0", cela signifie que tout modèle de $\mathrm{RT}^3_2 + \mathrm{RCA}_0$ est aussi un modèle de ACA_0.

Le théorème de Ramsey et ses conséquences sont connues pour être particulièrement difficiles à analyser d'un point de vue calculatoire. Jockusch [19] a montré que RT^n_k est équivalent à ACA_0 pour $n \geqslant 3$, impliquant donc que RT^n_k satisfait le phénomène du Club des Cinq. La question de savoir si RT^2_k implique ACA_0 est longtemps restée ouverte, avant d'être résolue par Seetapun [37] qui a montré que RT^2_k est strictement plus faible que ACA_0. Plus tard, Jockusch [19, 20] et Liu [23] ont montrés que RT^2_k est incomparable avec WKL_0, et donc que RT^2_k n'est pas linéairement ordonné avec les membres du Club des Cinq.

Afin d'avoir une meilleure compréhension du contenu calculatoire du théorème de Ramsey pour les pairs, Cholak, Jockusch et Slaman [3] l'ont décomposé en deux énoncés : le *théorème de Ramsey stable pour les pairs*, et le principe de *cohésion*. Une couleur pour les pairs $f : [\omega]^2 \to k$ est *stable* si pour tout $x \in \omega$, $\lim_y f(\{x,y\})$ existe. Un ensemble infini C est *cohésif* pour une séquence dénombrable d'ensembles $R_0, R_1, \dots$ si $C \subseteq^* R_i$ ou $C \subseteq^* \overline{R_i}$ pour tout $i \in \omega$, où $\subseteq^*$ signifie inclusion sauf pour un nombre fini d'éléments.

Statement (Théorème de Ramsey stable pour les pairs) : SRT^2_k: "Toute k-coloration stable de $[\omega]^2$ admet un ensemble homogène infini". ♢

Statement (Cohésion) : COH: "Toute séquence dénombrable d'ensemble admet un ensemble cohésif". ♢

Cholak, Jockusch et Slaman [3] et Mileti [25] ont démontré l'équivalence, relativement à RCA_0, entre RT^2_k et $\mathrm{SRT}^2_k + \mathrm{COH}$. Ils ont naturellement demandé si la décomposition est non-triviale, dans le sens où les deux énoncés SRT^2_k et COH sont strictement plus faibles que RT^2_k. Hirschfeldt, Jockusch, Kjoss-Hanssen, Lempp et Slaman [18] ont partiellement répondu à la question en montrant que COH est strictement plus faible que RT^2_2 relativement à RCA_0. La question de savoir si SRT^2_2 implique RT^2_2 relativement à RCA_0 est restée ouverte pendant longtemps. Comme RT^2_2 est équivalent à $\mathrm{SRT}^2_2 + \mathrm{COH}$, cette question est équivalente à celle de savoir si SRT^2_2 implique COH relativement à RCA_0.

D'un point de vue calculatoire, le théoreme de Ramsey stable pour les pairs et k couleurs est équivalent à l'énoncé combinatoirement plus simple appelé D^2_k.

Statement : D^n_k: "Pour toute k-partition Δ^0_n de ω, Il existe un sous-ensemble infini d'une des parties". ◊

Chong, Lempp et Yang [4], ont montré que SRT^2_k et D^2_k sont équivalent relativement à RCA_0. Le principe de cohésion admet aussi une caractérisation calculatoire intéressante. Jockusch et Stephan [21] ont montré que la séquence des ensembles primitifs récursifs est de difficulté maximale parmi les instances calculables de COH. Les ensembles cohésifs pour cette séquence sont appelés *p-cohesifs* et leurs degrés Turing sont précisément ceux dont le saut est PA relativement à $\emptyset'$, c'est à dire les degrés dont le saut peut calculer un chemin dans tout arbre binaire infini Δ^0_2. La question suivante est donc fortement liée à séparation de SRT^2_2 et COH.

Question : Y-a-t-il pour chaque ensemble Δ^0_2, un ensemble infini dans lui ou son complémentaire, dont le saut n'est pas de degré PA relativement à $\emptyset'$? ◊

Une approche naturelle pour séparer SRT^2_2 et RT^2_2 serait de prouver que chaque ensemble Δ^0_2 admet un ensemble infini G dans lui ou son complémentaire de degré low, c'est à dire avec $G' \leqslant_T \emptyset'$.

Mais Downey, Hirschfeldt, Lempp et Solomon [8] ont construit un ensemble Δ^0_2 avec aucun ensemble low infini dans lui ou son complémentaire. De manière surprenante, Chong, Slaman et Yang [5] ont résolu la question SRT^2_2 vs COH en construisant un modèle de $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ ne contenant que des ensembles low, et qui n'est donc pas un modèle de COH. La solution à cet apparent paradoxe vient du fait que le modèle est non-standard et ne satisfait pas l'induction Σ^0_2. Les ensembles de ce modèle sont low *à l'intérieur du modèle*, mais pas dans la méta-théorie. La construction de Downey, Hirschfeldt, Lempp et Solomon [8] requière l'induction Σ^0_2.

La preuve de Chong, Slaman et Yang [5] — remarquable pour sa sophistication et les nouvelles idées qu'elle contient — sépare formellement SRT^2_2 et COH relativement à RCA_0. La preuve n'est toutefois pas pleinement satisfaisante. D'abord elle laisse ouverte la question de savoir si le principe $(\forall k)\mathrm{SRT}^2_k$ implique COH, une question qui a été posée par Cholak, Jockusch et Slaman [3]. En effet, $(\forall k)\mathrm{SRT}^2_k$ implique l'induction Σ^0_2, et donc ne peut pas avoir de modèle ne contenant que des ensembles low. Ensuite, la séparation est faite par la partie du premier ordre des modèles, et il est naturel de demander si une séparation peut être obtenue basée sur la partie du second ordre. Chong, Slaman et Yang [5] ont donc naturellement posé la question suivante:

Question : Est-ce que tout ω-modèle de $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ est un modèle de COH? ◊

La question a eu un impact important dans le développement des mathématiques à rebours, et plus généralement de la calculabilité, pas seulement pour la question elle-même, mais pour toutes les questions qui lui sont liées, les nouvelles techniques et l'émulation intellectuelle générée dans la communauté. Plusieurs articles dédiés à cette question [2, 4, 7, 11, 10, 12, 17, 27, 32, 33] ont amené à la redécouverte des degrés Weihrauch par Dorais, Dzhafarov, Hirst, Mileti et Shafer [7], et à l'introduction de la réduction calculable par Dzhafarov [11]. Dzhafarov [11, 12] a obtenu des séparations partielles en montrant que COH n'est ni Weihrauch reductible, ni fortement calculatoirement réductible à SRT^2_2.

L'améalioration la plus récente est la preuve de Dzhafarov et Patey [10] qui ont montré que COH n'est pas Weihrauch réductible à SRT^2_2 même en autorisant un nombre fini de fonctionnelles Turing.

Nous présentons dans ce document le travail effectué avec Ludovic Patey [27, 29, 28], qui s'achève sur la construction d'un ω-modèle de $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ qui n'est pas un modèle de COH.

Contenu du document

Le lecteur confiant principalement intéressé par la séparation entre SRT^2_2 et RT^2_2 peut directement se rendre au Chapitre 5, qui a été écrit de manière à pouvoir être lu indépendamment du reste de ce document, même si les chapitres précédents aideront sans doute à la compréhension de cette construction difficile.

Pour séparer SRT^2_2 et RT^2_2, il est nécessaire d'étendre le forcing de Mathias calculable de Dzhafarov and Jockusch, afin d'avoir un contrôle fin sur la valeur de vérité les énoncés Σ^0_2. Cela a initialement été développé par Cholak, Jockusch et Slaman [3], puis raffiné successivement par Wang [41], Patey [33] et Monin et Patey [27].

Monin et Patey ont par la suite raffiné encore la technique [29] afin d'avoir un contrôle sur la valeur de vérité des énoncés Σ^0_α pour un ordinal calculable α. Des outils centraux introduits pour cela et utilisés tout le long de ce document sont les notions de classe "large" et classe "stable par partition", présentées dans le premier chapitre.

Ces notions sont ensuite utilisées dans le chapitre 2 et 3 pour créer une extension du forcing de Mathias afin de montrer les résultats suivants:

Theorem (M., Patey [29]): *Soit $m \geqslant 0$. Soit Z non $\emptyset^{(m)}$-calculable. Soit A un ensemble quelconque. Il existe un ensemble $G \in [A]^\omega \cup [\overline{A}]^\omega$ tel que Z n'est pas $G^{(m)}$-calculable.*

Theorem (M., Patey [29]): *Soit Z non Δ^1_1. Soit A un ensemble quelconque. Il existe un ensemble $G \in [A]^\omega \cup [\overline{A}]^\omega$ tel que Z n'est pas $\Delta^1_1(G)$ (et en particulier avec $\omega_1^G = \omega_1^{ck}$).*

Dans le chapitre 4 on prépare ensuite le lecteur à la lecture de la séparation entre SRT^2_2 et COH dans les ω-modèles, en montrant comment utiliser le forcing de Mathias pour créer des ensembles non cohésifs. Dans le chapitre 5 on prouve finalement les deux théorèmes suivants :

Theorem (M., Patey [28]): *Pour tout ensemble Z dont le saut n'est pas PA relativement à $\emptyset'$ et pour tout ensemble $\Delta^{0,Z}_2$ A, il y a un ensemble $G \in [A]^\omega \cup [\overline{A}]^\omega$ tel que $(G \oplus Z)'$ n'est pas PA relativement à $\emptyset'$.*

Le théorème peut ensuite être itéré pour construire un ω-modèle de $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ ne contenant aucun ensemble dont le saut est PA relativement à $\emptyset'$, ce qui conduit au théorème suivant:

Theorem (M., Patey [28]): *Il y a un ω-modèle de* $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ *qui n'est pas un modèle de* COH.

Cela répond à la question de Chong, Slaman et Yang [5], et aussi de Cholak, Jockusch et Slaman [3] car tout ω-modèle de SRT^2_2 est un modèle de $\forall k\ \mathrm{SRT}^2_k$.

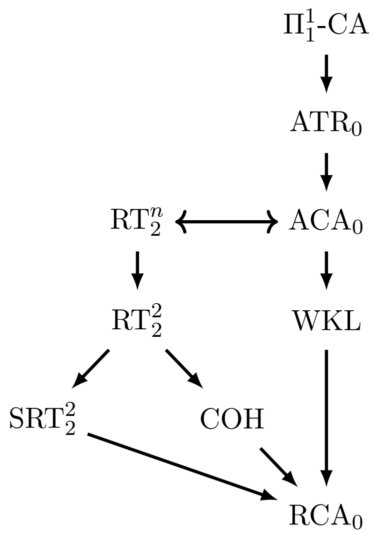

Figure 1: Un résumé des implications, relativement à RCA_0, entre différents énoncés de mathématiques à rebours, dans les ω-modèles. Toutes les implications sont strictes, et celles qui ne sont pas là sont des séparations.

Introduction

We present in this document various contributions to the computational study of Ramsey's theorem. The main result, presented in the last chapter, is an answer to the long standing open question of separating the reverse mathematics principles SRT^2_2 and RT^2_2 in ω-models.

Reverse mathematics

Reverse mathematics is a program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics. Its defining method can briefly be described as "going backwards from the theorems to the axioms", in contrast to the ordinary mathematical practice of deriving theorems from axioms.

Most reverse mathematics research focuses on subsystems of second-order arithmetic. The body of research in reverse mathematics has established that weak subsystems of second-order arithmetic suffice to formalize almost all undergraduate-level mathematics. In second-order arithmetic, all objects can be represented as either natural numbers or sets of natural numbers. For example, in order to prove theorems about real numbers, the real numbers can be represented as Cauchy sequences of rational numbers, each of which can be represented as a set of natural numbers.

Simpson [38] describes five particular subsystems of second-order arithmetic, that occur frequently in reverse mathematics. In order of increasing strength, these systems are named by the initialisms RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0.

1. RCA_0 (Recursive Comprehension Axiom) : It is the fragment of second-order arithmetic whose axioms are the axioms of Robinson arithmetic, induction for Σ^0_1 formulas, and comprehension for Δ^0_1 formulas. These axioms are necessary and sufficient to show the existance of all computable sets of numbers.

   The subsystem RCA_0 is the base system for reverse mathematics. Despite its seeming weakness (of not proving any non-computable sets exist), RCA_0 is sufficient to prove a number of classical theorems which, therefore, require only minimal logical strength.

2. WKL_0 (Weak König's Lemma) : The subsystem WKL_0 consists of RCA_0 plus the weak König's lemma, namely the statement that every infinite subtree of $2^{<\omega}$ has an infinite path.

   WKL_0 can prove many classical mathematical results which do not follow from RCA_0. For instance Gödel completeness theorem.

3. ACA_0 (Arithmetical Comprehension Axiom) : ACA_0 is RCA_0 plus the comprehension scheme for arithmetical formulas. That is, ACA_0 allows us to form the set of

natural numbers satisfying an arbitrary arithmetical formula (one with no quantification on sets, although possibly containing set parameters).

ACA_0 can be thought of as a framework of predicative mathematics, although there are predicatively provable theorems that are not provable in ACA_0. Most of the fundamental results about the natural numbers, and many other mathematical theorems, can be proven in this system.

4. ATR_0 (Arithmetical Transfinite Recursion) : ATR_0 is RCA_0 plus the possibility of defining sets by arithmetical transfinite induction on well-orders.

5. Π^1_1-CA_0 : Π^1_1-CA_0 (Π^1_1-Comprehension Axiom) is RCA_0 plus the axiom of comprehension for Π^1_1 formulas.

An early observation was that most mathematical theorems turn out to be equivalent, over RCA_0, to one of the five axiomatic systems above, which lead them to share the nickname of “Big Five”.

Models of subsystems of second order arithmetic have a *first-order part* : the integers, and a *second-order part* : the sets. A model in which the first-order part is merely ω is called an *ω-model*. Unless mentioned otherwise, every model considered in this document will be ω-models and then we often just say model to mean ω-models, considering it only as a set of reals, the first order part being implicit.

To separate two subsystems of second order arithmetic, By Gödel completeness theorem, it is necessary and sufficient to build a model of one of them which is not a model of the other one. Each of the big five subsystems differ already on ω-models. We will only be concern in this document by the three first principles, RCA_0, WKL_0 and ACA_0, for which we provide more details.

1. The models of RCA_0 are called *Turing ideals* : class of sets which are closed by Turing join and Turing reduction. The class of computable sets is the smallest such model.

2. The models of WKL_0 are called *Scott sets* : Turing ideals which are also models of weak König's lemma : for any infinite tree $T \subseteq 2^{<\omega}$ which is X-computable for X in the model, there must be a set $Y \in [T]$ which is in the model. It is easy to build an infinite computable tree which contains no computable infinite path, separating WKL_0 from RCA_0.

   Friedman also showed that RCA_0 and WKL_0 can only be separated on their second order parts : both system proves the same first-order sentences.

3. The models of ACA_0 are the Turing ideals which are also closed by Turing jump. By the low basis theorem, the class of all low sets is a model of WKL_0, but it is not closed by Turing jump and then provide a separation of ACA_0 from WKL_0.

   The first-order part of ACA_0 is exactly first-order Peano arithmetic; ACA_0 is a conservative extension of first-order Peano arithmetic. The two systems are provably (in a weak system) equiconsistent.

The Ramsey theorem

Among the theorems studied in reverse mathematics, Ramsey's theorem received a special attention from the community, since Ramsey's theorem for pairs historically was the first theorem known to escape the Big Five phenomenon. We give here a brief presentation of this theorem in our context, and the reader can refer to Hirschfeldt [16] for a detailed introduction to the reverse mathematics of Ramsey's theorem.

Given a set of integers X, $[X]^n$ denotes the set of all subsets of X of size n. For a coloring $f : [\omega]^n \to k$, a set of integers H is *homogeneous* if f is constant over $[H]^n$.

Statement (Ramsey's theorem) : RT^n_k: "Every k-coloring of $[\omega]^n$ admits an infinite homogeneous set". ◊

In what follows the principle RT^n_k is always considered relative to RCA_0 — like any reverse mathematic principle. For instance when we write "RT^3_2 implies ACA_0", it means any model of $\mathrm{RT}^3_2 + \mathrm{RCA}_0$ is also a model of ACA_0.

Ramsey's theorem and its consequences are notoriously hard to analyze from a computable-theoretic viewpoint. Jockusch [19] proved that RT^n_k is equivalent to ACA_0 whenever $n \geqslant 3$, thereby showing that RT^n_k satisfies the Big Five phenomenon. The question whether RT^2_k implies ACA_0 was a longstanding open question, until Seetapun [37] proved that RT^2_k is strictly weaker than ACA_0. Later, Jockusch [19, 20] and Liu [23] showed that RT^2_k is incomparable with WKL_0, and therefore that RT^2_k is not even linearly ordered with the Big Five.

In order to understand better the computational and proof-theoretic content of Ramsey's theorem for pairs, Cholak, Jockusch and Slaman [3] decomposed it into two statements, namely, stable Ramsey's theorem for pairs, and cohesiveness. A coloring of pairs $f : [\omega]^2 \to k$ is *stable* if for every $x \in \omega$, $\lim_y f(\{x, y\})$ exists. An infinite set C is *cohesive* for a countable sequence of sets $R_0, R_1, \dots$ if $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$ for every $i \in \omega$, where $\subseteq^*$ means inclusion but for finitely many elements.

Statement (Stable Ramsey's theorem for pairs) : SRT^2_k: "Every stable k-coloring of $[\omega]^2$ admits an infinite homogeneous set". ◊

Statement (Cohesiveness) : COH: "Every countable sequence of sets has a cohesive set". ◊

Cholak, Jockusch and Slaman [3] and Mileti [25] proved the equivalence over RCA_0 between RT^2_k and $\mathrm{SRT}^2_k + \mathrm{COH}$. They naturally wondered whether this decomposition is non-trivial, in the sense that both statements SRT^2_k and COH are strictly weaker than RT^2_k. Hirschfeldt, Jockusch, Kjoss-Hanssen, Lempp and Slaman [18] partially answered the question by proving that COH is strictly weaker than RT^2_2 over RCA_0. The question whether SRT^2_2 implies RT^2_2 over RCA_0 remained a long-standing open question. Since RT^2_2 is equivalent to $\mathrm{SRT}^2_2 + \mathrm{COH}$, this is equivalent to the question whether SRT^2_2 implies COH over RCA_0.

From a computability-theoretic viewpoint, stable Ramsey's theorem for pairs and k colors is equivalent to a combinatorially simpler statement called D^2_k.

Statement : D^n_k: "For every Δ^0_n k-partition of ω, there is an infinite subset of one of the parts". ◊

Chong, Lempp and Yang [4], proved that the computable equivalence between SRT^2_k and D^2_k also holds over RCA_0. The cohesiveness principle also admits a nice computability-theoretic characterization. Jockusch and Stephan [21] proved that the sequence of all primitive recursive sets is a maximally difficult instance of COH (among the computable ones). The cohesive sets for this sequence are called *p-cohesive* and their Turing degrees are precisely the ones whose jump is PA over $\emptyset'$, that is, the degrees whose jump can compute a path through any Δ^0_2 infinite binary tree. The following computable-theoretic question is therefore closely related to the previous question.

Question : Does every Δ^0_2 set has an infinite subset in it or its complement whose jump is not of PA degree over $\emptyset'$? $\diamond$

One natural approach to separate SRT^2_2 from RT^2_2 would be to prove that every Δ^0_2 set admits an infinite subset G in it or its complement of low degree, that is, $G' \leqslant_T \emptyset'$. However, Downey, Hirschfeldt, Lempp and Solomon [8] constructed a Δ^0_2 set with no low infinite subset of it or its complement. Very surprisingly, Chong, Slaman and Yang [5] answered the SRT^2_2 vs COH question by constructing a model of $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ with only low sets, which is not a model of COH. The solution to this apparent paradox was the use of a non-standard model of RCA_0 in which Σ^0_2 induction fails. The sets of this model are low *within the model*, but not low in the meta-theory. The construction of Downey, Hirschfeldt, Lempp and Solomon [8] requires Σ^0_2 induction to be carried out.

The proof of Chong, Slaman and Yang [5] — remarkable by its sophistication and striking new ideas — formally separates SRT^2_2 from COH over RCA_0. It remains however not fully satisfactory. First, it leaves open the question whether $(\forall k)\mathsf{SRT}^2_k$ implies COH which was also asked by Cholak, Jockusch and Slaman [3]. Indeed, $(\forall k)\mathsf{SRT}^2_k$ implies Σ^0_2 induction, and therefore cannot have any models with only low sets. Second, the separation is done by playing with the first order part of the models, and it is natural to ask if one could also achieve the separation based on the second order part. Chong, Slaman and Yang [5] naturally asked the following question:

Question : Is every ω-model of $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ a model of COH? $\diamond$

This question had an important impact in the development of reverse mathematics, and computability theory in general, not only by its self interest, but also by range of related questions, new techniques and intellectual emulation it generated in the community. Several articles are dedicated to this question [2, 4, 7, 11, 10, 12, 17, 27, 32, 33] and led to the rediscovery of Weihrauch degrees by Dorais, Dzhafarov, Hirst, Mileti and Shafer [7], and the design of the computable reduction by Dzhafarov [11]. Dzhafarov [11, 12] obtained partial separations by proving that COH is neither Weihrauch reducible, nor strongly computably reducible to SRT^2_2. The most recent improvement is a proof by Dzhafarov and Patey [10] proving that COH is not Weihrauch reducible to SRT^2_2 even when a finitely many Turing functionals are allowed.

We present in this document the work done with Ludovic Patey [27, 29, 28], ending with the construction of an ω-model of $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ which is not a model of COH.

Overview of the document

The confident reader who is mainly interested in the separation of SRT^2_2 from RT^2_2 may jump directly to Chapter 5 which has been written in a way so that it could be read independently from the rest of the document, although, the rest of the document would certainly help for a better understanding of this difficult construction.

In order to separate SRT^2_2 from RT^2_2, one shall extend the computable Mathias forcing of Dzhafarov and Jockusch to have a fine control the on truth of Σ^0_2 statements. This has been initially developed by Cholak, Jockusch and Slaman [3], and then successively refined by Wang [41], Patey [33] and Monin and Patey [27].

Monin and Patey then refined again the techniques [29] to have a tight control of Σ^0_α statements for any computable ordinal α. One central tool introduced for this and used all along this document, is the notion of *largeness* and *partition regular classes*, which are presented in Chapter 1.

These notions are then used in Chapter 2 and 3 to design an extension of Mathias forcing with which we show the following results :

Theorem (M., Patey [29]): *Let $m \geqslant 0$. Let Z be non $\emptyset^{(m)}$-computable. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not $G^{(m)}$-computable.*

Theorem (M., Patey [29]): *Let Z be non Δ^1_1. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not $\Delta^1_1(G)$ (and in particular with $\omega_1^G = \omega_1^{ck}$).*

In Chapter 4 we then prepare the reader to the separation of SRT^2_2 from COH in ω-models, by showing how to use Mathias forcing to build non-cohesive sets. In Chapter 5 we finally prove the two main theorems:

Theorem (M., Patey [28]): *For every set Z whose jump is not of PA degree over $\emptyset'$ and every $\Delta^{0,Z}_2$ set A, there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that $(G \oplus Z)'$ is not of PA degree over $\emptyset'$.*

This theorem can then be iterated to construct an ω-model of $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ containing no set whose jump is of PA degree over $\emptyset'$, from which we deduce the following theorem.

Theorem (M., Patey [28]): *There is an ω-model of* $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ *which is not a model of* COH.

This answers a question of Chong, Slaman and Yang [5], but also of Cholak, Jockusch and Slaman [3] since any ω-model of SRT^2_2 is a model of $\forall k\ \mathrm{SRT}^2_k$.

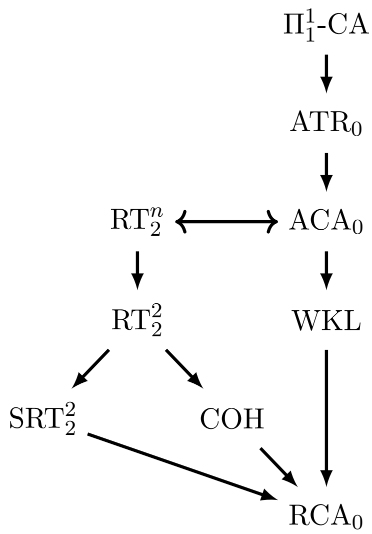

Figure 2: Summary diagram of implications between statements over RCA_0, and over ω-models. All the implications are strict, and the missing implications are separations.

Chapter 1

Background and main definitions

We assume the reader is familiar with the main computability notions. Here is a non exhaustive list of reference on the subject [6] [30] [9] [31].

1.1 Notations

We briefly specify here the notations we use, which should all be standard.

1.1.1 The Cantor space

We call *strings* finite sequences of zeros and ones. The empty word, denoted by ϵ is also considered to be a string. The space of strings is denoted by $2^{<\omega}$, and a string itself will be denoted by σ, τ or ρ. For a string σ, we denote the length of σ by $|\sigma|$. An infinite sequence of zeros and ones will be called a *set* or a *sequence* and we typically use letters X, Y or Z, to name sequences. The *Cantor space*, denoted by 2^ω is the set of all sequences.

For a string σ and a sequence X we write $\sigma \prec X$ and we say 'X *extends* σ' or that 'σ is a *prefix* of X', if the $|\sigma|$ first bits of X are equal to σ. Similarly, for two strings σ and τ, we say that $\sigma \preceq \tau$ if $|\sigma| \leqslant |\tau|$ and if the $|\sigma|$ first bits of τ are equal to σ. If we want the extension to be strict we write $\sigma \prec \tau$. If two strings σ and τ are such that $\sigma \npreceq \tau$ and $\tau \npreceq \sigma$, we say that σ and τ are *incomparable*, and we write $\sigma \perp \tau$. For a string σ, a sequence X, any n with $0 \leqslant n < |\sigma|$ and any m, we write $\sigma(n)$ and $X(m)$ to denote respectively the value of the n-th bit of σ and the value of the m-th bit of X (starting at position 0). For two strings σ, τ, we denote the concatenation of σ to τ by $\sigma\tau$. Finally, for an integer n, a string σ and a sequence X, we denote by $X\upharpoonright_n$ and $\sigma\upharpoonright_n$, respectively the n first bits of X and the n first bits of σ.

The elements of 2^ω are also considered to be sets of natural numbers, and the elements of $2^{<\omega}$ as finite sets of natural numbers. In that regards, Given $\sigma, \tau \in 2^{<\omega}$ and $X \in 2^\omega$, we write $\sigma \subseteq X$ to mean that $\sigma(i) = 1$ implies $X(i) = 1$ and we write $\sigma \subseteq \tau$ to mean that $\sigma(i) = 1$ implies $\tau(i) = 1$. Given $X, Y \in 2^\omega$ we write $X \subseteq^* Y$ is X is included in Y except for finitely many elements. Finally given $X \in 2^\omega$ and $n \in \omega$, we write $[X]^n$ to denote the set of subset of X of size n. We write $[X]^\omega$ to denote the set of infinite subsets of X.

1.1.2 The Baire space

As for the Cantor space, we call *string* a finite sequence of natural numbers (including the empty word ϵ), *sequence* or *function* an infinite one, and we define $\omega^{<\omega}$ to be the set of strings and ω^ω to be the set of sequences. In practice it will be in general clear when strings/sequences are meant to be strings/sequences of the Baire space rather than of the Cantor space, and when it might be ambiguous, we will always give precisions.

Elements of $\omega^{<\omega}$ will be usually denoted by σ, τ or ρ and elements of ω^ω will be usually denoted by f, g or h. For an integer n, a sequence f and strings σ, τ, the notions of length $|\sigma|$, extension/prefix $\sigma \prec X$, $\sigma \preccurlyeq \tau$, $\sigma \prec \tau$, comparability $\sigma \perp \tau$, n-th value $\sigma(n)$, $f(n)$, concatenation $\sigma\tau$, and restrictions $f\restriction_n$, $\sigma\restriction_n$, are as in the Cantor space.

1.1.3 Trees

Given a well-order $\leqslant$ on a set A, a tree T in A is a subset of A closed by predecessor : if $a \in T$ and $b \leqslant a$ then $b \in T$.

Most of the time we will consider that trees are subsets of $2^{<\omega}$ or $\omega^{<\omega}$ for the order $\preccurlyeq$ of prefix extension. Given a tree $T \subseteq 2^{<\omega}$ a *path* of T is a set $X \in 2^\omega$ such that $X\restriction_n \in T$ for every $n \in \omega$. We write $[T]$ for the set of paths of T.

1.1.4 Computability

Elements of ω will be usually be denoted by $a, b, c, d, e, i, j, k, l, m, n$, with e more specifically used for 'codes'. The letters r, s, t will usually refers to time of computations. We just recall here some standard notation which will be used in this thesis.

For any e we denote by $\varphi_e : \omega \to \omega$ the *computable function* of code e. If we allow a 'computable process' to access infinite objects as *oracle*, we then speak of *computable functional*. So for any $e \in \omega$, we will denote by $\Phi_e : 2^\omega \times \omega \to \omega$ the computable functional of code e. Sometimes it will happen that we want our functionals to have more than one oracle in input, with possibly some of them from the Baire space. When it is so we will always give precisions. For a given fixed oracle $X \in 2^\omega$, we denote by $\Phi_e^X : \omega \to \omega$ the curryfication of Φ_e applied to oracle X. We write $\Phi_e^X(n)\downarrow$ or sometimes $\Phi_e(X, n)\downarrow$ if the computation converges with oracle X and input n. We write $\Phi_e^X(n)\uparrow$ or sometimes $\Phi_e(X, n)\uparrow$ otherwise. Also for any $e \in \omega$, we denote by W_e the *computably enumerable* set of code e, that is the domain of φ_e. The notion relativizes and for $X \in \omega$, we denote W_e^X the domain of Φ_e^X. Note that we will not make any difference between W_e and $W_e^{0^\infty}$ (where 0^∞ denotes the sequence corresponding to the empty set of natural numbers).

We will often consider functionals $\Phi_e : 2^\omega \times \omega \to \omega$ as functions from 2^ω to ω^ω, or as functions from 2^ω to 2^ω. In this case we write $\Phi_e : 2^\omega \to \omega^\omega$ (respectively $\Phi_e : 2^\omega \to 2^\omega$) and we write $\Phi_e(X)$ to denote the image of Φ_e on the sequence X. Such a function Φ_e is defined on X when $\forall n\, \Phi_e(X, n)\downarrow$ (respectively when $\forall n\, \Phi_e(X, n)\downarrow \in \{0, 1\}$).

Quite often we will have to consider the running time of a given computation. So for a functional Φ_e, an oracle X and an integer n, we denote by $\Phi_{e,t}(X, n)$ or by $\Phi_e(X, n)[t]$ the result of the computation up to time t.

We will very often use computable bijections from $\omega \times \omega$ to ω or more generally from ω^n to ω. We denote such bijections by $\langle, \dots, \rangle$ and we write for example $\langle a, b\rangle$ for the result of the binary bijection on a and b. We give a first example of a the use of $\langle, \rangle$ by introducing for sequences $\{X_i\}_{i\in\omega}$, the notation $\oplus_{i\in\omega} X_i$, which denotes the sequence Z such that $Z(\langle i, j\rangle) = X_i(j)$. We also write $X \oplus Y$ to denote the sequence Z such that $Z(2i) = X(i)$ and $Z(2i+1) = Y(i)$.

1.2 Mathias forcing

All the techniques of this documents will be enhancement of Mathias forcing, that we give here :

Definition 1.2.1 (Mathias, see [22]) : A Mathias condition is given by (σ, X) where:

1. $\sigma \in 2^{<\omega}$
2. $X \in [\omega]^\omega$
3. $X \cap \{0, \dots, |\sigma|\} = \varnothing$

The partial order is defined by $(\tau, Y) \leqslant (\sigma, X)$ if

1. $\tau = \sigma\rho$ for $\rho \subseteq X$
2. $Y \subseteq X$ ◊

An infinite sequence of Mathias conditions $(\sigma_0, X_0) \geqslant (\sigma_1, X_1) \geqslant \dots$ ultimately builds a generic $G \in 2^\omega$, as the unique set such that $\sigma_n \leqslant G$ for every n. Due to the requirement that X is infinite for a condition (σ, X), a sufficiently generic set of conditions will guaranty that G is infinite. A Mathias conditions (σ, X) is then also a guaranty that up to finitely many elements, our generic will be a subset of X.

Mathias was able during his PhD thesis ([24] see [22]) to use his forcing to build a model of set theory in which the following generalization of Ramsey's theorem holds : For any coloring of the elements of $[\omega]^\omega$ in two colors, there is an infinite set $X \in [\omega]^\omega$ every infinite subset of which has the same color. Note that if the coloring is Borel, this is known as the Galvin-Příkrý theorem.

It is then not a surprise to see Mathias forcing necessary to study Ramsey's theorem from a computably theoretic perspective.

1.3 Computability theory

We give here the main theorems in computability theory that will be used in this document.

Theorem 1.3.1 (Low basis theorem (Jockusch, Soare [20])):
Let $X \in 2^\omega$. Let $\mathcal{P}$ be a non-empty $\Pi^0_1(X)$ class. The set X' uniformly computes a set Y' such that $Y \in \mathcal{P}$.

Note that a set X such that $X' \leqslant_T Y'$ is called *low relative to* Y.

Definition 1.3.2 : A set $X \in 2^\omega$ is PA(X) if X computes a function $f : \omega \to \omega$ such that $\Phi_n(X, n)\downarrow$ implies $f(n) \neq \Phi_n(X, n)$ for every n. ◊

Definition 1.3.3 : A set $X \in 2^\omega$ is p-cohesive if for every primitive recursive set Y we have $X \subseteq^* Y$ or $X \subseteq^* \omega - Y$. ◊

Theorem 1.3.4 (Jockusch, Stephan [21]):
The following are equivalent for a set X:

1. *X computes a p-cohesive set*
2. *X' is PA($\emptyset'$)*

Definition 1.3.5 : A *Scott set* is a set $\mathcal{M} \subseteq 2^\omega$ such that

1. For all $X \in \mathcal{M}$ if $X \geqslant_T Y$ then $Y \in \mathcal{M}$
2. For all $X, Y \in \mathcal{M}$ then $X \oplus Y \in \mathcal{M}$
3. For all $X \in \mathcal{M}$ if $T \subseteq 2^{<\omega}$ is an X-computable tree such that $[T]$ is non-empty then there is $Y \in [T]$ such that $Y \in \mathcal{M}$ ◊

1.4 Main definitions

1.4.1 Largeness and Partition regularity

We introduce two key notions which will be used all along this document in order to enhance Mathias forcing in various ways : largeness classes and partition regular classes. The later is well known and has been studied in the literature as a general combinatorial notion (see for instance [1] [36]). The less restrictive former notion was introduced in [27] [1] in order to design a forcing notion to show that for any set A there exists $G \in [A]^\omega \cup [\omega - A]^\omega$ which is not of high Turing degree.

Definition 1.4.1 : A *largeness* class is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that:

1. $\mathcal{L}$ is not empty
2. $\mathcal{L}$ is *upward closed* : If $X \in \mathcal{L}$ and $X \subseteq Y$, then $Y \in \mathcal{L}$
3. If $Y_0 \cup \dots \cup Y_k \supseteq \omega$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$ ◊

Note that given a set X and a set $Y_0 \cup \dots \cup Y_k \supseteq X$ we sometimes refer to $Y_0 \cup \dots \cup Y_k$ as a *k-cover* of X.

Definition 1.4.2 : A *partition regular* class is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that:

1. $\mathcal{L}$ is a largeness class
2. If $X \in \mathcal{L}$ and $Y_0 \cup \dots \cup Y_k \supseteq X$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$

Furthermore $\mathcal{L}$ is *proper* if $\mathcal{L} \neq 2^\omega$. ◊

In the literature, partition regular classes are usually defined without explicitly requiring (2) of Definition 1.4.1. However every studied example encountered by the author also satisfies upward closure with respect to set inclusion. In addition to that, the fact that Definition 1.4.2 contains (2) of Definition 1.4.1 brings the definition of partition regularity right next to another very famous and well studied notion : Proper partition regular classes are exactly the complements of proper set theoretic ideals : non-empty classes $\mathcal{I} \subseteq 2^\omega$ which are downward closed with respect to set inclusion, and which are closed by finite union. Indeed if $\mathcal{L} \neq 2^\omega$ is partition regular and $X, Y \notin \mathcal{L}$ then also $X \cup Y \notin \mathcal{L}$ by (2) of Definition 1.4.2. Conversely if $\mathcal{I}$ is an ideal and $X \notin \mathcal{I}$ with $Y_0 \cup \dots \cup Y_k \supseteq X$ then $Y_i \notin \mathcal{I}$ for some $i \leqslant k$ by the finite union property of ideals.

Definition 1.4.3 : A partition regular class $\mathcal{L}$ is *principal* if for some n we have

$$\mathcal{L} = \{X \in 2^\omega \ : \ n \in X\}$$

[1] under the name *acceptability class*

A partition regular class $\mathcal{L}$ is *non-trivial* if it contains no principal partition regular class. ◇

Proposition 1.4.4 : A partition regular class $\mathcal{L}$ contains only infinite sets iff it is non-trivial. ⋆

Proof: Suppose $\mathcal{L}$ contains a finite set $X = \{n_1, \dots, n_k\}$. Then in particular we have $\{n_1\} \cup \dots \cup \{n_k\} \supseteq X$. It follows that we must have $\{n_i\} \in \mathcal{L}$ for some $i \leqslant k$. Then any set X containing n_i is in $\mathcal{L}$, that is, we have $\{X \in 2^\omega : n \in X\} \subseteq \mathcal{L}$. Conversely it is clear that if $\mathcal{L}$ contains only infinite sets, then it is non-trivial. ∎

All the partition regular classes we manipulate in this document will be non-trivial.

Proposition 1.4.5 : Let $\mathcal{L}$ be a non-trivial partition regular class. Then $\mathcal{L}$ is closed by finite change of its elements. Furthermore if $\mathcal{L}$ is measurable it has measure 1. ⋆

Proof: Let $X \in \mathcal{L}$. By definition we have that any $Y \supseteq X$ also belongs to $\mathcal{L}$. Thus $\mathcal{L}$ is closed by finite addition. Consider now any $Y \subseteq X$ such that $|X - Y|$ is finite. We have in particular that $X = Y \cup \{n_0, \dots, n_k\}$ for some elements $n_0, \dots, n_k$. As $\mathcal{L}$ contains only infinite elements, we must have $Y \in \mathcal{L}$. Thus $\mathcal{L}$ is closed by finite suppression. We easily conclude that $\mathcal{L}$ is closed by finite changes.

If $\mathcal{L}$ is measurable, by the Kolmogorov 0-1 law it must be that $\mathcal{L}$ is of measure 0 or of measure 1. Suppose for contradiction that $\mathcal{L}$ is of measure 0. As $\mathcal{L}$ is measurable, if must be included in some Borel set $\mathcal{A}$ of measure 0. Let O be an oracle such that $\mathcal{A}$ is included in a $\Pi^0_2(O)$ set effectively of measure 0. Then no element of $\mathcal{L}$ is O-Martin-Löf random. Let Z be any O-Martin-Löf random set. We also have that $\overline{Z}$ is O-Martin-Löf random. Also $\omega \subseteq Z \cup \overline{Z}$. As $\omega \in \mathcal{L}$ we must have $Z \in \mathcal{L}$ or $\overline{Z} \in \mathcal{L}$, which is a contradiction. Thus $\mathcal{L}$ is not of measure 0 and therefore it is of measure 1. ∎

We now connect partition regular classes with largeness classes. Let us first make sure that if a set $\mathcal{A}$ contains a partition regular class, it contains one which contains all the others.

Proposition 1.4.6 : Suppose $\{\mathcal{L}_i\}_{i \in I}$ is an arbitrary non-empty collection of partition regular classes. Then $\bigcup_{i \in I} \mathcal{L}_i$ is a partition regular class. ⋆

Proof: It is clear that $\bigcup_{i \in I} \mathcal{L}_i$ is not empty. Let $X \in \bigcup_{i \in I} \mathcal{L}_i$. Let $Y \supseteq X$. There is i such that $X \in \mathcal{L}_i$. As $\mathcal{L}_i$ is partition regular we have $Y \in \mathcal{L}_i \subseteq \bigcup_{i \in I} \mathcal{L}_i$.

Let $X \in \bigcup_{i \in I} \mathcal{L}_i$. Let $Y_0 \cup \dots \cup Y_k \supseteq X$. There is i such that $X \in \mathcal{L}_i$. As $\mathcal{L}_i$ is partition regular we have that $Y_j \in \mathcal{L}_i \subseteq \bigcup_{i \in I} \mathcal{L}_i$ for some $j \leqslant k$. ∎

In particular for every class $\mathcal{A}$ such that $\mathcal{A}$ contains a partition regular class, there is a largest partition regular class included in $\mathcal{A}$. It leads to the following definition:

Definition 1.4.7 : Given a class $\mathcal{A} \subseteq 2^\omega$, let $\mathcal{L}(\mathcal{A})$ denote the largest partition regular subclass of $\mathcal{A}$. If $\mathcal{A}$ does not contain a partition regular class, let $\mathcal{L}(\mathcal{A})$ be the empty set. ◇

We now connect largeness classes to partition regular classes.

Lemma 1.4.8 : For any class $\mathcal{A} \subseteq 2^\omega$ the class $\mathcal{L}(\mathcal{A})$ equals:

$$\{X \in 2^\omega \ : \ \forall k \ \forall X_0 \cup \dots \cup X_k \supseteq X \ \exists i \leqslant k \ \text{ s.t. } X_i \in \mathcal{A}\}$$

Proof: For this proof we refer to $\mathcal{L}(\mathcal{A})$ as defined by this proposition, in order to show that it matches Definition 1.4.7. Note that by definition we must have $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$, as if $X \notin \mathcal{A}$ then itself as a 1-cover is not in $\mathcal{A}$.

Let us show that $\mathcal{L}(\mathcal{A})$ contains every partition regular class included in $\mathcal{A}$. Suppose $\mathcal{L} \subseteq \mathcal{A}$ is partition regular. Then given $X \in \mathcal{L}$, for every k and every $X_0 \cup \dots \cup X_k \supseteq X$ we have $X_i \in \mathcal{L} \subseteq \mathcal{A}$ for some $i \leqslant k$. It follows that $X \in \mathcal{L}(\mathcal{A})$ and thus that $\mathcal{L} \subseteq \mathcal{L}(\mathcal{A})$.

Let us show that if $\mathcal{L}(\mathcal{A})$ is non-empty, it is a partition regular class. Suppose $X \in \mathcal{L}(\mathcal{A})$. Let $Y \supseteq X$. Then for every k, every k-cover of Y is also a k-cover of X. As $X \in \mathcal{L}(\mathcal{A})$, one element of the k-cover belongs to $\mathcal{A}$. Thus for every k and every k-cover of Y, one element of the k-cover belongs to $\mathcal{A}$. It follows that $Y \in \mathcal{L}(\mathcal{A})$.

Let $X \in \mathcal{L}(\mathcal{A})$ and let $Y_0 \cup \dots \cup Y_k \supseteq X$ for some k. Let us show there is some $i \leqslant k$ such that $Y_i \in \mathcal{L}(\mathcal{A})$. Suppose for contradiction that this is not the case. In particular for every $i \leqslant k$ there are sets $Y_0^i, \dots Y_{k_i}^i \supseteq Y_i$ such that $\forall j \leqslant k_i$, we have $Y_j^i \notin \mathcal{A}$. In particular the sets $\{Y_j^i\}_{i \leqslant k, j \leqslant k_i}$ are a finite cover of X such that for every $i \leqslant k$ and every $j \leqslant k_i$ we have $Y_j^i \notin \mathcal{A}$. This contradicts that $X \in \mathcal{L}(\mathcal{A})$. Thus there must exists $i \leqslant k$ such that $Y_i \in \mathcal{L}(\mathcal{A})$. So if $\mathcal{L}(\mathcal{A})$ is non-empty it is a partition regular class. ∎

Corollary 1.4.9 : An upward closed class is a largeness class iff it contains a partition regular class.

Proof: Let $\mathcal{A}$ be an upward closed class. Suppose $\mathcal{A}$ is a largeness class. Then from the previous proposition we have $\omega \in \mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$. Thus $\mathcal{L}(\mathcal{A})$ is non-empty which means it is a partition regular class contained in $\mathcal{A}$.

Suppose now that $\mathcal{A}$ contains a partition regular class $\mathcal{L}$. Then $\mathcal{L} \subseteq \mathcal{L}(\mathcal{A})$ is non-empty and then $\omega \in \mathcal{L}(\mathcal{A})$ which implies by the previous proposition that $\mathcal{A}$ is a largeness class. ∎

Proposition 1.4.10 : Suppose $\{\mathcal{L}_n\}_{n\in\omega}$ is a collection of partition regular (resp. largeness) classes with $\mathcal{L}_{n+1} \subseteq \mathcal{L}_n$. Then $\bigcap_{n\in\omega} \mathcal{L}_n$ is a partition regular (resp. largeness) class. ⋆

Proof: For every n we have that $\omega \in \mathcal{L}_n$ because $\mathcal{L}_n$ is a largeness class. It follows that $\omega \in \bigcap_{n\in\omega} \mathcal{L}_n$. In particular $\bigcap_{n\in\omega} \mathcal{L}_n$ is not empty.

Suppose $X \in \bigcap_{n\in\omega} \mathcal{L}_n$. Let $Y \supseteq X$. For every n we have $X \in \mathcal{L}_n$ and thus $Y \in \mathcal{L}_n$ as $\mathcal{L}_n$ is a largeness class. Thus $Y \in \bigcap_{n\in\omega} \mathcal{L}_n$.

Suppose $X \in \bigcap_{n\in\omega} \mathcal{L}_n$ (resp. $\omega \in \bigcap_{n\in\omega} \mathcal{L}_n$). Let $Y_0 \cup \dots \cup Y_k \supseteq X$ (resp. $Y_0 \cup \dots \cup Y_k \supseteq \omega$). Suppose for contradiction that for every $i \leqslant k$ the set Y_i is not in $\bigcap_{n\in\omega} \mathcal{L}_n$. Thus there must be some n such that for every $i \leqslant k$ the set Y_i is not in $\mathcal{L}_n$. As $X \in \mathcal{L}_n$ (resp. $\omega \in \mathcal{L}_n$), it follows that $\mathcal{L}_n$ is not a partition regular class (resp. $\mathcal{L}_n$ is not a largeness class), which contradicts our hypothesis. ∎

1.4.2 Π^0_2 Partition regular classes

In this section all the partition regular classes we deal with are considered non-trivial.

Examples

We exclusively deal in this document with Π^0_2 (or $\boldsymbol{\Pi}^0_2$) partition regular classes, whose canonical examples are given in the following definition:

Definition 1.4.11 : For any infinite set X we define $\mathcal{L}_X$ as the $\Pi^0_2(X)$ partition regular class of the sets that intersect X infinitely often. $\diamond$

We exhibit here a few more Π^0_2 partition regular classes, in order to illustrate the diversity we have among them, even when we restrict the complexity to intersection of open sets.

Definition 1.4.12 : The complement of the summable ideal

$$\mathcal{L}_{1/n} = \left\{X \ : \ \sum_{n \in X} \frac{1}{1+n} = +\infty\right\}$$

is a non-trivial Π^0_2 partition regular class. $\diamond$

Definition 1.4.13 : The complement of the van der Waerden ideal:

$$\mathcal{L}_W = \{X \ : \ X \text{ contains arbitrarily long arithmetic progressions } \}$$

is a non-trivial Π^0_2 partition regular class. $\diamond$

These specific ideals bring us dangerously close to difficult open questions about natural numbers : The famous Erdös conjecture on arithmetic progressions states that $\mathcal{L}_{1/n} \subseteq \mathcal{L}_W$. We luckily here have no business with any of these specific propreties on natural numbers. We are indeed, as often in computability, more concern "globally" with all Π^0_2 partition regular classes.

Complexity

The first thing of interest for us is the low complexity of questions related to partition regularity.

Proposition 1.4.14 : Let $\mathcal{U}$ be a Σ^0_1 class. Then $\mathcal{L}(\mathcal{U})$ is a Π^0_2 class. $\star$

Proof: By Lemma 1.4.8 we have

$$\mathcal{L}(\mathcal{U}) = \{X \in 2^\omega \ : \ \forall k \ \forall X_0 \cup \dots \cup X_k \supseteq X \ \exists i \leqslant k \ X_i \in \mathcal{U}\}$$

Once k is fixed, if for all $X_0 \cup \dots \cup X_k \supseteq X$ there exists $i \leqslant k$ such that $X_i \in \mathcal{U}$, then by compactness there must exists a finite prefix $\sigma \prec X$ such that for every k-cover $\tau_0 \cup \dots \cup \tau_k \supseteq \sigma$ we must have $[\tau_i] \subseteq \mathcal{U}$ for some $i \leqslant k$. It follows that for k fixed the class

$$\{X \in 2^\omega \ : \ \forall X_0 \cup \dots \cup X_k \supseteq X \ \exists i \leqslant k \ X_i \in \mathcal{U}\}$$

is a Σ^0_1 class. Thus $\mathcal{L}(\mathcal{U})$ is a Π^0_2 class. ∎

Corollary 1.4.15 : Let $\mathcal{U}$ be an upward closed Σ^0_1 class. The sentence "$\mathcal{U}$ is a largeness class" is Π^0_2.

Proof: By Corollary 1.4.9 the open set $\mathcal{U}$ is a largeness class iff $\omega \in \mathcal{L}(\mathcal{U})$, which is by the previous proposition a Π^0_2 sentence. ■

Minimal largeness classes

For this section we suppose that $\{\mathcal{U}_e\}_{e \in \omega}$ is an enumeration of all the Σ^0_1 upward closed open sets.

Definition 1.4.16 : A largeness class $\bigcap_{e \in C} \mathcal{U}_e$ (for any set C) is *minimal* if for every Σ^0_1 largeness class $\mathcal{U}$, we have $\mathcal{U} \cap \bigcap_{e \in C} \mathcal{U}_e \subsetneq \bigcap_{e \in C} \mathcal{U}_e$ implies that $\mathcal{U} \cap \bigcap_{e \in C} \mathcal{U}_e$ is not a largeness class. ◊

The canonical construction of a minimal largeness class is done with a gready algorithm. This is made explicit with the next proposition.

Definition 1.4.17 : A presentation C of a largeness class $\bigcap_{e \in C} \mathcal{U}_e$ is *minimal* if $\bigcap_{e \in C} \mathcal{U}_e$ is minimal. It is *syntactically minimal* if for every Σ^0_1 largeness class $\mathcal{U}_e$ we have $e \notin C$ implies that $\mathcal{U}_e \cap \bigcap_{e \in C} \mathcal{U}_e$ is not a largeness class. ◊

Note that using Lemma 1.4.8, every minimal largeness class is a partition regular class. We suppose the minimal largeness classes we work with are always non-trivial partition regular classes.

Proposition 1.4.18 : There is a syntactically minimal presentation of a largeness class which is computable in $\emptyset''$. Furthermore every syntactically minimal presentation of a largeness class computes $\emptyset''$. ⋆

Proof: Let a be an index for the Σ^0_1 class of all sets containing at least two elements. One can build a syntactically minimal presentation C of a largeness class by a gready algorithm using $\emptyset''$: We start with $C_0 = \mathcal{U}_a$. For every new index $n+1$, we ask if $C_n \cap \mathcal{U}_{n+1}$ is a largeness class, which is by Corollary 1.4.9 a Π^0_2 question. If it is the case we set $C_{n+1} = C_n \cup \{n+1\}$. Otherwise we set $C_{n+1} = C_n$. It is clear from Proposition 1.4.10 that $C = \bigcup_n C_n$ is a $\emptyset''$-computable syntactically minimal presentation of a largeness class. Note that $\mathcal{U}_C \subseteq \mathcal{U}_a$ also ensure that $\mathcal{U}_C$ is non-trivial.

Suppose now that C is a (non-trivial) syntactically minimal presentation of a largeness class. Let b be any index such that $\mathcal{U}_b = \mathcal{U}_a$. In particular we must have $\bigcap_{e \in C} \mathcal{U}_e \subseteq \mathcal{U}_b$ and thus that $\mathcal{U}_b \cap \bigcap_{e \in C} \mathcal{U}_e$ is a largeness class and thus that $b \in C$. Consider any Π^0_2 formula $\forall x\ \exists y\ \phi(x, y)$. We uniformly build the index b of the following Σ^0_1 class : for every x, once we find y_t for every $t \leqslant x$ such that $\phi(t, y_t)$ is true, we put in $\mathcal{U}_b$ all the strings of length x with at least two 1's.

Suppose the formula is true, thus we have that $\mathcal{U}_b$ contains exactly the sets with at least two elements and thus that $b \in C$. Otherwise there is some x such that $\mathcal{U}_b$ does not contain the set $0^x 1^\infty$. Note that every non-trivial partition regular class must contains all the co-finite elements. Thus $\mathcal{U}_b$ contains no non-trivial partition regular class and thus $b \notin C$. It follows that C can decide every Π^0_2 formula and then compute $\emptyset''$. ■

It is not clear whether every minimal presentation (not necessarily syntactically minimal) of a largeness class computes $\emptyset''$. All we can obtain here is that such presentations must compute p-cohesive sets. To prove it we need the following lemma :

Lemma 1.4.19 : Let $\mathcal{A}$ be a largeness class. Let $X_0 \cup \dots \cup X_l \supseteq \omega$. Then $\mathcal{A} \cap \mathcal{L}_{X_i}$ is a largeness class for some $i \leqslant l$. ⋆

Proof: As $\mathcal{A}$ is a largeness class then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$ is a partition regular class. There must be $i \leqslant l$ such that $X_i \in \mathcal{L}(\mathcal{A})$. Let us show that $\mathcal{A} \cap \mathcal{L}_{X_i}$ is a largeness class. It is clearly non-empty and upward closed. Let $Y_0 \cup \dots \cup Y_k \supseteq \omega$. In particular we have $(Y_0 \cap X_i) \cup \dots \cup (Y_k \cap X_i) \supseteq X_i$. As $X_i \in \mathcal{L}(\mathcal{A})$ we must have $Y_j \cap X_i \in \mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$ for some $j \leqslant k$, and clearly $Y_j \cap X_i \in \mathcal{L}_{X_i}$. Thus $Y_j \cap X_i \in \mathcal{A} \cap \mathcal{L}_{X_i}$ and thus $Y_j \in \mathcal{A} \cap \mathcal{L}_{X_i}$. It follows that $\mathcal{A} \cap \mathcal{L}_{X_i}$ is a largeness class. ∎

Proposition 1.4.20 : Every minimal presentation of a largeness class computes a p-cohesive set. ⋆

Proof: Let $\bigcap_{e \in C} \mathcal{U}_e$ be a minimal largeness class. In particular for any computable set X we must have, using Lemma 1.4.19 that $\mathcal{L}_X \cap \bigcap_{e \in C} \mathcal{U}_e$ is a largeness class, or that $\mathcal{L}_{\overline{X}} \cap \bigcap_{e \in C} \mathcal{U}_e$ is a largeness class. Suppose without loss of generality that $\mathcal{L}_X \cap \bigcap_{e \in C} \mathcal{U}_e$ is a largeness class. Then by minimality we must have $\bigcap_{e \in C} \mathcal{U}_e \subseteq \mathcal{L}_X$. In particular we have $X \in \bigcap_{e \in C} \mathcal{U}_e$ and $\overline{X} \notin \bigcap_{e \in C} \mathcal{U}_e$. It follows that for any computable set X we have $X \in \bigcap_{e \in C} \mathcal{U}_e$ or $\overline{X} \in \bigcap_{e \in C} \mathcal{U}_e$, but not both. Let us now compute, using C', a total function $f : \omega \to \{0,1\}$ such that $f(e) \neq \Phi_e(\emptyset', e)$.

For any e let $X_e = \{s \in \omega \ : \ \Phi_e(\emptyset', e)[s] \downarrow = 0\}$. Suppose that $\Phi_e(\emptyset', e) \downarrow = i$ for $i \in \{0,1\}$. If $\Phi_e(\emptyset', e) \downarrow = 0$, then $\overline{X_e}$ is finite and thus $X_e \in \bigcap_{e \in C} \mathcal{U}_e$ and $\overline{X_e} \notin \bigcap_{e \in C} \mathcal{U}_e$. If $\Phi_e(\emptyset', e) \downarrow = 1$, then X_e is finite and thus $\overline{X_e} \in \bigcap_{e \in C} \mathcal{U}_e$ and $X_e \notin \bigcap_{e \in C} \mathcal{U}_e$.

The function f is defined as follow : for any e, we look for the first $n \in C$ such that $X_e \notin \mathcal{U}_n$ or $\overline{X_e} \notin \mathcal{U}_n$. If $X_e \notin \mathcal{U}_n$ then we set $f(e) = 0$ (Note that we must have in this case $\Phi_e(\emptyset', e) \neq 0$). If $\overline{X_e} \notin \mathcal{U}_n$ then we set $f(e) = 1$ (Note that we must have in this case $\Phi_e(\emptyset', e) \neq 1$). It is clear that f is $\mathrm{DNC}_2^{\emptyset'}$. As f is C'-computable we have that C computes a p-cohesive set. ∎

For the developments to come, we would ideally need to work with largeness classes with a presentation computable in some set which is $\mathrm{PA}(\emptyset')$. We however still do not know is such a presentations exist. To overcome this we will soon introduce the notion of cohesive largeness classes.

Question 1.4.1 : Let C be a presentation of a minimal largeness class $\bigcap_{e \in C} \mathcal{U}_e$. Must we have $C \geqslant_T \emptyset''$? Is there such a presentation which is computable in a PA over $\emptyset'$? ◇

Minimality with respect to a set

In order to overcome the necessity of working with a minimal largeness class having a presentation computable in a $\mathrm{PA}(\emptyset')$, we use a slightly weaker notion which will reveal itself sufficient through Corollary 1.4.24 : the notion of cohesive largeness classes.

Before we introduce it, we slightly enhance Definition 1.4.16 to consider largeness classes which are minimal with respect to every element in a countable set $\mathcal{M}$. Also in order to work with these countable sets, we need to introduce notations to improve readability. The countable classes $\mathcal{M} = \{X_0, X_1, \dots\}$ that we will use will come together with presentations $M = \oplus_{n \in \omega} X_n$. Given such a presentation se way that e is an M-index for X_e. We also say that an *index* for a $\Sigma^0_1(X)$ class $\mathcal{U}$ for some $X \in \mathcal{M}$ is given by the pair $\langle a, b \rangle$ where $\mathcal{U} = \mathcal{U}_a^{X_b}$.

Notation

Given a countable set $\mathcal{M}$ presented by $M = \oplus_{n\in\omega} X_n$, given a set $C \subseteq \omega^2$ of indices, we write $\mathcal{U}_C^{\mathcal{M}}$ for the set

$$\bigcap_{\langle a,b\rangle \in C} \mathcal{U}_a^{X_b}$$

We are now ready to define largeness classes which are minimal with respect to a countable set.

Definition 1.4.21 : Let $\mathcal{M}$ be a countable set. A largeness class $\mathcal{U}_C^{\mathcal{M}}$ for some $C \subseteq \omega^2$ is $\mathcal{M}$-*minimal* if for every $X \in \mathcal{M}$ and for every $\Sigma_1^0(X)$ largeness class $\mathcal{U}$, we have $\mathcal{U} \cap \mathcal{U}_C^{\mathcal{M}} \subsetneq \bigcap_{e\in C} \mathcal{U}_e$ implies that $\mathcal{U} \cap \mathcal{U}_C^{\mathcal{M}}$ is not a largeness class. ◊

Cohesive largeness classes

We now turn to our solution to get around the necessity of having minimal largeness classes which are computable in some PA($\emptyset'$).

Definition 1.4.22 : A largeness class $\mathcal{L}$ is cohesive if for any computable $X \in 2^\omega$ we have $\mathcal{L} \subseteq \mathcal{L}_X$ or $\mathcal{L} \subseteq \mathcal{L}_{\overline{X}}$. Given a countable set $\mathcal{M}$, a largeness class $\mathcal{L}$ is $\mathcal{M}$-cohesive if for any $X \in \mathcal{M}$ we have $\mathcal{L} \subseteq \mathcal{L}_X$ or $\mathcal{L} \subseteq \mathcal{L}_{\overline{X}}$. ◊

If $\mathcal{L}$ is an $\mathcal{M}$-cohesive largeness class, note that given any finite covering $Y_0 \cup \cdots \cup Y_k \supseteq \omega$ with $Y_0 \oplus \cdots \oplus Y_k \in \mathcal{M}$, there must be $i \leqslant k$ such that $\mathcal{L} \subseteq \mathcal{L}_{Y_i}$.

Lemma 1.4.23 : Let $\mathcal{M}$ be a countable Scott set. Let $\mathcal{U}_C^{\mathcal{M}}$ be an $\mathcal{M}$-cohesive largeness class. Let $\mathcal{V}_1, \mathcal{V}_2$ be two $\Sigma_1^0(X)$ largeness classes for some $X \in \mathcal{M}$. Suppose $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_1$ and $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_2$ are both largeness classes. Then $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_1 \cap \mathcal{V}_2$ is a largeness class. ⋆

Proof: Suppose for contradiction that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_1 \cap \mathcal{V}_2$ is not a largeness class. There exists then a finite set $F \subseteq C$ such that $\bigcap_{\langle a,b\rangle\in C} \mathcal{U}_a^{X_b} \cap \mathcal{V}_1 \cap \mathcal{V}_2$ is not a largeness class. Let $X \in \mathcal{M}$ and $\mathcal{U}$ be a $\Sigma_1^0(X)$ class equal to $\bigcap_{\langle a,b\rangle\in F} \mathcal{U}_a^{X_b} \cap \mathcal{V}_1 \cap \mathcal{V}_2$. We have a cover $Y_0 \cup \cdots \cup Y_k \supseteq \omega$ such that $Y_i \notin \mathcal{U}$ for every $i \leqslant k$. As $\mathcal{M}$ is a Scott set there is such a cover in $\mathcal{M}$. As $\mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive class, there must be $i \leqslant k$ such that $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{Y_i}$. It follows that $Y_i \in \mathcal{U}_C^{\mathcal{M}}$ and $j \neq i$ implies $Y_j \notin \mathcal{U}_C^{\mathcal{M}}$. As both $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_1$ and $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_2$ are largeness class it must be that $Y_i \in \mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_1$ and $Y_i \in \mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_2$ (as it cannot be the case for $j \neq i$), which contradicts that $Y_i \notin \mathcal{U}$. Thus $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{V}_1 \cap \mathcal{V}_2$ is a largeness class. ■

Corollary 1.4.24 : Let $\mathcal{M}$ be a countable Scott set. Let $\mathcal{L}$ be an $\mathcal{M}$-cohesive largeness class. There is a unique $\mathcal{M}$-minimal largeness subclass of $\mathcal{L}$.

Proof: It is clear by combining the previous lemma with Proposition 1.4.10. ■

Notation

Given a countable Scott set $\mathcal{M}$, given a cohesive largeness class $\mathcal{U}_C^{\mathcal{M}}$, we write $\langle \mathcal{U}_C^{\mathcal{M}} \rangle$ for the unique minimal largeness subclass of $\mathcal{U}_C^{\mathcal{M}}$.

Chapter 2

Arithmetic cone avoidance

2.1 Introduction

The goal of this chapter is to show the following theorems:

Theorem 2.1.1 (M., Patey [29]):
Let $m \geqslant 0$. Let Z be non $\emptyset^{(m)}$-computable. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not $G^{(m)}$-computable.

Theorem 2.1.2 (M., Patey [29]):
Let Z be non arithmetically definable. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not arithmetically definable relative to G.

Dzhafarov and Jockusch showed [13] that given any non-computable set Z and given any set A, there is an infinite subset $G \subseteq A$ or $G \subseteq \overline{A}$ which does not compute Z. Note that even if A is infinite, leaving the possibility that $G \subseteq \overline{A}$ is important : given any set Z, any infinite subset of $A = \{Z\restriction_n : n \in \omega\}$ can compute Z (considering $A \subseteq \omega$ via a computable encoding of strings into integers).

In order to show their result, Dzhafarov and Jockusch used computable Mathias forcing, with the particularity of building two generics : one as an infinite subset of A and one as an infinite subset of $\overline{A}$. At the end of the construction Dzhafarov and Jockusch then use what is referred to in the literature as a *pairing argument* to show that at least one of the generic does not compute Z.

The author of this document together with L. Patey then designed in [27] an enhancement of Dzhafarov and Jockusch's techniques, inspired by the second jump control of [3], in order to show that for any non-Δ^0_2 set Z and any set A, there is an infinite subset $G \subseteq A$ or $G \subseteq \overline{A}$ whose jump does not compute Z. Here again two generics are built, and a pairing argument is used to show that there is at least one generic whose jump does not compute Z.

The techniques used by the author and L. Patey could not be extended in order to show that for any non-Δ^0_n set Z and any set A, there is an infinite subset $G \subseteq A$ or $G \subseteq \overline{A}$ such that $G^{(n-1)}$ does not compute Z. The encountered difficulty was overcome by the same authors in [29], where they also show how to get around the pairing argument : only one generic is built in whichever among A or $\overline{A}$ satisfies some specific property which makes the construction work. Note that Hirschfeldt [18] previously found a way to get around the pairing by giving two different constructions, one if A or $\overline{A}$ is hyperimmune relative to some countable Scott set and one if it is not the case. The solution we present here is on the other hand uniform.

2.2 Preliminaries

In order to define our forcing conditions, we first need to construct a few things.

Proposition 2.2.1 : There is a sequence of sets $\{M_n\}_{n<\omega}$ such that:

1. M_n codes for a countable Scott set $\mathcal{M}_n$
2. $\emptyset^{(n)}$ is uniformly coded by an element of $\mathcal{M}_n$
3. Each M_n' is uniformly computable in $\emptyset^{(n+1)}$ ⋆

PROOF: Let us show the following: there is a functional $\Phi : 2^\omega \to 2^\omega$ such that for any oracle X, we have that $M' = \Phi(X')$ is such that $M = \oplus_{n\in\omega} X_n$ codes for a Scott set $\mathcal{M}$ with $X_0 = X$.

Fix a uniformly computable enumeration $\mathcal{C}_0^Y, \mathcal{C}_1^Y, \dots$ of all non-empty $\Pi_1^0(Y)$ classes. Let $\mathcal{D}_X$ be the $\Pi_1^0(X)$ class of all $\bigoplus_n Y_n$ such that $Y_0 = X$ and for every $n = \langle a, b\rangle \in \omega$, $Y_{n+1} \in \mathcal{C}_a^{\bigoplus_{j\leq b} Y_j}$. Note that this $\Pi_1^0(X)$ class is uniform in X and any member of $\mathcal{D}_X$ is a code of a Scott set whose first element is X. Using the Low basis theorem [20], there is a Turing functional Φ such that $\Phi(X')$ is the jump of a member of $\mathcal{D}_X$ for any X.

Using this function Φ, it is clear that uniformly in $\emptyset^{(n+1)}$ one can compute the jump of a set M_n coding for a Scott set $\mathcal{M}_n$ and containing $\emptyset^{(n)}$ as its first element. ■

In order to use the same forcing to show both Theorem 2.1.1 and Theorem 2.1.2, we need to make sure that as long as a fixed X is not computable from $\emptyset^{(n)}$, then we can also build our sequence $\{M_n\}_{n<\omega}$, making sure that X is not computable from M_n.

Proposition 2.2.2 : Let $n \in \omega$ be such that X is not computable from $\emptyset^{(n)}$. There is then a sequence $\{M_n\}_{n<\omega}$ satisfying (1) (2) and (3) of Proposition 2.2.1 with in addition that X is not computable from M_n. ⋆

PROOF: We decompose into two cases. Suppose first that X is not computable from $\emptyset^{(n+1)}$. Then as M_n is computable from $\emptyset^{(n+1)}$ it is clear that M_n does not compute X without any addition to the proof of Proposition 2.2.1.

Suppose now that X is computable from $\emptyset^{(n+1)}$. We only sketch here the idea, more details being available in Theorem 4.1 of [18]: We have to choose M_n carefully, not only using the Low Basis theorem, but also combining it with an effective version of the cone avoidance basis theorem for Π_1^0 class. We describe how to do one step. Let $\mathcal{P}_m$ be a non-empty $\Pi_1^0(\emptyset^{(n)})$ class and Φ_e be a functional. We look for the first a such that $\mathcal{P}_{m+1} = \{Y \in \mathcal{P} : \Phi_e(Y, a) \neq X(a)\}$ is non-empty. Note first that as X is not computable from $\emptyset^{(n)}$ we must find such an a. Note also that the process of checking if $\{Y \in \mathcal{P} : \Phi_e(Y, a) \neq X(a)\}$ is non-empty is $\emptyset^{(n+1)}$ computable as X is $\emptyset^{(n+1)}$-computable and as $\mathcal{P}$ is $\Pi_1^0(\emptyset^{(n)})$. It follows that this steps can be uniformly interspersed in the construction of the low basis theorem. ■

Let us assume that $\{\mathcal{M}_n\}_{n<\omega}$ is a sequence which verifies Proposition 2.2.1. Recall the notation $\langle\mathcal{U}_C^{\mathcal{M}}\rangle$ from the previous section : the unique minimal largeness subclass of an $\mathcal{M}$-cohesive largeness class.

Proposition 2.2.3 : There is a sequence of sets $\{C_n\}_{n\in\omega}$ such that:

1. $\mathcal{U}_{C_n}^{\mathcal{M}_n}$ is an $\mathcal{M}_n$-cohesive largeness class
2. $\mathcal{U}_{C_{n+1}}^{\mathcal{M}_{n+1}} \subseteq \langle\mathcal{U}_{C_n}^{\mathcal{M}_n}\rangle$
3. Each C_n is coded by an element of $\mathcal{M}_{n+1}$ uniformly in n and M_{n+1}. ⋆

In order to prove Proposition 2.2.3 we use the two following uniformity lemmas, which will also be helpful later to continue the sequence of Proposition 2.2.3 through the computable ordinals (see Proposition 3.2.2).

Lemma 2.2.4 : There is a functional $\Phi : 2^\omega \times \omega \to 2^\omega$ such that for any set M coding for a Scott set $\mathcal{M}$, for any e such that $C = \Phi_e(M'')$ is such that $\mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive largeness class, $D = \Phi(M'', e)$ is such that $C \subseteq D$ and $\mathcal{U}_D^{\mathcal{M}} = \langle\mathcal{U}_C^{\mathcal{M}}\rangle$. ⋆

Proof: Say $\mathcal{M} = \{X_0, X_1, \dots\}$ with $M = \bigoplus_i X_i$. Let $\{\langle e_t, i_t\rangle\}_{t\in\omega}$ be an enumeration of $\omega\times\omega$. Suppose that at stage t a finite set $D^t \subseteq \{\langle e_0, i_0\rangle, \dots, \langle e_t, i_t\rangle\}$ has been defined such that $\mathcal{U}_{D^t}^{\mathcal{M}} \cap \mathcal{U}_C^{\mathcal{M}}$ is a largeness class and such that for any $s \leqslant t$, $\langle e_s, i_s\rangle \notin D^t$ implies that $\mathcal{U}_{e_s}^{X_{i_s}} \cap \mathcal{U}_{D^t}^{\mathcal{M}} \cap \mathcal{U}_C^{\mathcal{M}}$ is not a largeness class.

Then at stage $t+1$, we ask M'' if $\mathcal{U}_{e_{t+1}}^{X_{i_{t+1}}} \cap \mathcal{U}_{D^t}^{\mathcal{M}} \cap \mathcal{U}_C^{\mathcal{M}}$ is a largeness class. If so we define $D^{t+1} = D^t \cup \{\langle e_{t+1}, i_{t+1}\rangle\}$. Otherwise we define $D^{t+1} = D^t$. Then $D = C \cup \bigcup_t D^t$ is uniformly M''-computable and $\mathcal{U}_D^{\mathcal{M}}$ equals $\langle\mathcal{U}_C^{\mathcal{M}}\rangle$. ∎

Lemma 2.2.5 : There is a functional $\Phi : 2^\omega\times\omega\times\omega \to \omega$ such that for any set M coding for a Scott set $\mathcal{M}$, for any set N coding for a Scott set $\mathcal{N}$ such that $M' \in \mathcal{N}$ with N-index i_M, for any $C \in \mathcal{N}$ with N-index i_C, such that $\mathcal{U}_C^{\mathcal{M}}$ is a partition regular class, $\Phi(N, i_M, i_C)$ is an N-index for $D \supseteq C$ such that $\mathcal{U}_D^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive largeness class. ⋆

Proof: The functional Φ does the following : It looks for M' at index i_M inside $\mathcal{N}$. From M' it computes $M = \oplus_n X_n$. It then computes with $M' + C$ the tree T containing all the elements σ such that

$$\left(\bigcap_{\sigma(i)=0} 2^\omega - X_i\right) \cap \left(\bigcap_{\sigma(i)=1} X_i\right) \in \bigcap_{\langle e,j\rangle\in C\restriction|\sigma|} \mathcal{U}_e^{X_j}$$

Clearly $[T]$ is not empty. The functional Φ then finds an N-index for an element $Y \in [T]$. For $\sigma \prec Y$ let $X_\sigma = (\bigcap_{\sigma(i)=0}(2^\omega - X_i)) \cap (\bigcap_{\sigma(i)=0} X_i)$. We must have for every $\sigma \prec Y$ that $X_\sigma \in \mathcal{U}_C^{\mathcal{M}}$. It follows as $\mathcal{U}_C^{\mathcal{M}}$ is partition regular, that for every $\sigma \prec Y$, $\mathcal{L}_{X_\sigma} \cap \mathcal{U}_C^{\mathcal{M}}$ is a largeness class. Thus $\bigcap_{\sigma\prec Y} \mathcal{L}_{X_\sigma} \cap \mathcal{U}_C^{\mathcal{M}}$ is an $\mathcal{M}$-cohesive largeness class. Also $M \oplus Y \oplus C$ uniformly computes a set D such that $\mathcal{U}_D^{\mathcal{M}} = \bigcap_{\sigma\prec Y} \mathcal{L}_{X_\sigma} \cap \mathcal{U}_C^{\mathcal{M}}$. The function Φ then returns an N-index for D. ∎

Proof (Proof of Proposition 2.2.3): Suppose that stage n we have defined C_n verifying (1)(2) and (3). Let us define C_{n+1}.

Note that the set C_n is coded by an element of $\mathcal{M}_{n+1}$, and thus that C_n is computable in $\emptyset^{(n+2)}$ and then computable in M_n''. Using Lemma 2.2.4 we define $D_n \supseteq C_n$ to be such that $\mathcal{U}_{D_n}^{\mathcal{M}_n} = \langle \mathcal{U}_{C_n}^{\mathcal{M}_n} \rangle$ and such that D_n is uniformly M_n''-computable. We define E_{n+1} to be the transfer of the M_n-indices constituting D_n into M_{n+1}-indices, using that M_n is an element of M_{n+1}. So we have $\mathcal{U}_{E_{n+1}}^{\mathcal{M}_{n+1}} = \mathcal{U}_{D_n}^{\mathcal{M}_n}$.

Note that as E_{n+1} is computable in $M_n'' \oplus M_{n+1}$ and thus in $\emptyset^{((n+1)+1)}$. It is then coded by an element of $\mathcal{M}_{(n+1)+1}$. Note also that $\mathcal{U}_{E_{n+1}}^{\mathcal{M}_{n+1}}$ is partition regular as it equals $\langle \mathcal{U}_{C_n}^{\mathcal{M}_n} \rangle$. Using Lemma 2.2.5 we uniformly find an $\mathcal{M}_{(n+1)+1}$-index of $C_{n+1} \supseteq E_{n+1}$ to be such that $\mathcal{U}_{C_{n+1}}^{\mathcal{M}_{n+1}}$ is an $\mathcal{M}_{n+1}$-cohesive largeness class. ■

2.3 The forcing

We now describe the forcing that will be used to show Theorem 2.1.1 and Theorem 2.1.2. Let us fix any set Z and let $A^0 \cup A^1 \supseteq \omega$ be any sets.

Using Proposition 2.2.2, let us fix a sequence $\{M_n\}_{n<\omega}$ verifying Proposition 2.2.1 and such that as long as Z is not computable from $\emptyset^{(n)}$, then Z is not computable from M_n. Let us fix a sequence $\{C_n\}_{n<\omega}$ verifying Proposition 2.2.3. Let $\mathcal{S} = \bigcap_{n<\omega} \mathcal{U}_{C_n}^{\mathcal{M}_n} = \bigcap_{n<\omega} \langle \mathcal{U}_{C_n}^{\mathcal{M}_n} \rangle$. Note that as an intersection of partition regular class, S is also a partition regular class.

As $\mathcal{S}$ is a largeness class, there must be some $i < 2$ such that $A^i \in \mathcal{S}$. Let then $A = A^i$ for some i such that $A^i \in \mathcal{S}$. Note that this is where we get around the disjunctive requirement : inside whichever belongs to S among A^0 or A^1, we guaranty the possibility of building an infinite subset G such that as long as Z is not computable from $\emptyset^{(n)}$, it is not computable from $G^{(n)}$.

Definition 2.3.1 : Let $\mathbb{P}_\omega$ be the set of conditions (σ, X) such that:

1. (σ, X) is a Mathias condition
2. $\sigma \subseteq A$
3. $X \subseteq A$
4. $X \in \mathcal{S}$

Given two conditions $(\sigma, X), (\tau, Y) \in \mathbb{P}_\omega$ we let $(\sigma, X) \preccurlyeq (\tau, Y)$ be the usual Mathias extension, that is, $\sigma \succcurlyeq \tau$, $X \subseteq Y$ and $\sigma - \tau \subseteq Y$. ◊

We now define an abstract forcing question for Σ_n^0 formulas. The Δ_0 formulas we manipulate have one set parameter. Also given a Σ_n^0 formulas, such as for instance $\exists x\ \forall y\ \Phi(G, x, y)$ were Φ is Δ_0, we often consider the Σ_n^0 class of elements of 2^ω making the formula true, rather than the formula itself, as done in the following definition.

Definition 2.3.2 : Let $\sigma \in 2^{<\omega}$. Given a Σ_1^0 class $\mathcal{U}$, let $\sigma \operatorname{?\vdash} \mathcal{U}$ holds if

$$\{Y\ :\ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\}\ [\sigma \cup \tau] \subseteq \mathcal{U}\} \cap \mathcal{U}_{C_0}^{\mathcal{M}_0}$$

is a largeness class. Then inductively, given a Σ_m^0 class $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ with $1 < m < \omega$, we let $\sigma \operatorname{?\vdash} \mathcal{B}$ holds if

$$\{Y\ :\ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\}\ \exists n\ \sigma \cup \tau \operatorname{?\nvdash} 2^\omega - \mathcal{B}_n\} \cap \mathcal{U}_{C_{m-1}}^{\mathcal{M}_{m-1}}$$

is a largeness class.

For a condition $p = (\sigma, X) \in \mathbb{P}_\omega$ and a Σ^0_m set $\mathcal{B}$ for some m, we write $p \,?\!\vdash \mathcal{B}$ if $\sigma \,?\!\vdash \mathcal{B}$. $\diamond$

We shall now study the effectivity of the relation $?\!\vdash$. To do so we introduce the following notation.

Definition 2.3.3 : Let $\sigma \in 2^{<\omega}$. Given a Σ^0_1 class $\mathcal{B}$, we write $\mathcal{U}(\mathcal{B}, \sigma)$ for the open set:

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ [\sigma \cup \tau] \subseteq \mathcal{B}\}$$

Given a Σ^0_m class $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$, we write $\mathcal{U}(\mathcal{B}, \sigma)$ for the open set:

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \,?\!\nvdash 2^\omega - \mathcal{B}_n\}$$

Let us now study the complexity of the relation $?\!\vdash$ together with the complexity of the sets $\mathcal{U}(\mathcal{B}, \sigma)$.

Proposition 2.3.4 : Let $\sigma \in 2^{<\omega}$. Let $\mathcal{B}$ be a Σ^0_m class for $0 < m < \omega$

1. The set $\mathcal{U}(\mathcal{B}, \sigma)$ is an upward-closed $\Sigma^0_1(C_{m-2} \oplus \emptyset^{(m-1)})$ open set if $m > 1$ and an upward-closed Σ^0_1 open set if $m = 1$.

2. The relation $\sigma \,?\!\vdash \mathcal{B}$ is $\Pi^0_1(C_{m-1} \oplus \emptyset^{(m)})$.

This is uniform in σ and a code for the class $\mathcal{B}$. $\star$

Proof: This is done by induction on the effective Borel codes. We start with $m = 1$. Let $\mathcal{V}$ be a Σ^0_1 class and $\sigma \in 2^{<\omega}$. It is clear that

$$\mathcal{U}(\mathcal{V}, \sigma) = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ [\sigma \cup \tau] \subseteq \mathcal{V}\}$$

is an upward closed Σ^0_1 class. Then $\sigma \,?\!\vdash \mathcal{V}$ iff $\mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}^{\mathcal{M}_0}_{C_0}$ is a largeness class, that is, iff for every finite set $F \subseteq C_0$, the class $\mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}^{\mathcal{M}_0}_F$ is a largeness class. By Corollary 1.4.9, for each $F \subseteq C_0$, the statement is $\Pi^0_2(M_0)$ uniformly in F, and thus $\Pi^0_1(M'_0)$ uniformly in F. It is then $\Pi^0_1(\emptyset')$ uniformly in F. Thus the whole statement is $\Pi^0_1(C_0 \oplus \emptyset')$.

Suppose (1) and (2) are true for m, every Σ^0_m class and every σ. Let $\sigma \in 2^{<\omega}$ and let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ be a Σ^0_{m+1} class. Let

$$\mathcal{U}(\mathcal{B}, \sigma) = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \,?\!\nvdash 2^\omega - \mathcal{B}_n\}$$

Let us show (1). For each $n \in \omega$, the class $2^\omega - \mathcal{B}_n$ is a Σ^0_m class uniformly in $\sigma \cup \tau$ and in a code for $\mathcal{B}_n$. By induction hypothesis, the relation $\sigma \cup \tau \,?\!\nvdash 2^\omega - \mathcal{B}_n$ is $\Sigma^0_1(C_{m-1} \oplus \emptyset^{(m)})$. It follows that $\mathcal{U}(\mathcal{B}, \sigma)$ is an upward closed $\Sigma^0_1(C_{m-1} \oplus \emptyset^{(m)})$ class.

Let us now show (2). We have that $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_m}_{C_m}$ is a largeness class if for all $F \subseteq C_m$, the class $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_m}_F$ is a largeness class. By Corollary 1.4.9, it is a $\Pi^0_2(M_m)$ statement uniformly in F and then a $\Pi^0_1(M'_m)$ statement uniformly in F and then a $\Pi^0_1(\emptyset^{(m+1)})$ statement uniformly in F. It follows that the statement "$\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}^{\mathcal{M}_m}_{C_m}$ is a largeness class" is $\Pi^0_1(C_m \oplus \emptyset^{(m+1)})$. ∎

We now define the forcing relation. Again the relation is defined on Σ^0_m sets rather than defined on Σ^0_m.

Definition 2.3.5 : Let $(\sigma, X) \in \mathbb{P}_\omega$. Let $\mathcal{U}$ be a Σ^0_1 class. We define

$$\begin{array}{lcccl} (\sigma, X) & \Vdash & \mathcal{U} & \leftrightarrow & [\sigma] \subseteq \mathcal{U} \\ (\sigma, X) & \Vdash & 2^\omega - \mathcal{U} & \leftrightarrow & \forall \tau \subseteq X \ [\sigma \cup \tau] \nsubseteq \mathcal{U} \end{array}$$

Then inductively for Σ^0_m classes $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ for $m > 1$, we define:

$$\begin{array}{lcccl} (\sigma, X) & \Vdash & \mathcal{B} & \leftrightarrow & \exists n \ (\sigma, X) \Vdash \mathcal{B}_n \\ (\sigma, X) & \Vdash & 2^\omega - \mathcal{B} & \leftrightarrow & \forall n \ \forall \tau \subseteq X \ \sigma \cup \tau \ ?\!\vdash 2^\omega - \mathcal{B}_n \end{array}$$

We now show a couple of useful and classical properties on the forcing relation.

Lemma 2.3.6 : Let $p \in \mathbb{P}_\omega$. Let $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_n$ be a Π^0_m class for $m > 1$. Then $p \Vdash \bigcap_{n<\omega} \mathcal{B}_n$ iff for every $n \in \omega$ and every $q \leqslant p$ we have $q \ ?\!\vdash \mathcal{B}_n$. ⋆

Proof: Suppose $p \Vdash \bigcap_{n<\omega} \mathcal{B}_n$ with $p = (\sigma, X)$. By definition of the forcing relation it is clear that for every n and every $q \leqslant p$ we have $q \ ?\!\vdash \mathcal{B}_n$. Suppose now that for every n and every $q \leqslant p$ we have $q \ ?\!\vdash \mathcal{B}_n$. Given any $\tau \subseteq X$ we have that $(\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\})$ is a valid extension of p for which we have $\sigma \cup \tau \ ?\!\vdash \mathcal{B}_n$ for every n. It follows that $p \Vdash \bigcap_{n<\omega} \mathcal{B}_n$.■

Proposition 2.3.7 : Let $p \in \mathbb{P}_\omega$. Let $\mathcal{B}$ be a Σ^0_m class for some $m > 0$. If $p \Vdash \mathcal{B}$ and $q \leqslant p$ then $q \Vdash \mathcal{B}$. ⋆

Proof: It is clear for Σ^0_1 and Π^0_1 classes. We proceed by induction on m. For $m > 1$ suppose $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ is a Σ^0_m class. By definition, there is some $n \in \omega$ such that $p \Vdash \mathcal{B}_n$. As $\mathcal{B}_n$ is a Π^0_{m-1}, by induction hypothesis, $q \Vdash \mathcal{B}_n$ and thus $q \Vdash \mathcal{B}$.

Suppose now $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_n$ is a Π^0_m class. By Lemma 2.3.6, for all $n \in \omega$ and all $r \leqslant p$, $r \ ?\!\vdash \mathcal{B}_n$. Thus if $q \leqslant p$, also for all n and all $r \leqslant q$, $r \ ?\!\vdash \mathcal{B}_n$. It follows that $q \Vdash \bigcap_{n<\omega} \mathcal{B}_n$. ■

We now show a key lemma, showing that the forcing question $?\!\vdash$, computationally simple, can decide if the corresponding formula can actually be forced, or if it its negation which can be forced. This lemma can be considered as the core of the proof.

Lemma 2.3.8 : Let $p \in \mathbb{P}_\omega$ with $p = (\sigma, X)$. Let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ be a Σ^0_m class for $m > 0$.

1. Suppose $p \ ?\!\vdash \mathcal{B}$. Then there exists $q \leqslant p$ such that $q \Vdash \mathcal{B}$.
2. Suppose $p \ ?\!\nvdash \mathcal{B}$. Then there exists $q \leqslant p$ such that $q \Vdash 2^\omega - \mathcal{B}$. ⋆

Proof: Let $p \in \mathbb{P}_\omega$. We start with $m = 1$. Let $\mathcal{V}$ be a Σ^0_1 class and suppose $p \ ?\!\vdash \mathcal{V}$. Let

$$\mathcal{U}(\mathcal{V}, \sigma) = \{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ [\sigma \cup \tau] \subseteq \mathcal{V}\}$$

The class $\mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}^{\mathcal{M}_0}_{C_0}$ is a largeness class. As $\mathcal{U}^{\mathcal{M}_0}_{C_0}$ is $\mathcal{M}_0$-cohesive, then $\langle \mathcal{U}^{\mathcal{M}_0}_{C_0} \rangle \subseteq \mathcal{U}(\mathcal{V}, \sigma)$. As $X \in \mathcal{S} \subseteq \langle \mathcal{U}^{\mathcal{M}_0}_{C_0} \rangle \subseteq \mathcal{U}(\mathcal{V}, \sigma)$, there is $\tau \subseteq X$ such that $[\sigma \cup \tau] \subseteq \mathcal{V}$. As $\mathcal{S}$ contains only infinite sets and is partition regular, $X - \{0, \dots, |\sigma \cup \tau|\} \in \mathcal{S}$. Then $(\sigma \cup \tau, X - \{0, \dots, \sigma \cup \tau\})$ is a valid extension of (σ, X) such that $(\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\}) \Vdash \mathcal{U}$.

Suppose now that $\sigma \ ?\!\nvdash \mathcal{U}$. The class $\mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}^{\mathcal{M}_0}_{C_0}$ is not a largeness class. It follows that there is a k-cover $Y_0 \cup \dots \cup Y_k \supseteq \omega$ such that $Y_i \notin \mathcal{U}(\mathcal{V}, \sigma) \cap \mathcal{U}^{\mathcal{M}_0}_{C_0}$ for each $i \leqslant k$. As

$\mathcal{S}$ is partition regular and as $X \in \mathcal{S}$ we have some $i \leqslant k$ such that $Y_i \cap X \in \mathcal{S} \subseteq \mathcal{U}_{C_0}^{\mathcal{M}_0}$. It follows that $Y_i \cap X \notin \mathcal{U}(\mathcal{V}, \sigma)$. Note that $(\sigma, Y_i \cap X)$ is a valid extension of (σ, X) for which $(\sigma, Y_i \cap X) \Vdash 2^\omega - \mathcal{V}$.

Suppose now $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ is a Σ^0_{m+1} class for $m > 0$. Suppose $\sigma\, ?{\vdash} \bigcup_{n<\omega} \mathcal{B}_n$. Let

$$\mathcal{U}(\mathcal{B}, \sigma) = \{Y \ :\ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\}\ \exists n\ \sigma \cup \tau\, ?{\nvdash}\, 2^\omega - \mathcal{B}_n\}$$

By definition, the class $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}_{C_m}^{\mathcal{M}_m}$ is a largeness class. As $\mathcal{U}_{C_m}^{\mathcal{M}_m}$ is $\mathcal{M}_m$-cohesive and as, by Proposition 2.3.4, the set $\mathcal{U}(\mathcal{B}, \sigma)$ is a $\Sigma^0_1(Y)$ for some $Y \in \mathcal{M}_m$, then $\langle \mathcal{U}_{C_m}^{\mathcal{M}_m} \rangle \subseteq \mathcal{U}(\mathcal{B}, \sigma)$. As $X \in \mathcal{S} \subseteq \langle \mathcal{U}_{C_m}^{\mathcal{M}_m} \rangle \subseteq \mathcal{U}(\mathcal{B}, \sigma)$, there is $\tau \subseteq X$ such that $\sigma \cup \tau\, ?{\nvdash}\, 2^\omega - \mathcal{B}_n$ for some n. Note that as $\mathcal{S}$ contains only infinite sets and is partition regular we have $X - \{0, \dots, |\sigma \cup \tau|\} \in \mathcal{S}$. Also $(\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\})$ is a valid extension of (σ, X) such that $(\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\})\, ?{\nvdash}\, 2^\omega - \mathcal{B}_n$. By induction hypothesis we have some $q \preccurlyeq (\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\})$ such that $q \Vdash \mathcal{B}_n$. It follows that $q \Vdash \mathcal{B}$.

Suppose now $\sigma\, ?{\nvdash} \bigcup_{n<\omega} \mathcal{B}_n$. It follows that $\mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}_{C_m}^{\mathcal{M}_m}$ is not a largeness class. It follows that there is a k-cover $Y_0 \cup \dots \cup Y_k \supseteq \omega$ such that $Y_i \notin \mathcal{U}(\mathcal{B}, \sigma) \cap \mathcal{U}_{C_m}^{\mathcal{M}_m}$ for each $i \leqslant k$. As $\mathcal{S}$ is partition regular and as $X \in \mathcal{S}$, there is some $i \leqslant k$ such that $Y_i \cap X \in \mathcal{S} \subseteq \mathcal{U}_{C_m}^{\mathcal{M}_m}$. It follows that $Y_i \cap X \notin \mathcal{U}(\mathcal{B}, \sigma)$. It means that for every $\tau \subseteq Y_i \cap X$ and every $n \in \omega$, $\sigma \cup \tau\, ?{\vdash}\, 2^\omega - \mathcal{B}_n$. It follows that $(\sigma, Y_i \cap X) \Vdash \bigcap_{n<\omega} 2^\omega - \mathcal{B}_n$. ∎

Proposition 2.3.9 : Let $\mathcal{F} \subseteq \mathbb{P}_\omega$ be a sufficiently generic filter. Then there is a unique set $G_\mathcal{F} \in 2^\omega$ such that for every $(\sigma, X) \in \mathcal{F}$ we have $\sigma \prec G_\mathcal{F}$. ⋆

PROOF: Trivial. ∎

Notation

Let $\mathcal{F} \subseteq \mathbb{P}_\omega$ be a sufficiently generic filter. We write $G_\mathcal{F} \in 2^\omega$ for the set of the previous proposition.

We now show that forcing implies truth.

Theorem 2.3.10:
Let $\mathcal{F} \subseteq \mathbb{P}_\omega$ be a generic enough filter. Let $p \in \mathcal{F}$. Let $\mathcal{B}$ be a Σ^0_m class for $m > 0$. Suppose $p \Vdash \mathcal{B}$. Then $G_\mathcal{F} \in \mathcal{B}$. Suppose $p \Vdash 2^\omega - \mathcal{B}$. Then $G_\mathcal{F} \in 2^\omega - \mathcal{B}$.

PROOF: By induction on m. Let $p \in \mathbb{P}_\omega$ with $p = (\sigma, X)$. We start with $m = 1$. Let $\mathcal{U}$ be a Σ^0_1 class. Suppose $p \Vdash \mathcal{U}$, that is $[\sigma] \subseteq \mathcal{U}$. Then clearly $G_\mathcal{F} \in \mathcal{U}$. Suppose now $p \Vdash 2^\omega - \mathcal{U}$, that is, $[\sigma \cup \tau] \nsubseteq \mathcal{U}$ for all $\tau \subseteq X$. Then also $G_\mathcal{F} \in 2^\omega - \mathcal{U}$ easily.

Let now $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ be a Σ^0_{m+1} class. Suppose $p \Vdash \bigcup_{n<\omega} \mathcal{B}_n$. Then there exists n such that $p \Vdash \mathcal{B}_n$. By induction hypothesis we have if $\mathcal{F}$ is sufficiently generic, then $G_\mathcal{F} \in \mathcal{B}_n \subseteq \bigcup_{n<\omega} \mathcal{B}_n$.

Let now $\mathcal{B}$ be a Π^0_{m+1} class. Suppose $p \Vdash \bigcap_{n<\omega} \mathcal{B}_n$. Then by Lemma 2.3.6 for every n and every $q \preccurlyeq p$, $q\, ?{\vdash}\, \mathcal{B}_n$. From Lemma 2.3.8, for every $n \in \omega$ and every $q \preccurlyeq p$, there is some $r \preccurlyeq q$ such that $r \Vdash \mathcal{B}_n$. It follows that for every n, the set $\{r \in \mathbb{P}_\omega \ :\ r \Vdash \mathcal{B}_n\}$ is dense below p. If $\mathcal{F}$ is sufficiently generic, for every $n \in \omega$, there is some $r \in \mathcal{F}$ such that $r \Vdash \mathcal{B}_n$. By induction hypothesis, if $\mathcal{F}$ is sufficiently generic, then for every $n \in \omega$, $G_\mathcal{F} \in \mathcal{B}_n$. It follows that $G_\mathcal{F} \in \bigcap_{n<\omega} \mathcal{B}_n$. ∎

2.4 Preservation of arithmetic reductions

We now turn to the proof of Theorem 2.1.1 together with Theorem 2.1.2.

Proposition 2.4.1 : *Let Φ be a functional. Let $n, i \in \omega$. Let $m \geqslant 0$. The set $\{X : \exists t\ \Phi(X^{(m)}, n)[t] \downarrow = i\}$ is a Σ^0_{m+1} class.* ⋆

Proof: In case this is not obvious, a generalization will be proved with Proposition 3.1.2.■

Theorem (2.1.2): *Let Z be non arithmetically definable. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not arithmetically definable relative to G.*

Proof: Let Φ be a functional. Let $m \geqslant 0$. Let $\mathcal{B}^n = \{X : \Phi(X^{(m)}, n) \downarrow\}$. We want to show that $Z \neq \{n : G_{\mathcal{F}}^{(m)} \in \mathcal{B}^n\}$. From Proposition 2.4.1, $\mathcal{B}^n$ is Σ^0_{m+1}.

Let $p \in \mathbb{P}_\omega$ be a condition. From Proposition 2.3.4, the set $\{n : p \mathrel{?\vdash} \mathcal{B}^n\}$ is $\Pi^0_1(C_m \oplus \emptyset^{(m+1)})$. As Z is not $\Pi^0_1(C_m \oplus \emptyset^{(m+1)})$, then there is some $n \in Z$ such that $p \mathrel{?\nvdash} \mathcal{B}^n$ or some $n \notin Z$ such that $p \mathrel{?\vdash} \mathcal{B}^n$. In the first case, by Lemma 2.3.8, there is an extension $q \leqslant p$ such that $q \Vdash 2^\omega - \mathcal{B}^n$ for some $n \in Z$. In the second case, by Lemma 2.3.8, there is an extension $q \leqslant p$ such that $q \Vdash \mathcal{B}^n$ for some $n \notin Z$. By Theorem 2.3.10, in the first case $\Phi(G_{\mathcal{F}}^{(m)}, n) \uparrow$ holds for some $n \in Z$, and in the second case, $\Phi(G_{\mathcal{F}}^{(m)}, n) \downarrow$ holds for some $n \notin Z$.

If $\mathcal{F}$ is sufficiently generic, this is true for any m and any functional Φ. It follows that for any m the set Z is not $\Sigma^0_1(G_{\mathcal{F}}^{(m)})$ and thus not $\Delta^0_1(G_{\mathcal{F}}^{(m)})$. ■

We now turn to the proof of Theorem 2.1.1. which is a bit more complicated, due to the fact that the complexity of the relation $?\vdash$ is too big to have a direct simple proof. Let us take the example of $m = 0$. Z is not computable and the forcing question for Σ^0_1 sentences is $\Pi^0_1(C_0 \oplus \emptyset')$ for C_0 low relative to $\emptyset'$. In the standard proof of cone avoidance from Dzhafarov and Jockusch, the forcing question for Σ^0_1 sentences is $\Sigma^0_1(X)$ for X which does not compute Z, and this is the right complexity to have a direct proof.

Here the question is too complex (Z could for instance be computable in C_0) and forces to have extra-complexity in the proof of cone avoidance. As the full proof may be a bit abstract at first, we start with the case $m = 0$.

Theorem (2.1.1 for $m = 0$): *Let Z be non-computable. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not G-computable.*

Proof: Recall that our set Z was fixed with in particular Z not $\Delta^0_1(M_0)$. Let Φ be a functional. Let $\mathcal{B}^{0,n} = \{X : \Phi(X, n) \downarrow = 0\}$ and let $\mathcal{B}^{1,n} = \{X : \Phi(X, n) \downarrow = 1\}$. We want to show that $Z \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{1,n}\}$ or $\omega - Z \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{0,n}\}$ whenever $G_{\mathcal{F}}$ is generic enough. For each n the sets $\mathcal{B}^{0,n}$ and $\mathcal{B}^{1,n}$ are Σ^0_1 classes.

Let $p = (\sigma, X)$ be a forcing condition. We basically want to find $q \leqslant p$ such that $q \Vdash \mathcal{B}^{0,n}$ for some $n \in Z$ or such that $q \Vdash \mathcal{B}^{1,n}$ for some $n \notin Z$. By Theorem 2.3.10 the result would follow. Let us show we can always find such an extension q.

Suppose first that $\mathcal{A} \cap \mathcal{U}_{C_0}^{\mathcal{M}_0}$ is a largeness class, where

$$\mathcal{A} = \{Y : \exists \tau_0, \tau_1 \subseteq Y - \{0, \ldots, |\sigma|\}\ \exists n\ [\sigma \cup \tau_0] \subseteq \mathcal{B}^{0,n} \wedge [\sigma \cup \tau_1] \subseteq \mathcal{B}^{1,n}\}$$

As $\mathcal{U}_{C_0}^{\mathcal{M}_0}$ is $\mathcal{M}_0$-cohesive we must have $\langle \mathcal{U}_{C_0}^{\mathcal{M}_0} \rangle \subseteq \mathcal{A}$ and thus $X \in \langle \mathcal{U}_{C_0}^{\mathcal{M}_0} \rangle \subseteq \mathcal{A}$. Then there is $\tau_0, \tau_1 \subseteq X$ and n such that $[\sigma \cup \tau_0] \subseteq \mathcal{B}^{0,n}$ and $[\sigma \cup \tau_1] \subseteq \mathcal{B}^{1,n}$. If $n \in Z$ we let $q = (\sigma \cup \tau_0, X - \{0, \dots, |\sigma \cup \tau_0|\})$ and if $n \notin Z$ we let $q = (\sigma \cup \tau_1, X - \{0, \dots, |\sigma \cup \tau_1|\})$. We have $q \leqslant p$. In the first case $q \Vdash \mathcal{B}^{0,n}$ and in the second case $q \Vdash \mathcal{B}^{1,n}$. For a sufficiently generic filter $\mathcal{F}$ containing q, in the first case we have $G_{\mathcal{F}} \in \mathcal{B}^{0,n}$ for $n \in Z$ and then $G_{\mathcal{F}} \notin \mathcal{B}^{1,n}$ for $n \in Z$. Then $Z \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{1,n}\}$. Symmetrically in the second case we have $\omega - Z \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{0,n}\}$.

Suppose now that $\mathcal{A} \cap \mathcal{U}_{C_0}^{\mathcal{M}_0}$ is not a largeness class. Let

$$F_0 = \{n : \forall q \leqslant p \ \exists r \leqslant q \ r \Vdash \mathcal{B}^{0,n}\} \quad \text{and} \quad F_1 = \{n : \forall q \leqslant p \ \exists r \leqslant q \ r \Vdash \mathcal{B}^{1,n}\}$$

Suppose first that $F_1 \neq Z$ or $F_0 \neq \omega - Z$. Suppose first $F_1 \neq Z$. If there is n such that $n \notin Z$ and $n \in F_1$, then $r \Vdash \mathcal{B}^{1,n}$ for some $r \leqslant p$ and for a sufficiently generic filter $\mathcal{F}$ containing r we have $Z \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{1,n}\}$. If there is n such that $n \in Z$ and $n \notin F_1$, then there exists $q \leqslant p$ such that for all $r \leqslant q$ we have $r \nVdash \mathcal{B}^{1,n}$. Thus there must be $r \leqslant q$ such that $r \Vdash 2^\omega - \mathcal{B}^{1,n}$. It follows that for a sufficiently generic filter $\mathcal{F}$ containing r we have $Z \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{1,n}\}$.

Suppose now $F_0 \neq \omega - Z$. Symmetrically we have a condition $r \leqslant p$ such that $\omega - Z \neq \{n : G_{\mathcal{F}} \in \mathcal{B}^{0,n}\}$ for a sufficiently generic filter $\mathcal{F}$ containing r. Suppose now for contradiction that:

(1) $F_1 = Z$ and $F_0 = \omega - Z$

As $\mathcal{A} \cap \mathcal{U}_{C_0}^{\mathcal{M}_0}$ is not a largeness class, there must be a cover $Y_0 \cup \dots \cup Y_k \in \mathcal{M}_0$ such that $Y_i \notin \mathcal{A} \cap \mathcal{U}_{C_0}^{\mathcal{M}_0}$ for every $i \leqslant k$.

As $\mathcal{S} \subseteq \langle \mathcal{U}_{C_0}^{\mathcal{M}_0} \rangle$ is partition regular there must be $i \leqslant k$ such that $X \cap Y_i \in \mathcal{S} \subseteq \langle \mathcal{U}_{C_0}^{\mathcal{M}_0} \rangle$. We also have $Y_i \in \mathcal{U}_{C_0}^{\mathcal{M}_0}$ and then $Y_i \notin \mathcal{A}$. Thus

(2) For all n, for all $\tau_0, \tau_1 \subseteq Y_i - \{0, \dots, |\sigma|\}$ the following holds:

$$[\sigma \cup \tau_0] \nsubseteq \mathcal{B}^{0,n} \text{ or } [\sigma \cup \tau_1] \nsubseteq \mathcal{B}^{1,n}$$

We shall now argue that for all $n \in Z$ there exists $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ such that $[\sigma \cup \tau] \subseteq \mathcal{B}^{1,n}$. Let $n \in Z$. If not then for every $r \leqslant (\sigma, X \cap Y_i)$ we have $r \nVdash \mathcal{B}^{1,n}$ which contradicts (1).

Symmetrically, we show that for all $n \in \omega - Z$ there exists $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ such that $[\sigma \cup \tau] \subseteq \mathcal{B}^{0,n}$. Therefore, for every $n \in Z$ we have using (2) that:

1. There exists some $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ such that $[\sigma \cup \tau] \subseteq \mathcal{B}^{1,n}$
2. For all $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ we have $[\sigma \cup \tau] \nsubseteq \mathcal{B}^{0,n}$

Symmetrically, for every $n \notin Z$ we prove, using (2), that:

1. There exists some $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ such that $[\sigma \cup \tau] \subseteq \mathcal{B}^{0,n}$
2. For all $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ we have $[\sigma \cup \tau] \nsubseteq \mathcal{B}^{1,n}$

We can now compute Z as follows : For each $n \in \omega$, look for some $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ such that either $[\sigma \cup \tau] \subseteq \mathcal{B}^{0,n}$ or $[\sigma \cup \tau] \subseteq \mathcal{B}^{1,n}$. This is a $\Sigma_1^0(M_0)$ event. Thus Z is $\Delta_1^0(M_0)$, which is a contradiction. ■

We now turn to the full proof, which is essentially the same

Theorem (2.1.1): *Let $m \geqslant 0$. Let Z be non $\emptyset^{(m)}$-computable. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not $G^{(m)}$-computable.*

Proof: The case $m = 0$ has been handled. We suppose now $m > 0$. Recall that our set Z was fixed with in particular Z not $\Delta^0_1(M_m)$. Let Φ be a functional. Let $\mathcal{B}^{0,n} = \{X : \Phi(X^{(m)}, n) \downarrow = 0\}$ and let $\mathcal{B}^{1,n} = \{X : \Phi(X^{(m)}, n) \downarrow = 1\}$. We want to show that $Z \neq \{n : G_\mathcal{F} \in \mathcal{B}^{1,n}\}$ or $\omega - Z \neq \{n : G_\mathcal{F} \in \mathcal{B}^{0,n}\}$ whenever $G_\mathcal{F}$ is generic enough. From Proposition 2.4.1, for each n the sets $\mathcal{B}^{0,n}$ and $\mathcal{B}^{1,n}$ are Σ^0_{m+1} classes.

Let $p = (\sigma, X)$ be a forcing condition. We basically want to find $q \leqslant p$ such that $q \Vdash \mathcal{B}^{0,n}$ for some $n \in Z$ or such that $q \Vdash \mathcal{B}^{1,n}$ for some $n \notin Z$. By Theorem 2.3.10 the result would follow. Let us show we can always find such an extension q.

For each $n \in \omega$, let $\mathcal{B}^{0,n} = \bigcup_{a \in \omega} \mathcal{B}^{0,n}_a$ and $\mathcal{B}^{1,n} = \bigcup_{b \in \omega} \mathcal{B}^{1,n}_b$. Suppose first that $\mathcal{A} \cap \mathcal{U}^{\mathcal{M}_m}_{C_m}$ is a largeness class, where

$$\mathcal{A} = \{Y : \exists \tau_0, \tau_1 \subseteq Y - \{0, \dots, |\sigma|\} \ \exists \langle n, a, b\rangle \ \sigma \cup \tau_0 \,?\!\nvdash 2^\omega - \mathcal{B}^{0,n}_a \wedge \sigma \cup \tau_1 \,?\!\nvdash 2^\omega - \mathcal{B}^{1,n}_b\}$$

As $\mathcal{U}^{\mathcal{M}_m}_{C_m}$ is $\mathcal{M}_m$-cohesive we must have $\langle \mathcal{U}^{\mathcal{M}_m}_{C_m} \rangle \subseteq \mathcal{A}$ and thus $X \in \langle \mathcal{U}^{\mathcal{M}_m}_{C_m} \rangle \subseteq \mathcal{A}$. Then there is $\tau_0, \tau_1 \subseteq X$ and n, a, b such that $\sigma \cup \tau_0 \,?\!\nvdash 2^\omega - \mathcal{B}^{0,n}_a$ and $\sigma \cup \tau_1 \,?\!\nvdash 2^\omega - \mathcal{B}^{1,n}_b$. If $n \in Z$ we let $q = (\sigma \cup \tau_0, X - \{0, \dots, |\sigma \cup \tau_0|\})$ and if $n \notin Z$ we let $q = (\sigma \cup \tau_1, X - \{0, \dots, |\sigma \cup \tau_1|\})$. We have $q \leqslant p$. By Lemma 2.3.8, in the first case we have some $r \leqslant q$ such that $r \Vdash \mathcal{B}^{0,n}$ and in the second case some $r \leqslant q$ such that $r \Vdash \mathcal{B}^{1,n}$. For a sufficiently generic filter $\mathcal{F}$ containing r, in the first case we have $G_\mathcal{F} \in \mathcal{B}^{0,n}$ for $n \in Z$ and then $G_\mathcal{F} \notin \mathcal{B}^{1,n}$ for $n \in Z$. Then $Z \neq \{n : G_\mathcal{F} \in \mathcal{B}^{1,n}\}$. Symmetrically in the second case we have $\omega - Z \neq \{n : G_\mathcal{F} \in \mathcal{B}^{0,n}\}$.

Suppose now that $\mathcal{A} \cap \mathcal{U}^{\mathcal{M}_m}_{C_m}$ is not a largeness class. Let

$$F_0 = \{n : \forall q \leqslant p \ \exists r \leqslant q \ r \Vdash \mathcal{B}^{0,n}\} \quad \text{and} \quad F_1 = \{n : \forall q \leqslant p \ \exists r \leqslant q \ r \Vdash \mathcal{B}^{1,n}\}$$

Suppose first $F_1 \neq Z$ or $F_0 \neq \omega - Z$. Suppose first $F_1 \neq Z$. If there is n such that $n \notin Z$ and $n \in F_1$, then $r \Vdash \mathcal{B}^{1,n}$ for some $r \leqslant p$ and we have $Z \neq \{n : G_\mathcal{F} \in \mathcal{B}^{1,n}\}$ for a sufficiently generic filter $\mathcal{F}$ containing r. If there is n such that $n \in Z$ and $n \notin F_1$, then for some $q \leqslant p$ and all $r \leqslant q$ we have $r \nVdash \mathcal{B}^{1,n}$. Thus there must be $r \leqslant q$ such that $r \Vdash 2^\omega - \mathcal{B}^{1,n}$. It follows that $Z \neq \{n : G_\mathcal{F} \in \mathcal{B}^{1,n}\}$ for a sufficiently generic filter $\mathcal{F}$ containing r.

Suppose now $F_0 \neq \omega - Z$. Symmetrically there is a condition $r \leqslant p$ such that $\omega - Z \neq \{n : G_\mathcal{F} \in \mathcal{B}^{0,n}\}$ for a sufficiently generic filter $\mathcal{F}$ containing r. Suppose now for contradiction that:

(1) We have $F_1 = Z$ and $F_0 = \omega - Z$

As $\mathcal{A} \cap \mathcal{U}^{\mathcal{M}_m}_{C_m}$ is not a largeness class, there must be a cover $Y_0 \cup \dots \cup Y_k \in \mathcal{M}_m$ such that $Y_i \notin \mathcal{A} \cap \mathcal{U}^{\mathcal{M}_m}_{C_m}$ for every $i \leqslant k$.

As $\mathcal{S} \subseteq \langle \mathcal{U}^{\mathcal{M}_m}_{C_m} \rangle$ is partition regular there must be $i \leqslant k$ such that $X \cap Y_i \in \mathcal{S} \subseteq \langle \mathcal{U}^{\mathcal{M}_m}_{C_m} \rangle$. We also have $Y_i \in \mathcal{U}^{\mathcal{M}_m}_{C_m}$ and then $Y_i \notin \mathcal{A}$. Thus

(2) For all n, a, b, for all $\tau_0, \tau_1 \subseteq Y_i - \{0, \dots, |\sigma|\}$ the following holds:

$$\sigma \cup \tau_0 \,?\!\vdash 2^\omega - \mathcal{B}^{0,n}_a \text{ or } \sigma \cup \tau_1 \,?\!\vdash 2^\omega - \mathcal{B}^{1,n}_b$$

We shall now argue that for all $n \in Z$ there exists $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ together with b such that $\sigma \cup \tau \,?\!\nvdash 2^\omega - \mathcal{B}^{1,n}_b$. Let $n \in Z$ and suppose otherwise. Then for every $\tau \subseteq X \cap Y_i$ and every b we have $\sigma \cup \tau \,?\!\vdash 2^\omega - \mathcal{B}^{1,n}_b$, which by definition means $(\sigma, X \cap Y_i) \Vdash 2^\omega - \mathcal{B}^{1,n}$

which contradicts (1). It follows that there exists $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ together with b such that $\sigma \cup \tau \mathrel{?}\nvdash 2^\omega - \mathcal{B}_b^{1,n}$.

Symmetrically, we show that for all $n \in \omega - Z$ there exists $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ together with a such that $\sigma \cup \tau \mathrel{?}\nvdash 2^\omega - \mathcal{B}_a^{0,n}$. Therefore, for every $n \in Z$ we have using (2) that:

1. There exists some $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ and $b \in \omega$ such that $\sigma \cup \tau \mathrel{?}\nvdash 2^\omega - \mathcal{B}_b^{1,n}$
2. For all $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ and for all $a \in \omega$, $\sigma \cup \tau \mathrel{?}\vdash 2^\omega - \mathcal{B}_a^{0,n}$

Symmetrically, for every $n \notin Z$ we prove, using (2), that:

1. There exists some $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ and $a \in \omega$ such that $\sigma \cup \tau \mathrel{?}\nvdash 2^\omega - \mathcal{B}_a^{0,n}$
2. For all $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ and for all $b \in \omega$, $\sigma \cup \tau \mathrel{?}\vdash 2^\omega - \mathcal{B}_b^{1,n}$

We can now compute Z as follows : For each $n \in \omega$, look for some $\tau \subseteq Y_i - \{0, \dots, |\sigma|\}$ and some $c \in \omega$ such that either $\sigma \cup \tau \mathrel{?}\nvdash 2^\omega - \mathcal{B}_c^{0,n}$ or $\sigma \cup \tau \mathrel{?}\nvdash 2^\omega - \mathcal{B}_c^{1,n}$. This is a $\Sigma_1^0(M_m)$ event. Thus Z is $\Delta_1^0(M_m)$, which is a contradiction. ■

Chapter 3

Hyperarithmetic cone avoidance

The goal of this section if to show the following theorem:

Theorem 3.0.1 (M., Patey [29]):
Let Z be non Δ^1_1. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not $\Delta^1_1(G)$ (and in particular with $\omega_1^G = \omega_1^{ck}$).

3.1 Background

We start with a short background on higher recursion theory.

3.1.1 Computable ordinals

We let ω_1^{ck} denote the first non-computable ordinal. There is a Π^1_1 set $\mathcal{O}_1 \subseteq \omega$ such that each $o \in \mathcal{O}_1$ codes for an ordinal $\alpha < \omega_1^{ck}$ and each ordinal $\alpha < \omega_1^{ck}$ has a unique code in $\mathcal{O}_1$. Furthermore given that $o \in \mathcal{O}_1$, one can computably recognize if o codes for 0, if o codes for a successor ordinal $\alpha + 1$, in which case we can uniformly and computably produce a code in $\mathcal{O}_1$ for α, and if o codes for a limit ordinal $\sup_n \beta_n$, in which case we can uniformly and computably produce for each n codes in $\mathcal{O}_1$ for β_n. See [35] for more details about $\mathcal{O}_1$. In this section, we manipulate each ordinal $\alpha < \omega_1^{ck}$ via its respective code in $\mathcal{O}_1$. To simplify the reading, we use the notation α instead of the code for α.

3.1.2 The effective Borel sets

We also use codes for effective Borel subsets of ω or of 2^ω : For $\alpha < \omega_1^{ck}$ a code for a $\Sigma^0_{\alpha+1}$ set $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ is the code of a function that effectively enumerates codes for each Π^0_α set $\mathcal{B}_n$. A code for a $\Pi^0_{\alpha+1}$ set $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_n$ is the code of a function that effectively enumerates codes for each Σ^0_α set $\mathcal{B}_n$. For $\alpha = \sup_n \beta_n$ limit a code of a Σ^0_α set $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ is the code of a function that effectively enumerate codes for each $\Pi^0_{\beta_n}$ set $\mathcal{B}_{\beta_n}$ with $\sup_n \beta_n = \alpha$. The code of a Π^0_α set $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$ is the code of a function that effectively enumerate codes for each $\Sigma^0_{\beta_n}$ set $\mathcal{B}_{\beta_n}$ with $\sup_n \beta_n = \alpha$. We also assume the codes for effective Borel sets include some information so that we can computably distinguish Π^0_α from Σ^0_α codes as well as distinguish if $\alpha = 1$, if α is successor or if it is limit.

3.1.3 The iterated jumps

We use such codes to iterate the jump through the ordinals:

1. $\emptyset^{(0)} = \varnothing$
2. $\emptyset^{(\alpha+1)} = (\emptyset^{(\alpha)})'$
3. $\emptyset^{(\sup_n \alpha_n)} = \oplus_{n\in\omega}\emptyset^{(\alpha_n)}$

Note that for $n < \omega$ the set $\emptyset^{(n)}$ is Σ^0_n and complete for Σ^0_n questions. Above the first limit ordinal the situation is slightly different : $\emptyset^{(\omega)}$ is Δ^0_ω and not Σ^0_ω. Also given $\alpha \geqslant \omega$ we have that $\emptyset^{(\alpha+1)}$ is Σ^0_α and complete for Σ^0_α questions.

Proposition 3.1.1 : Let $n \in \omega$.

1. Let $m > 0$. The set $\{X \ : \ n \in X^{(m)}\}$ is a Σ^0_m class.
2. Let α be limit. The set $\{X \ : \ n \in X^{(\alpha)}\}$ is a Δ^0_β class for some $\beta < \alpha$.
3. Let $\alpha = \beta + 1$ with $\beta \geqslant \omega$. The set $\{X \ : \ n \in X^{(\alpha)}\}$ is a Σ^0_β class. ⋆

Proof: The set $\{X \ : \ n \in X'\}$ is clearly Σ^0_1. Let $m > 1$. the set $\{X \ : \ n \in X^{(m)}\}$ equals

$$\bigcup_{\{\sigma \,:\, \Phi_n(\sigma,n)\downarrow\}} \bigcap_{\{i \,:\, \sigma(i)=0\}} \{X \ : \ i \notin X^{(m-1)}\} \cap \bigcap_{\{i \,:\, \sigma(i)=1\}} \{X \ : \ i \in X^{(m-1)}\}$$

This is by induction a Σ^0_m set.

Let α be limit. Let p_1, p_2 be projections of the pairing function, that is, $x = \langle p_1(x), p_2(x)\rangle$. Then $\{X \ : \ n \in X^{(\alpha)}\}$ equals $\{X \ : \ p_1(n) \in X^{(p_2(n))}\}$, which is a Δ^0_β set for $\beta < \alpha$.

Let $\alpha = \beta + 1$. The set $\{X \ : \ n \in X^{(\beta+1)}\}$ equals

$$\bigcup_{\{\sigma \,:\, \Phi_n(\sigma,n)\downarrow\}} \bigcap_{\{i \,:\, \sigma(i)=0\}} \{X \ : \ i \notin X^{(\beta)}\} \cap \bigcap_{\{i \,:\, \sigma(i)=1\}} \{X \ : \ i \in X^{(\beta)}\}$$

This is by induction a Σ^0_β class. ∎

Proposition 3.1.2 : Let Φ be a functional. Let $n, i \in \omega$.

1. Let $m > 0$. The set $\{X \ : \ \exists t \ \Phi(X^{(m)}, n)[t] \downarrow = i\}$ is a Σ^0_{m+1} class.
2. Let $\alpha \geqslant \omega$. The set $\{X \ : \ \exists t \ \Phi(X^{(\alpha)}, n)[t] \downarrow = i\}$ is a Σ^0_α class. ⋆

Proof: Trivial using Proposition 3.1.1 ∎

3.1.4 Π^1_1 and Σ^1_1 sets of integers

We previously mentioned a Π^1_1 set $\mathcal{O}_1$ of unique notations for ordinals. This set is included in Kleene's $\mathcal{O}$, the set of all the constructible codes for the computable ordinals. Given an ordinal $\alpha < \omega_1^{ck}$, let $\mathcal{O}_{<\alpha}$ denote the elements of $\mathcal{O}$ which code for an ordinal strictly smaller than α. Each $\mathcal{O}_{<\alpha}$ is Δ^1_1 uniformly in α (it actually always is a $\Sigma^0_{\alpha+1}$ set [26]). It is well-known that $\mathcal{O}$ is a Π^1_1-complete set [35], that is, for any Π^1_1 set $B \subseteq \omega$ there is a computable function $f : \omega \to \omega$ such that $n \in B \leftrightarrow f(n) \in \mathcal{O}$. For such a Π^1_1 set B let us define $B_\alpha = \{n \ : \ f(n) \in \mathcal{O}_{<\alpha}\}$. In particular, each B_α is Δ^1_1 uniformly in α and $B = \bigcup_{\alpha<\omega_1^{ck}} B_\alpha$. In particular B is a $\Sigma^0_{\omega_1^{ck}}$ set. Note that contrary to Σ^0_α sets for $\alpha < \omega_1^{ck}$, the $\Sigma^0_{\omega_1^{ck}}$ sets are not described with a computable code, but rather with a Π^1_1 set of codes for all the Π^0_α that constitutes it. For $B = \bigcup_{\alpha<\omega_1^{ck}} B_\alpha$ a $\Sigma^0_{\omega_1^{ck}}$ set, with a little hack, we can even make sure that at most one new element appears in each B_α. For this reason, we often see Π^1_1 sets as enumerable along the computable ordinals.

By complementation a Σ^1_1 set $B \subseteq \omega$ can be seen as co-enumerable along the computable ordinals and we have $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ where each B_α is Δ^1_1 uniformly in α. We also say in this case that B is $\Pi^0_{\omega_1^{ck}}$.

3.1.5 Σ^1_1-boundedness

A central theorem when working with Σ^1_1 and Π^1_1 sets is Σ^1_1-boundedness:

Theorem 3.1.3 (Σ^1_1-boundedness [39]):
Let B be a Σ^1_1 set of codes for ordinals, then the supremum of the ordinals coded by elements of B is strictly smaller than ω_1^{ck}.

We mostly here use the following corollary:

Corollary 3.1.4 : Let $f : \omega \to \omega_1^{ck}$ be a total Π^1_1 function. Then $\sup_n f(n) = \alpha < \omega_1^{ck}$.

Note that $f : \omega \to \omega_1^{ck}$ means the range of f is a subset of $\mathcal{O}_1$. The corollary comes from the fact that if f is total, then it becomes Δ^1_1 and its range is then a Σ^1_1 set of codes for ordinals. As an example we apply here Σ^1_1-boundedness to show a simple fact that will be needed later : adding an ω-bounded quantifier to a $\Sigma^0_{\omega_1^{ck}}$ or a $\Pi^0_{\omega_1^{ck}}$ set does not change its complexity.

Lemma 3.1.5 : Every $\Sigma^0_{\omega_1^{ck}+1}$ set of integers is $\Pi^0_{\omega_1^{ck}}$. ⋆

Proof: Let B be $\Sigma^0_{\omega_1^{ck}+1}$, that is, $B = \bigcup_{n\in\omega} \bigcap_{\alpha\in\omega_1^{ck}} B_{n,\alpha}$ where each $B_{n,\alpha}$ is Σ^0_α uniformly in n and α. Then B is $\Pi^0_{\omega_1^{ck}}$ via the following equality : $\bigcup_{n\in\omega} \bigcap_{\alpha\in\omega_1^{ck}} B_{n,\alpha} = \bigcap_{\alpha\in\omega_1^{ck}} \bigcup_{n\in\omega} \bigcap_{\beta\in\alpha} B_{n,\beta}$. The inclustion $\bigcup_{n\in\omega} \bigcap_{\alpha\in\omega_1^{ck}} B_{n,\alpha} \subseteq \bigcap_{\alpha\in\omega_1^{ck}} \bigcup_{n\in\omega} \bigcap_{\beta\in\alpha} B_{n,\beta}$ is clear. For the other inclusion, suppose that $m \notin \bigcup_{n\in\omega} \bigcap_{\alpha\in\omega_1^{ck}} B_{n,\alpha}$, then the function $f : \omega \to \omega_1^{ck}$ which to n associates the smallest α such that $m \notin \bigcap_{\beta\in\alpha} B_{n,\beta}$ is total. By Σ^1_1-boundedness there must be some α such that for every n the integer m is in no set $\bigcap_{\beta\in\alpha} B_{n,\beta}$. ∎

3.1.6 Π^1_1 and Σ^1_1 sets of reals

Given $X \in 2^\omega$ we let $\mathcal{O}^X$ be the set of X-constructible codes for X-computable ordinals. We let $\omega_1^X \geqslant \omega_1^{ck}$ be the smallest non X-computable ordinal. For $\alpha < \omega_1^X$, we let $\mathcal{O}^X_{<\alpha}$ be the elements of $\mathcal{O}^X$ coding for an ordinal strictly smaller than α.

One can show that a set $\mathcal{B} \subseteq 2^\omega$ is Π^1_1 iff there exists some $e \in \omega$ such that $\mathcal{B} = \{X : e \in \mathcal{O}^X\}$, that is, $\mathcal{B}$ is the set of elements relative to which e codes for an X-computable ordinal. In particular, $\mathcal{B} = \bigcup_{\alpha<\omega_1}\{X : e \in \mathcal{O}^X_{<\alpha}\}$. Note that the union may go up to ω_1, indeed, Π^1_1 sets of reals are not necessarily Borel.

A Π^1_1 set of particular interest is the set of element X such that $\omega_1^X > \omega_1^{ck}$. The set is Borel, but not effectively. One can even prove that it contains no non-empty Σ^1_1 subset : this is known as the Gandy Basis theorem (see Sacks [35, III.1.5]):

Theorem 3.1.6 (Gandy Basis theorem):
Let $\mathcal{B} \subseteq 2^\omega$ be a non-empty Σ^1_1 set. Then there exists $X \in \mathcal{B}$ such that $\omega_1^X = \omega_1^{ck}$.

3.1.7 The general strategy to show hyperarithmetic cone avoidance

Let Z be non Δ^1_1. Our goal is to build a generic $G \subseteq A$ or $G \subseteq \omega - A$ such that Z is not $\Delta^1_1(G)$. This is done in two steps: first show that Z is not $G^{(\alpha)}$-computable for any $\alpha < \omega_1^{ck}$ and second show that $\omega_1^G = \omega_1^{ck}$, so in particular we cannot have that Z is $G^{(\alpha)}$-computable for $\omega_1^{ck} \leqslant \alpha < \omega_1^G$.

The first part is simply an iteration of the forcing through the computable ordinals, and raises no particular issue. This is done in Section 3.2.

The second part is a little bit trickier but still follows a canonical technic, which has often been used, up to some cosmetic changes in its presentation, to show this kind of preservation theorem (see for instance [15], [34] or [40]) : Suppose $\omega_1^G > \omega_1^{ck}$, in particular there is an element $e \in \mathcal{O}^G$ which codes for ω_1^{ck}, that is e is the code of a functional with $\forall n\ \Phi_e(G,n) \downarrow\in \mathcal{O}^G_{<\omega_1^{ck}}$ with $\sup_n |\Phi_e(G,n)| = \omega_1^{ck}$ where $|\Phi_e(G,n)|$ is the ordinal coded by $\Phi_e(G,n)$. All we have to do is to show that such a code e does not exist. Given e we show that one of the following holds:

1. $\exists n\ \forall \alpha < \omega_1^{ck}\ \Phi_e(G,n) \notin \mathcal{O}^G_{<\alpha}$
2. $\exists \alpha < \omega_1^{ck}\ \forall n\ \Phi_e(G,n) \in \mathcal{O}^G_{<\alpha}$

Each set $\{X : \Phi_e(X,n) \notin \mathcal{O}^X_{<\alpha}\}$ is Δ^1_1 uniformly in α. It follows that the set $\{X : \exists n\ \forall \alpha < \omega_1^{ck}\ \Phi_e(X,n) \notin \mathcal{O}^X_{<\alpha}\}$ is a $\Sigma^0_{\omega_1^{ck}+1}$ set of reals. Contrary to $\Sigma^0_{\omega_1^{ck}+1}$ sets of integers, such sets cannot be simplified. We are then required to extend our forcing questions in order to control the truth of $\Sigma^0_{\omega_1^{ck}+1}$-statements. This is what will be done in Section 3.4.

3.2 Preliminaries

We start by making sure that we can extend Proposition 2.2.1 through the computable ordinals.

Proposition 3.2.1 : There is a sequence of sets $\{M_\alpha\}_{\alpha<\omega_1^{ck}}$ such that:

1. M_α codes for a countable Scott set $\mathcal{M}_\alpha$
2. $\emptyset^{(\alpha)}$ is uniformly coded by an element of $\mathcal{M}_\alpha$

3. Each M'_α is uniformly computable in $\emptyset^{(\alpha+1)}$ ⋆

Proof: In the proof of Proposition 2.2.1 we show how to build a functional $\Phi : 2^\omega \to 2^\omega$ such that for any oracle X, we have that $M' = \Phi(X')$ is such that $M = \oplus_{n\in\omega} X_n$ codes for a Scott set $\mathcal{M}$ with $X_0 = X$.

We simply use here this functionnal with any $\emptyset^{(\alpha+1)}$ for $\alpha < \omega_1^{ck}$. ∎

Note $\emptyset^{(\beta)}$ is computable in $\emptyset^{(\alpha)}$ for $\beta < \alpha$ in a uniform way : there is a unique computable function $f(\emptyset^{(\alpha)}, \alpha, \beta)$ which outputs $\emptyset^{(\beta)}$ for every $\beta < \alpha$. Also Proposition 3.2.1 implies that M_β is computable in $\emptyset^{(\alpha)}$ for $\beta < \alpha$ and similarly, the computation is uniform in β, α.

We now turn to an extention of Proposition 2.2.3 to the computable ordinals, for which we reuse Lemma 2.2.4 and Lemma 2.2.5.

Proposition 3.2.2 : There is a sequence of sets $\{C_\alpha\}_{\alpha<\omega_1^{ck}}$ such that:

1. $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is an $\mathcal{M}_\alpha$-cohesive largeness class
2. $\beta < \alpha$ implies $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta} \rangle$
3. Each C_α is coded by an element of $\mathcal{M}_{\alpha+1}$ uniformly in α and $M_{\alpha+1}$. ⋆

Proof: Let X_i^α be the element of $\mathcal{M}_\alpha$ of code i, so that each $M_\alpha = \oplus_i X_i^\alpha$. Let us argue that there is a computable function $f : \omega_1^{ck} \times \omega_1^{ck} \times \omega$ such that whenever $\beta < \alpha$, then $X_i^\beta = X_{f(\alpha,\beta,i)}^\alpha$: Given an ordinal α the function f considers the M_α-code of $\emptyset^{(\alpha)}$ (which is uniformly coded in $\mathcal{M}_\alpha$) and uses it to produce an M_α-code of $M_\beta = \oplus_i X_i^\beta$ (as M_β is computable in $\emptyset^{(\alpha)}$, uniformly in β, α) and then returns an M_α-code of X_i^β. Given $\beta < \alpha$ and $C \subseteq \omega^2$, we then let $g(\alpha, \beta, C) = \{\langle e, f(\alpha,\beta,i)\rangle : \langle e, i\rangle \in C\}$. In particular, $\mathcal{U}_{g(\alpha,\beta,C)}^{\mathcal{M}_\alpha} = \mathcal{U}_C^{\mathcal{M}_\beta}$.

Suppose that stage α we have defined by induction sets C_β for each $\beta < \alpha$, verifying (1)(2) and (3). Let us proceed and define C_α.

Suppose first that $\alpha = \beta + 1$ is successor. Note that the set C_β is coded by an element of $\mathcal{M}_{\beta+1}$ uniformly in β, and thus that C_β is uniformly computable in $\emptyset^{(\beta+2)}$ and then uniformly computable in M''_β. Using Lemma 2.2.4 we define $D_\beta \supseteq C_\beta$ to be such that $\mathcal{U}_{D_\beta}^{\mathcal{M}_\beta} = \langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta} \rangle$ and such that D_β is uniformly M''_β-computable. We define E_α to be $g(\alpha, \beta, D_\beta)$, so that $\mathcal{U}_{E_\alpha}^{\mathcal{M}_\alpha} = \mathcal{U}_{D_\beta}^{\mathcal{M}_\beta}$. Note that as E_α is uniformly computable in M''_β and thus in $\emptyset^{(\alpha+1)}$, it is uniformly coded by an element of $\mathcal{M}_{\alpha+1}$. Note also that $\mathcal{U}_{E_\alpha}^{\mathcal{M}_\alpha}$ is partition regular as it equals $\langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta} \rangle$. Using Lemma 2.2.5 we uniformly find an $\mathcal{M}_{\alpha+1}$-index of $C_\alpha \supseteq E_\alpha$ to be such that $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is an $\mathcal{M}_\alpha$-cohesive largeness class.

At limit stage $\alpha = \sup_n \beta_n$, each set C_{β_n} is coded by an element of $\mathcal{M}_{\beta_n+1}$ uniformly in β_n and that $\mathcal{M}_{\beta_n+1}$ is uniformly computable in $\emptyset^{(\alpha)}$. It follows that $\bigcup_n C_{\beta_n}$ is uniformly computable in $\emptyset^{(\alpha+1)}$. We define D_α to be $\bigcup_n g(\alpha, \beta_n, C_{\beta_n})$. Note that D_α is uniformly computable in $\emptyset^{(\alpha+1)}$ and thus coded by an element of $\mathcal{M}_{\alpha+1}$ uniformly in α. Note also that $\mathcal{U}_{D_\alpha}^{M_\alpha} = \bigcap_{n\in\omega} \mathcal{U}_{C_{\beta_n}}^{M_{\beta_n}} = \bigcap_{n\in\omega} \langle \mathcal{U}_{C_{\beta_n}}^{M_{\beta_n}} \rangle$. As an intersection of partition regular class, $\mathcal{U}_{D_\alpha}^{M_\alpha}$ is partition regular. Using Lemma 2.2.5 there is a set $C_\alpha \supseteq D_\alpha$ such that $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is $\mathcal{M}_\alpha$-cohesive and such that C_α is uniformly coded by an element of $\mathcal{M}_{\alpha+1}$. ∎

3.3 The forcing

Let $A^0 \cup A^1 = \omega$. Let $Z \in 2^\omega$ be non-hyperarithmetic. From now on, fix sequences $\{\mathcal{M}_\alpha\}_{\alpha<\omega_1^{ck}}$ and $\{C_\alpha\}_{\alpha<\omega_1^{ck}}$ which verify Proposition 3.2.1 and Proposition 3.2.2, respectively. Recall that in order to show $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$ we need to decide the truth of $\Sigma^0_{\omega_1^{ck}+1}$ statments. To do so it is not enough to work with the partition regular class $\bigcap_{\beta<\omega_1^{ck}} \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}$. We need something a bit more restrictive. We do not give details right away about that, and assume we work within some partition regular class $\mathcal{S} \subseteq \bigcap_{\beta<\omega_1^{ck}} \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta}$. The details on $\mathcal{S}$ are given in Section 3.4.

Note that there must be $i < 2$ such that $A^i \in \mathcal{S}$. Let then $A = A^i$ for some i such that $A^i \in \mathcal{S}$.

Definition 3.3.1 : Let $\mathbb{P}_{\omega_1^{ck}}$ be the set of conditions (σ, X) such that:

1. (σ, X) is a Mathias condition
2. $\sigma \subseteq A$
3. $X \subseteq A$
4. $X \in \mathcal{S}$

Given two conditions $(\sigma, X), (\tau, Y) \in \mathbb{P}_{\omega_1^{ck}}$ we let $(\sigma, X) \leqslant (\tau, Y)$ be the usual Mathias extension, that is, $\sigma \succcurlyeq \tau$, $X \subseteq Y$ and $\sigma - \tau \subseteq Y$. $\diamond$

Note that we have $\mathbb{P}_{\omega_1^{ck}} \subseteq \mathbb{P}_\omega$. Recall the relation

$$\sigma \mathrel{?}\!\vdash \mathcal{B}$$

that was defined in Definition 2.3.2 for Σ^0_m sets $\mathcal{B}$. We now extend this definition to Σ^0_α sets for $\alpha \geqslant \omega$

Definition 3.3.2 : Let $\sigma \in 2^{<\omega}$. Let α with $\omega \leqslant \alpha < \omega_1^{ck}$. Given a Σ^0_α class $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$, we define $\sigma \mathrel{?}\!\vdash \mathcal{B}$ if

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_n}\} \cap \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$$

is a largeness class.

For a condition $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$ and an effectively Borel set $\mathcal{B}$, we write $p \mathrel{?}\!\vdash \mathcal{B}$ if $\sigma \mathrel{?}\!\vdash \mathcal{B}$. $\diamond$

We now extend Definition 2.3.3 in a straightforward way to effective transfinite Borel sets.

Definition 3.3.3 : Let $\sigma \in 2^{<\omega}$. Given a Σ^0_1 class $\mathcal{B}$, we write $\mathcal{U}(\mathcal{B}, \sigma)$ for the open set:

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ [\sigma \cup \tau] \subseteq \mathcal{B}\}$$

Given a Σ^0_α class $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ for $1 < \alpha < \omega_1^{ck}$ we write $\mathcal{U}(\mathcal{B}, \sigma)$ for the open set:

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?}\!\nvdash 2^\omega - \mathcal{B}_{\beta_n}\}$$

Proposition 2.3.4 settled the complexity of the relation $?\!\vdash$ by showing that it is $\Pi^0_1(C_{m-1} \oplus \emptyset^{(m)})$ for a Σ^0_m class. We extend here the proposition for Σ^0_α classes. Note that in the following one might have the false impression that we loose one jump compare to Proposition 2.3.4. This is due to the fact that for $\alpha \geqslant \omega$ the Σ^0_α-complete set is $\emptyset^{(\alpha+1)}$ and not $\emptyset^{(\alpha)}$.

Proposition 3.3.4 : Let $\sigma \in 2^{<\omega}$. Let $\mathcal{B}$ be a Σ^0_α class for $\alpha \geqslant \omega$.

1. The set $\mathcal{U}(\mathcal{B},\sigma)$ is an upward closed $\Sigma^0_1(C_{\alpha-1} \oplus \emptyset^{(\alpha)})$ open set if α is successor and an upward closed $\Sigma^0_1(\emptyset^{(\alpha)})$ open set if α is limit.
2. The relation $\sigma \mathrel{?\vdash} \mathcal{B}$ is $\Pi^0_1(C_\alpha \oplus \emptyset^{(\alpha+1)})$.

This is uniform in σ and a code for the class $\mathcal{B}$. ⋆

Proof: This is done by induction on the effective Borel codes. Let $\omega \leqslant \alpha < \omega_1^{ck}$. Suppose (1) and (2) are true for any $\omega \leqslant \beta < \alpha$. Let $\sigma \in 2^{<\omega}$ and let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ be a Σ^0_α class. Let

$$\mathcal{U}(\mathcal{B},\sigma) = \{Y \ : \ \exists \tau \subseteq Y - \{0,\dots,|\sigma|\}\ \exists n\ \sigma \cup \tau \mathrel{?\nvdash} 2^\omega - \mathcal{B}_{\beta_n}\}$$

Let us show (1). Suppose first α is limit. For each $n \in \omega$, the class $2^\omega - \mathcal{B}_{\beta_n}$ is a $\Sigma^0_{\beta_n}$ class uniformly in $\sigma \cup \tau$ and in a code for $\mathcal{B}_{\beta_n}$. By induction hypothesis, or by Proposition 2.3.4 in case $\alpha = \omega$, the relation $\sigma \cup \tau \mathrel{?\nvdash} 2^\omega - \mathcal{B}_{\beta_n}$ is, in any case, $\Sigma^0_1(\emptyset^{(\beta_n+2)})$ and thus $\Sigma^0_1(\emptyset^{(\alpha)})$. It follows that $\mathcal{U}(\mathcal{B},\sigma)$ is an upward-closed $\Sigma^0_1(\emptyset^{(\alpha)})$ open set.

Suppose now $\alpha \geqslant \omega$ with $\alpha = \beta + 1$. For each n we have that $2^\omega - \mathcal{B}_{\beta_n}$ is a Σ^0_β class uniformly in n. By induction hypothesis, the relation $\sigma \cup \tau \mathrel{?\nvdash} 2^\omega - \mathcal{B}_{\beta_n}$ is $\Sigma^0_1(C_\beta \oplus \emptyset^{(\beta+1)})$. It follows that $\mathcal{U}(\mathcal{B},\sigma)$ is an upward closed $\Sigma^0_1(C_{\alpha-1} \oplus \emptyset^{(\alpha)})$ class.

Let us now show (2). Suppose $\alpha \geqslant \omega$ successor or limit. Then $\mathcal{U}(\mathcal{B},\sigma) \cap \mathcal{U}^{\mathcal{M}_\alpha}_{C_\alpha}$ is a largeness class if for all $F \subseteq C_\alpha$, the class $\mathcal{U}(\mathcal{B},\sigma) \cap \mathcal{U}^{\mathcal{M}_\alpha}_F$ is a largeness class. It is a $\Pi^0_2(M_\alpha)$ statement uniformly in F and then a $\Pi^0_1(M'_\alpha)$ statement uniformly in F and then a $\Pi^0_1(\emptyset^{(\alpha+1)})$ statement uniformly in F. It follows that the statement $\mathcal{U}(\mathcal{B},\sigma) \cap \mathcal{U}^{\mathcal{M}_\alpha}_{C_\alpha}$ is a largeness class is $\Pi^0_1(C_\alpha \oplus \emptyset^{(\alpha+1)})$. ■

We finally extend the forcing relation of Definition 2.3.5 to the transfinite

Definition 3.3.5 : Let $(\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. Let $\mathcal{U}$ be a Σ^0_1 class. We define

$$\begin{array}{lcccl} (\sigma,X) & \Vdash & \mathcal{U} & \leftrightarrow & [\sigma] \subseteq \mathcal{U} \\ (\sigma,X) & \Vdash & 2^\omega - \mathcal{U} & \leftrightarrow & \forall \tau \subseteq X\ [\sigma \cup \tau] \nsubseteq \mathcal{U} \end{array}$$

Then inductively for Σ^0_α classes $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ with $0 < \alpha < \omega_1^{ck}$, we define:

$$\begin{array}{lcccl} (\sigma,X) & \Vdash & \mathcal{B} & \leftrightarrow & \exists n\ (\sigma,X) \Vdash \mathcal{B}_{\beta_n} \\ (\sigma,X) & \Vdash & 2^\omega - \mathcal{B} & \leftrightarrow & \forall n\ \forall \tau \subseteq X\ \sigma \cup \tau \mathrel{?\vdash} 2^\omega - \mathcal{B}_{\beta_n} \end{array}$$

Note that the relation $\Vdash$ does not change compare to the arithmetical case : the definition goes through exactly the same way in the transfinite. It is the same for the relation $?\vdash$. For these reasons the following lemmas and propositions and theorems are all proved exactly the same way as for the arithmetical case, only now our set $\mathcal{S}$ is included in $\bigcap_{\beta<\omega_1^{ck}} \mathcal{U}^{\mathcal{M}_\beta}_{C_\beta}$ and not just in $\bigcap_{m<\omega} \mathcal{U}^{\mathcal{M}_m}_{C_m}$.

Lemma 3.3.6 : Let $p \in \mathbb{P}_{\omega_1^{ck}}$. Let $\mathcal{B} = \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$ be a Π^0_α class. Then $p \Vdash \bigcap_{n<\omega} \mathcal{B}_{\beta_n}$ iff for every $n \in \omega$ and every $q \leqslant p$ we have $q \,?\!\vdash \mathcal{B}_{\beta_n}$. ⋆

PROOF: Same as Lemma 2.3.6. ∎

Proposition 3.3.7 : Let $p \in \mathbb{P}_{\omega_1^{ck}}$. Let $\mathcal{B}$ be an effectively Borel set. If $p \Vdash \mathcal{B}$ and $q \leqslant p$ then $q \Vdash \mathcal{B}$. ⋆

PROOF: Same as Proposition 2.3.7. ∎

Lemma 3.3.8 : Let $p \in \mathbb{P}_{\omega_1^{ck}}$. Let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ be a Σ^0_α class for $0 < \alpha < \omega_1^{ck}$.

1. Suppose $p \,?\!\vdash \mathcal{B}$. Then there exists $q \leqslant p$ such that $q \Vdash \mathcal{B}$.
2. Suppose $p \,?\!\nvdash \mathcal{B}$. Then there exists $q \leqslant p$ such that $q \Vdash 2^\omega - \mathcal{B}$. ⋆

PROOF: Same as Lemma 2.3.8. ∎

Notation

Let $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ be a sufficiently generic filter. We write $G_\mathcal{F} \in 2^\omega$ for the unique set such that $\sigma \prec G_\mathcal{F}$ for $(\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$.

Theorem 3.3.9:
Let $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ be a generic enough filter. Let $p \in \mathcal{F}$. Let $\mathcal{B}_\alpha = \bigcup_{n<\omega} \mathcal{B}_{\beta_n}$ be a Σ^0_α class for $0 < \alpha < \omega_1^{ck}$. Suppose $p \Vdash \mathcal{B}_\alpha$. Then $G_\mathcal{F} \in \mathcal{B}_\alpha$. Suppose $p \Vdash 2^\omega - \mathcal{B}_\alpha$. Then $G_\mathcal{F} \in 2^\omega - \mathcal{B}_\alpha$.

PROOF: Same as Theorem 2.3.10. ∎

We now have all the necessary parts to extend Theorem 2.1.1 in the transfinite.

Theorem 3.3.10:
Let $\alpha \leqslant \omega_1^{ck}$ be a limit ordinal. Suppose Z is not $\Delta^0_1(\emptyset^{(\beta)})$ for every $\beta < \alpha$. Let $\mathcal{F}$ be a sufficiently generic filter. Then for every $\beta < \alpha$, Z is not $\Delta^0_1(G_\mathcal{F}^{(\beta)})$.

PROOF: Let Φ be a functional and $\beta < \alpha$. Let $\mathcal{B}^n = \{X \ : \Phi(X^{(\beta)}, n)\downarrow\}$. We want to show that $Z \neq \{n \ : G_\mathcal{F}^{(\beta)} \in \mathcal{B}^n\}$. From Proposition 3.1.2, $\mathcal{B}^n$ is a $\Sigma^0_{\beta+1}$ set for each $n \in \omega$ (Σ^0_β if $\beta \geqslant \omega$ and $\Sigma^0_{\beta+1}$ if $\beta < \omega$).

Let $p \in \mathbb{P}_{\omega_1^{ck}}$ be a condition. From Proposition 3.3.4, the set $\{n \ : \ p \,?\!\vdash \mathcal{B}^n\}$ is $\Pi^0_1(\emptyset^{(\beta+3)})$. As Z is not $\Pi^0_1(\emptyset^{(\beta+3)})$, then there is some $n \in Z$ such that $p \,?\!\nvdash \mathcal{B}^n$ or some $n \notin Z$ such that $p ?\vdash \mathcal{B}^n$. In the first case, there is an extension $q \leqslant p$ such that $q \Vdash 2^\omega - \mathcal{B}^n$ for some $n \in Z$. In the second case, there is an extension $q \leqslant p$ such that $q \Vdash \mathcal{B}^n$ for some $n \notin Z$. By Theorem 3.3.9, in the first case $\Phi(G_{\mathcal{F}}^{(\beta)}, n) \uparrow$ holds for some $n \in Z$, and in the second case, $\Phi(G_{\mathcal{F}}^{(\beta)}, n) \downarrow$ holds for some $n \notin Z$.

If $\mathcal{F}$ is sufficiently generic, this is true for any $\beta < \alpha$ and any functional Φ. It follows that for any ordinal β the set Z is not $\Sigma_1^0(G_{\mathcal{F}}^{(\beta)})$ and thus not $\Delta_1^0(G_{\mathcal{F}}^{(\beta)})$. ∎

Note that an extension of Theorem 2.1.2 in the transfinite would also be possible : Given Z non $\Delta_1^0(\emptyset^{(\alpha)})$, one can find $\mathcal{G}_{\mathcal{F}} \subseteq A$ such that Z is not $\Delta_1^0(G_{\mathcal{F}}^{(\alpha)})$. We however do not need to have that level of precision to show hyperarithmetic cone avoidance. There is on the other hand a new difficulty. We also need to show that for sufficiently generic filters $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$, the generic $G_{\mathcal{F}}$ does not collapse ω_1^{ck}. This is done in the next section.

3.4 Preservation of ω_1^{ck}

Definition 3.4.1 : Let Γ be a class of complexity. A largeness class $\mathcal{A}$ is Γ-minimal, if for every Γ-open set $\mathcal{U}$ we have $\mathcal{A} \cap \mathcal{U}$ is a largeness class implies $\mathcal{A} \subseteq \mathcal{U}$. ◊

Proposition 3.4.2 : The class $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is Δ_1^1-minimal. ⋆

Proof: For every $\alpha < \omega_1^{ck}$ we have that $\emptyset^{(\alpha)} \in \mathcal{M}_\alpha$ and $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \langle \mathcal{C}_\alpha \rangle$ where $\langle \mathcal{C}_\alpha \rangle$ is $\mathcal{M}_\alpha$-minimal. As $\emptyset^{(\alpha)} \in \mathcal{M}_\alpha$ we also have that $\langle \mathcal{C}_\alpha \rangle$ is minimal for $\Sigma_1^0(\emptyset^{(\alpha)})$ open sets. It follows that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is Δ_1^1-minimal. ∎

As discussed previously, in order to show preservation of ω_1^{ck} we will need to force $\Sigma_{\omega_1^{ck}+1}^0$ statements. Given a $\Sigma_{\omega_1^{ck}}^0$ class $\mathcal{B} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ and $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$, we will in particular need to ask questions of the form : is

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists \alpha < \omega_1^{ck} \ \sigma \cup \tau \, ?\nVdash 2^\omega - \mathcal{B}_\alpha\} \cap \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$$

a largeness class ?

The problem is that this question is already definitionally too complex. It is of the form $\forall \alpha < \omega_1^{ck} \ \exists \alpha < \omega_1^{ck} \dots$ and we cannot afford such an alternation of two transfinite quantifiers to preserve ω_1^{ck}. We will use a trick in order to overcome this difficulty : the use of a Δ_1^1-cohesive set $C \in \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ with $\omega_1^C = \omega_1^{ck}$. Then instead of asking if $\mathcal{U} \cap \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is a largeness class for a Π_1^1-open set $\mathcal{U}$, it will be enough, through Corollary 3.4.8, to ask if $\mathcal{U} \cap \mathcal{L}_C$ is a largness class (recall the class $\mathcal{L}_C$ of elements intersecting C infinitely often), while Corollary 3.4.7 will make sure that this question has the right complexity.

Proposition 3.4.3 : There is a set $C \in \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ such that C is Δ_1^1-cohesive and $\omega_1^C = \omega_1^{ck}$. ⋆

Proof: Let us argue that for any partition regular class $\bigcap_{n<\omega} \mathcal{U}_n$ where each $\mathcal{U}_n$ is open, not necessarily effectively of uniformly, there is a Δ_1^1-cohesive C in $\bigcap_{n<\omega} \mathcal{U}_n$. This is done by Mathias forcing with conditions (σ, X) such that X is Δ_1^1 with $X \in \bigcap_{n<\omega} \mathcal{U}_n$. Given a condition (σ, X) and n we can force the generic to be in $\mathcal{U}_n$ as follows : As

$X \in \mathcal{U}_n$ we must have that $\sigma \cup X \in \mathcal{U}_n$ because $\mathcal{U}_n$ is upward closed. Thus there must be $\tau \subseteq X - \{0, \dots, |\sigma|\}$ such that $[\sigma \cup \tau] \subseteq \mathcal{U}_n$. As $\bigcap_{n<\omega} \mathcal{U}_n$ contains only infinite set we must have $X - \{0, \dots, |\sigma \cup \tau|\} \in \bigcap_{n<\omega} \mathcal{U}_n$. Thus $(\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\})$ is a valid extension. Let now Y be Δ_1^1. We can force the generic to be included in Y or $\omega - Y$ up to finitely many elements as follow : We have $X \cap Y \in \bigcap_{n<\omega} \mathcal{U}_n$ or $X \cap (\omega - Y) \in \bigcap_{n<\omega} \mathcal{U}_n$. Then $(\sigma, X \cap Y)$ or $(\sigma, X \cap (\omega - Y))$ is a valid extension. Thus any partition regular class $\bigcap_{n<\omega} \mathcal{U}_n$ contains a Δ_1^1-cohesive set. In particular we have that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ contains a Δ_1^1-cohesive set.

We also have that the set $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is a Σ_1^1 class and that the class of Δ_1^1-cohesive sets is a Σ_1^1 class. As their intersection is non-empty, by the Σ_1^1-basis theorem it must contains C with $\omega_1^C = \omega_1^{ck}$. ∎

From now on we fix a Δ_1^1-cohesive set $C \in \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ with $\omega_1^C = \omega_1^{ck}$.

Lemma 3.4.4 : Let $\mathcal{U}$ be a Δ_1^1 open set. Suppose $\mathcal{L}_C \cap \mathcal{U}$ is a largeness class. Then $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \mathcal{U}$ ⋆

PROOF: Suppose $\mathcal{L}_C \cap \mathcal{U}$ is a largeness class. Let us show that $\mathcal{U} \cap \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is a largeness class. Suppose first for contradiction that it is not. Then there is a Δ_1^1 cover $Y_0 \cup \dots \cup Y_k \supseteq \omega$ together with a Δ_1^1 open largeness class $\mathcal{V} \supseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ such that $Y_i \notin \mathcal{U} \cap \mathcal{V}$ for every $i \leqslant k$. As each Y_i is Δ_1^1, there is some $i \leqslant k$ such that $C \subseteq^* Y_i$. Note also that since $C \in \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$, then $C \in \mathcal{L}(\mathcal{V})$ and thus $\mathcal{L}_C \cap \mathcal{V}$ is a largeness class. It follows that $Y_j \in \mathcal{L}_C \cap \mathcal{V}$ for some $j \leqslant k$. As $j \neq i$ implies $|Y_j \cap C| < \infty$, then $Y_i \in \mathcal{L}_C \cap \mathcal{V}$ and thus $Y_i \in \mathcal{V}$. As $\mathcal{L}_C \cap \mathcal{U}$ is a largeness class then by a similar argument, $Y_i \in \mathcal{L}_C \cap \mathcal{U}$ and thus $Y_i \in \mathcal{U}$. It follows that $Y_i \in \mathcal{U} \cap \mathcal{V}$, contradicting our hypothesis. Thus $\mathcal{U} \cap \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is a largeness class.

Now from Proposition 3.4.2 we have that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is minimal for Δ_1^1 open sets, then $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \mathcal{U}$. ∎

Definition 3.4.5 : Let $\mathcal{B} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ be a $\Sigma_{\omega_1^{ck}}^0$ class. Let $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. We define $p \mathrel{?{\vdash}} \mathcal{B}$ if the set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists \alpha < \omega_1^{ck} \ \sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_\alpha\} \cap \mathcal{L}_C$$

is a largeness class. ◇

Given a $\Sigma_{\omega_1^{ck}}^0$ class $\mathcal{B} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ the following set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\} \ \exists \alpha < \omega_1^{ck} \ \sigma \cup \tau \mathrel{?{\nvdash}} 2^\omega - \mathcal{B}_\alpha\}$$

is a Π_1^1 open set, that is an open set $\bigcup_{\sigma \in B} [\sigma]$ where $B = \bigcup_{\alpha<\omega_1^{ck}} B_\alpha$ is a Π_1^1 set of strings. We also suppose that each B_α is $\emptyset^{(\alpha)}$-computable and that $\{B_\alpha\}_{\alpha<\omega_1^{ck}}$ is increasing. Given such a set $\mathcal{U}$ described by $U \subseteq \omega$ we write $\mathcal{U}_\alpha$ for the Δ_1^1 open set $\bigcup_{\sigma \in U_\alpha} [\sigma]$.

Proposition 3.4.6 : Let $\mathcal{U}$ be an upward-closed Π_1^1 open set. The class $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class iff there exists some $\alpha < \omega_1^{ck}$ such that $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class. ⋆

PROOF: Suppose $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class. Then clearly $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. Suppose now that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. For each n let $\mathcal{U}_n^C$ be the $\Sigma_1^0(C)$ open set such that $\mathcal{L}_C = \bigcap_n \mathcal{U}_n^C$ with $\mathcal{U}_{n+1}^C \subseteq \mathcal{U}_n^C$. We have

$$\forall n\ \forall k\ \exists \alpha\ \forall Y_0 \cup \cdots \cup Y_k\ \exists i \leqslant k\ \exists \sigma \subseteq Y_i\ [\sigma] \subseteq \mathcal{U}_\alpha \cap \mathcal{U}_n^C$$

Note that given k and α the predicate $P_\alpha^{n,k} \equiv \forall Y_0 \cup \cdots \cup Y_k\ \exists i \leqslant k\ \exists \sigma \subseteq Y_i\ [\sigma] \subseteq \mathcal{U}_\alpha \cap \mathcal{U}_n^C$ is $\Sigma_1^0(C \oplus \emptyset^{(\alpha+1)})$ uniformly in n, k and α. Thus the function $f : \omega^2 \to \omega_1^{ck}$ which to n, k associates the smallest α such that $P_\alpha^{n,k}$ is true is a total $\Pi_1^1(C)$ function. By Σ_1^1-boundedness we have $\beta = \sup_{n,k} f(n,k) < \omega_1^C = \omega_1^{ck}$. It follows that

$$\forall n\ \forall k\ \forall Y_0 \cup \cdots \cup Y_{k-1}\ \exists i < k\ \exists \sigma \subseteq Y_i\ [\sigma] \subseteq \mathcal{U}_\beta \cap \mathcal{U}_n^C$$

Then $\mathcal{U}_\beta \subseteq \mathcal{U}$ is such that $\mathcal{U}_\beta \cap \mathcal{L}_C$ is a largeness class. ∎

Corollary 3.4.7 : Let $\mathcal{B} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ be a $\Sigma^0_{\omega_1^{ck}}$ class. Let $(\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. The relation $p\,?\!\vdash \mathcal{B}$ is $\Sigma^0_{\omega_1^{ck}}(C)$

PROOF: The relation $p\,?\!\vdash \mathcal{B}$ is equivalent to

$$\exists \alpha < \omega_1^{ck}\ \{Y\ :\ \exists \tau \subseteq Y - \{0,\dots,|\sigma|\}\ \sigma \cup \tau\ ?\!\nvdash 2^\omega - \mathcal{B}_\alpha\} \cap \mathcal{L}_C$$

is a largeness class. ∎

Corollary 3.4.8 : The class $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is minimal for Π_1^1 open sets $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class.

PROOF: Given a Π_1^1-open set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class, there must be $\alpha < \omega_1^{ck}$ such that $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class. By Lemma 3.4.4 it must be that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \mathcal{U}_\alpha$. ∎

Definition 3.4.9 : Let $\mathcal{B} = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ be a $\Pi^0_{\omega_1^{ck}}$ class. Let $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. We define $p \Vdash \mathcal{B}$ if for every $\tau \subseteq X - \{0,\dots,|\sigma|\}$ and for every $\alpha < \omega_1^{ck}$ we have $\sigma \cup \tau\ ?\!\vdash \mathcal{B}_\alpha$ ◊

Proposition 3.4.10 : Let $\mathcal{B} = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_\alpha$ be a $\Pi^0_{\omega_1^{ck}}$ class. Let $\mathcal{F}$ be sufficiently generic with $p \in \mathcal{F}$. Let $p \Vdash \mathcal{B}$. Then $G_\mathcal{F} \in \mathcal{B}$. ⋆

PROOF: Using Lemma 3.3.8, for every α and every $q \leqslant p$, there is some $r \leqslant q$ such that $r \Vdash \mathcal{B}_\alpha$. Thus for every α the set $\{r\ :\ r \Vdash \mathcal{B}_\alpha\}$ is dense below p. It follows from Theorem 3.3.9 that if $\mathcal{F}$ is sufficiently generic, $G_\mathcal{F} \in \mathcal{B}$. ∎

We now increase the complexity by one notch to reach what we need for preservation of ω_1^{ck}.

Definition 3.4.11 : Let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ be a $\Sigma^0_{\omega_1^{ck}+1}$ class where each $\Pi^0_{\omega_1^{ck}}$ set $\mathcal{B}_n = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$. We define $p\,?\!\vdash \mathcal{B}$ if the set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0,\dots,|\sigma|\}\ \exists n\ \sigma\cup\tau\,?\!\nvdash 2^\omega - \mathcal{B}_n\} \cap \mathcal{L}_C$$

is a largeness class. ◇

Given a $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ with $\mathcal{B}_n = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$, the following set

$$\mathcal{U} = \{Y \ : \ \exists \tau \subseteq Y - \{0,\dots,|\sigma|\}\ \exists n\ \sigma\cup\tau\,?\!\nvdash 2^\omega - \mathcal{B}_n\}$$

is a $\Sigma^1_1(C)$ open set, that is an open set $\mathcal{U} = \bigcup_{\sigma\in B}[\sigma]$ where $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ is a $\Sigma^1_1(C)$ set of strings. We furthermore assume that $\{B_\alpha\}_{\alpha<\omega_1^{ck}}$ is decreasing. We then write $\mathcal{U}_\alpha$ for the $\Delta^1_1(C)$-open set $\bigcup_{\sigma\in B_\alpha}[\sigma]$.

Computability theorists have a strong habits of working with enumerable open sets. With that respect, Σ^1_1-open sets, that is, co-enumerable along the computable ordinals, are strange objects to consider. Note that given such an open set we have $\mathcal{U} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha$, but not necessarily equality. However the elements X of $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha - \mathcal{U}$ are all such that $\omega_1^X > \omega_1^{ck}$. It is in particular a meager and nullset.

Let us detail a little bit the set $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ that we can consider so that $\mathcal{U} = \bigcup_{\sigma\in B}[\sigma]$. To ease the notation we introduce the following definition, in the same spirit as $\mathcal{U}(\mathcal{B},\sigma)$ defined previously:

Definition 3.4.12 : Let $\mathcal{B}$ be a Σ^0_α class. We define $\mathcal{V}(\mathcal{B},\sigma)$ to be the set

$$\{Y \ : \ \exists \tau \subseteq Y - \{0,\dots,|\sigma|\}\ \sigma\cup\tau\,?\!\nvdash \mathcal{B}\}$$

Given a $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ with $\mathcal{B}_n = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$, given

$$\mathcal{U} = \{Y \ : \ \exists \tau \subseteq Y - \{0,\dots,|\sigma|\}\ \exists n\ \sigma\cup\tau\,?\!\nvdash 2^\omega - \mathcal{B}_n\}$$

we have by Corollary 3.4.7 that $\mathcal{U}$ equals:

$$\{Y \ : \ \exists \tau \subseteq Y - \{0,\dots,|\sigma|\}\ \exists n\ \forall \alpha < \omega_1^{ck}\ \mathcal{V}(2^\omega - \mathcal{B}_{n,\alpha}, \sigma\cup\tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

Let

$$B = \{\tau \text{ with } \tau(i) = 0 \text{ for } i < |\sigma| : \exists n\ \forall \alpha < \omega_1^{ck}\ \mathcal{V}(2^\omega - \mathcal{B}_{n,\alpha}, \sigma\cup\tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

Let

$$B_\alpha = \{\tau \text{ with } \tau(i) = 0 \text{ for } i < |\sigma| : \exists n\ \forall \beta < \alpha\ \mathcal{V}(2^\omega - \mathcal{B}_{n,\beta}, \sigma\cup\tau) \cap \mathcal{L}_C \text{ is not a largeness class}\}$$

By Σ^1_1-boundedness (see Lemma 3.1.5) we have that $B = \bigcap_\alpha B_\alpha$. We also have $\mathcal{U} = \bigcup_{\sigma\in B}[\sigma]$.

We now show the core lemma that will be used to show $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$ for $\mathcal{F}$ a sufficiently generic filter:

Lemma 3.4.13 : Let $B = \bigcap_{\alpha<\omega_1^{ck}} B_\alpha$ be a $\Sigma^1_1(C)$ set of strings where each B_α is $\Delta^1_1(C)$ uniformly in α and where $\beta < \alpha$ implies $B_\alpha \subseteq B_\beta$. Let $\mathcal{U} = \bigcup_{\sigma\in B}[\sigma]$ be a $\Sigma^1_1(C)$ upward closed open set with $\mathcal{U}_\alpha = \bigcup_{\sigma\in B_\alpha}[\sigma]$ a $\Delta^1_1(C)$ upward closed open set. We have $\mathcal{U} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha$. Furthermore, $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class iff for every $\alpha < \omega_1^{ck}$, $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class. ⋆

Proof: It is clear that $\mathcal{U} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha$. Also it is clear that if $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class, then also $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_\alpha \cap \mathcal{L}_C$ is a largeness class.

Suppose $\mathcal{U} \cap \mathcal{L}_C$ is not a largeness class. Then there is a cover $Y_0 \cup \cdots \cup Y_k \supseteq \omega$ with $Y_i \notin \mathcal{U} \cap \mathcal{L}_C$ for every $i \leqslant k$. There must be a $\Sigma_1^0(C)$ open set $\mathcal{V} \supseteq \mathcal{L}_C$ such that $Y_i \notin \mathcal{U} \cap \mathcal{V}$ for every $i \leqslant k$.

Let $f : \omega \to \omega_1^{ck}$ be the function which on n finds a cover $\sigma_0 \cup \cdots \cup \sigma_k \supseteq \{0, \ldots, n\}$ and α such that for $i \leqslant k$ and every $\tau \preccurlyeq \sigma_i$ we have $[\tau] \subseteq \mathcal{V}$ implies $\tau \notin B_\alpha$. As $\mathcal{U} \cap \mathcal{V}$ is not a largeness class, f is a total $\Pi_1^1(C)$ function. By Σ_1^1-boundedness, $\beta = \sup_n f(n) < \omega_1^C = \omega_1^{ck}$. By compactness, there is a cover $Y_0 \cup \cdots \cup Y_k$ such that for every $i \leqslant k$ if $Y_i \in \mathcal{V}$ then for every $\tau \prec Y_i$, $\tau \notin B_\beta$ and thus $Y_i \notin \mathcal{U}_\beta$.

It follows that $\mathcal{U}_\beta \cap \mathcal{L}_C$ is not a largeness class. ■

Corollary 3.4.14 : $\mathcal{L}_C$ contains a unique largeness subclass, which is minimal for both Π_1^1 and $\Sigma_1^1(C)$-open sets $\mathcal{U}$.

Proof: Suppose $\mathcal{U}_0, \mathcal{U}_1$ are two $\Sigma_1^1(C)$ open sets with $\mathcal{U}_i = \bigcup_{\sigma \in B_i} [\sigma]$ and $\mathcal{U}_{i,\alpha} = \bigcup_{\sigma \in B_{i,\alpha}} [\sigma]$. for $i < 2$. Suppose also $\mathcal{U}_0 \cap \mathcal{L}_C$ and $\mathcal{U}_1 \cap \mathcal{L}_C$ are largeness classes. By Lemma 3.4.13 it follows that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{0,\alpha} \cap \mathcal{L}_C$ and $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{1,\alpha} \cap \mathcal{L}_C$ are largeness classes. By Lemma 3.4.4 it follows that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{0,\alpha}$ and $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{1,\alpha}$.

As $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{0,\alpha} \cap \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{1,\alpha} = \bigcap_{\alpha<\omega_1^{ck}} (\mathcal{U}_{0,\alpha} \cap \mathcal{U}_{1,\alpha})$ then $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \bigcap_{\alpha<\omega_1^{ck}} (\mathcal{U}_{0,\alpha} \cap \mathcal{U}_{1,\alpha})$. As $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \cap \mathcal{L}_C$ is a largeness class then $\bigcap_{\alpha<\omega_1^{ck}} (\mathcal{U}_{0,\alpha} \cap \mathcal{U}_{1,\alpha}) \cap \mathcal{L}_C$ is a largeness class. Thus by Lemma 3.4.13 the set $\mathcal{U}_0 \cap \mathcal{U}_1$ is a largeness class.

It follow that the intersection $\mathcal{I}$ of every $\Sigma_1^1(C)$ open set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class, is a largeness class. Furthermore as $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \cap \mathcal{L}_C$ is a largeness class for every α, the class $\mathcal{I}$ must be included in $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$. Also from Corollary 3.4.8 the class $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is minimal for Π_1^1-open sets $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. It follows that the class $\mathcal{I} \cap \mathcal{L}_C$ is minimal for $\Sigma_1^1(C)$ and Π_1^1 open sets. ■

We can now detail the class $\mathcal{S}$ involved in the definition of $\mathbb{P}_{\omega_1^{ck}}$: Let $\mathcal{S}$ be the unique largeness class included in $\mathcal{L}_C$ which is minimal for $\Sigma_1^1(C)$ and Π_1^1 open sets, that is, $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ intersected with every $\Sigma_1^1(C)$ open set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. Note that $\mathcal{S}$ must be partition regular.

Lemma 3.4.15 : Consider a $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ with $\Pi^0_{\omega_1^{ck}}$ set $\mathcal{B}_n = \bigcap_{\alpha\in\omega_1^{ck}} \mathcal{B}_{n,\alpha}$. Let $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. Suppose $\sigma \mathrel{?}\vdash \mathcal{B}$. Then there is a condition $q \preccurlyeq p$ together with some n such that $q \Vdash \bigcap_{\alpha<\omega_1^{ck}} \mathcal{B}_{n,\alpha}$ ⋆

Proof: Let

$$\mathcal{U} = \{Y \ : \ \exists \tau \subseteq Y - \{0, \ldots, |\sigma|\} \ \exists n \ \sigma \cup \tau \mathrel{?}\nvdash 2^\omega - \mathcal{B}_n\}$$

The class $\mathcal{U}$ is a $\Sigma_1^1(C)$-open set and $\mathcal{U} \cap \mathcal{L}_C$ is a largeness class. By definition of $\mathcal{S}$, $\mathcal{S} \subseteq \mathcal{U}$. As $X \in \mathcal{S} \subseteq \mathcal{U}$ there is some $\tau \subseteq X - \{0, \ldots, |\sigma|\}$ and some n such that $\sigma \cup \tau \mathrel{?}\nvdash 2^\omega - \mathcal{B}_n$. Let now

$$\mathcal{V} = \{Y \ : \ \exists \rho \subseteq Y - \{0, \ldots, |\sigma \cup \tau|\} \ \exists \alpha \ \sigma \cup \tau \cup \rho \mathrel{?}\nvdash \mathcal{B}_{n,\alpha}\}$$

As $\sigma \cup \tau \mathrel{?}\nvdash \bigcup_{\alpha\in\omega_1^{ck}} 2^\omega - \mathcal{B}_{n,\alpha}$ then $\mathcal{V} \cap \mathcal{L}_C$ is not a largeness class. Thus there is a cover $Y_0 \cup \cdots \cup Y_k \supseteq \omega$ such that $Y_i \notin \mathcal{V} \cap \mathcal{L}_C$ for every $i \leqslant k$. As $\mathcal{V} \cap \mathcal{L}_C$ is upward-closed,

$X \cap Y_i \notin \mathcal{V} \cap \mathcal{L}_C$ for every $i \leqslant k$. As $\mathcal{S} \subseteq \mathcal{L}_C$ is partition regular, there is some $i \leqslant k$ such that $X \cap Y_i \in \mathcal{S} \subseteq \mathcal{L}_C$. Therefore we must have $X \cap Y_i \notin \mathcal{V}$ and thus

$$\forall \rho \subseteq X \cap Y_i - \{0, \ldots, |\sigma \cup \tau|\}\ \forall \alpha\ \sigma \cup \tau \cup \rho\ ?\!\vdash \mathcal{B}_{n,\alpha}$$

Thus $(\sigma \cup \tau, X \cap Y_i)$ is an extension of (σ, X) such that:

$$(\sigma \cup \tau, X \cap Y_i) \Vdash \bigcap_{\alpha < \omega_1^{ck}} \mathcal{B}_{n,\alpha}$$

Lemma 3.4.16 : Consider a $\Sigma^0_{\omega_1^{ck}+1}$ class $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$ with $\Pi^0_{\omega_1^{ck}}$ set $\mathcal{B}_n = \bigcap_{\alpha < \omega_1^{ck}} \mathcal{B}_{n,\alpha}$. Let $p = (\sigma, X) \in \mathbb{P}_{\omega_1^{ck}}$. Suppose $\sigma\ ?\!\nvdash \mathcal{B}$. Then there is a condition $q \preccurlyeq p$ together with some $\beta < \omega_1^{ck}$ such that $q \Vdash \bigcap_{n \in \omega} \bigcup_{\alpha < \beta} 2^\omega - \mathcal{B}_{n,\alpha}$. ⋆

PROOF: Let

$$\mathcal{U} = \{Y\ :\ \exists \tau \subseteq Y - \{0, \ldots, |\sigma|\}\ \exists n\ \sigma \cup \tau\ ?\!\nvdash 2^\omega - \mathcal{B}_n\}$$

The class $\mathcal{U}$ is a $\Sigma^1_1(C)$-open set and $\mathcal{U} \cap \mathcal{L}_C$ is not a largeness class. Let us recall Definition 3.4.12 together with the notation coming after it: $\mathcal{V}(\mathcal{B}, \sigma)$ is the set

$$\{Y\ :\ \exists \tau \subseteq Y - \{0, \ldots, |\sigma|\}\ \sigma \cup \tau\ ?\!\nvdash \mathcal{B}\}$$

Together with

$B = \{\tau$ with $\tau(i) = 0$ for $i < |\sigma| :\ \exists n\ \forall \alpha < \omega_1^{ck}\ \mathcal{V}(\mathcal{B}_{n,\alpha}, \sigma \cup \tau) \cap \mathcal{L}_C$ is not a largeness class$\}$

with $B = \bigcap_{\alpha < \omega_1^{ck}} B_\alpha$ such that

$B_\alpha = \{\tau$ with $\tau(i) = 0$ for $i < |\sigma| :\ \exists n\ \forall \beta < \alpha\ \mathcal{V}(\mathcal{B}_{n,\beta}, \sigma \cup \tau) \cap \mathcal{L}_C$ is not a largeness class$\}$

and with $\mathcal{U} = \bigcup_{\sigma \in B} [\sigma]$.

Using Lemma 3.4.13, there is some $\alpha < \omega_1^{ck}$ — that we can suppose limit — such that the set

$\mathcal{U}_\alpha = \{Y\ :\ \exists \tau \subseteq Y - \{0, \ldots, |\sigma|\}\ \exists n\ \forall \beta < \alpha\ \mathcal{V}(\mathcal{B}_{n,\beta}, \sigma \cup \tau) \cap \mathcal{L}_C$ is not a largeness class$\}$

is such that $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is not a largeness class. Thus there is a cover $Y_0 \cup \cdots \cup Y_{k-1} \supseteq \omega$ such that $Y_i \notin \mathcal{U}_\alpha \cap \mathcal{L}_C$ for every $i < k$. As $\mathcal{U}_\alpha \cap \mathcal{L}_C$ is upward-closed, then also $X \cap Y_i \notin \mathcal{U}_\alpha \cap \mathcal{L}_C$ for every $i < k$. As $X \in \mathcal{S} \subseteq \mathcal{L}_C$ and as $\mathcal{S}$ is partition regular, there is some $i < k$ such that $X \cap Y_i \in \mathcal{S} \subseteq \mathcal{L}_C$. It follows that $X \cap Y_i \notin \mathcal{U}_\alpha$ and thus that:

$$\forall \tau \subseteq X \cap Y_i - \{0, \ldots, |\sigma|\}\ \forall n\ \exists \beta < \alpha\ \mathcal{V}(\mathcal{B}_{n,\beta}, \sigma \cup \tau) \cap \mathcal{L}_C \text{ is a largeness class}$$

Let $\{\beta_m\}_{m \in \omega}$ be such that $\sup_m \beta_m = \alpha$. Let $\tau \subseteq X \cap Y_i - \{0, \ldots, |\sigma|\}$ and $n \in \omega$. We have for some m that $\mathcal{V}(\mathcal{B}_{n,\beta_m}, \sigma \cup \tau) \cap \mathcal{L}_C$ is a largeness class. Then the set

$$\{Y\ :\ \exists \rho \subseteq Y - \{0, \ldots, |\sigma \cup \tau|\}\ \exists m\ \sigma \cup \tau \cup \rho\ ?\!\nvdash \mathcal{B}_{n,\beta_m}\} \cap \mathcal{L}_C$$

is a largeness class and then

$$\{Y\ :\ \exists \rho \subseteq Y - \{0, \ldots, |\sigma \cup \tau|\}\ \exists m\ \sigma \cup \tau \cup \rho\ ?\!\nvdash \mathcal{B}_{n,\beta_m}\} \cap \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$$

is a largeness class and thus $\sigma \cup \tau\ ?\!\vdash \bigcup_m 2^\omega - \mathcal{B}_{n,\beta_m}$. As this is true for every n and every $\tau \subseteq X \cap Y_i - \{0, \ldots, |\sigma|\}$ it follows that $(\sigma, X \cap Y_i)$ is an extension of (σ, X) such that

$$(\sigma, X \cap Y_i) \Vdash \bigcap_{n \in \omega} \bigcup_{\beta < \alpha} 2^\omega - \mathcal{B}_{n,\beta}$$

This concludes the proof. ■

We now show that if $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ is sufficiently generic, then $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$. We use the following fact : If $\omega_1^G > \omega_1^{ck}$, then in particular some G-computable ordinal must code for ω_1^{ck}, that is, there must be a G-computable function Φ such that for every n, $\Phi(G,n)$ codes, relative to G, for an ordinal smaller than ω_1^{ck} and with $\sup_n |\Phi(G,n)| = \omega_1^{ck}$. We show that this never happens by forcing that for every functional Φ either for some n, $\Phi(G,n)$ does not code for an ordinal smaller than ω_1^{ck}, or there is an ordinal $\alpha < \omega_1^{ck}$ such that $\Phi(G,n)$ always codes for some ordinal smaller than α.

Given G and α let $\mathcal{O}_\alpha^G$ be the set of G-codes for ordinals smaller than α. For $\alpha < \omega_1^{ck}$, the class $\{G \ : \ n \in \mathcal{O}_\alpha^G\}$ is Δ_1^1 uniformly in α and n.

Theorem 3.4.17:
Suppose $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ is sufficiently generic. Then $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$.

PROOF: Let $p \in \mathbb{P}_{\omega_1^{ck}}$ be a condition. Given a functional $\Phi : 2^\omega \times \omega \to \omega$, let

$$\mathcal{B} = \{X \ : \ \exists n \ \forall \alpha < \omega_1^{ck} \ \Phi(X,n) \notin \mathcal{O}_\alpha^X\}$$

Suppose $p \,?\!\vdash \mathcal{B}$. Then from Lemma 3.4.15, there is an extension $q \leqslant p$ and some n such that

$$q \Vdash \{X \ : \ \forall \alpha < \omega_1^{ck} \ \Phi(X,n) \notin \mathcal{O}_\alpha^X\}$$

It follows from Proposition 3.4.10 that if $\mathcal{F}$ is sufficiently generic, for every $\alpha < \omega_1^{ck}$, $\Phi(G_\mathcal{F},n) \notin \mathcal{O}_\alpha^{G_\mathcal{F}}$. Suppose now $p \,?\!\nvdash \mathcal{B}$. Then from Lemma 3.4.16, there is an extension $q \leqslant p$ and some $\alpha < \omega_1^{ck}$ such that

$$q \Vdash \{X \ : \ \forall n \ \Phi(X,n) \in \mathcal{O}_\alpha^X\}$$

It follows from Theorem 3.3.9 that if $\mathcal{F}$ is sufficiently generic, $\sup_n \Phi(G_\mathcal{F},n) \leqslant \alpha$. ∎

We can finally show the desired theorem.

Theorem (3.0.1): *Let Z be non Δ_1^1. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not $\Delta_1^1(G)$ (and in particular with $\omega_1^G = \omega_1^{ck}$).*

PROOF: Let $\mathcal{F} \subseteq \mathbb{P}_{\omega_1^{ck}}$ be sufficiently generic. Then from Theorem 3.3.10 the set Z is not $\Delta_1^0(G_\mathcal{F}^{(\alpha)})$ for $\alpha < \omega_1^{ck}$. From Theorem 3.4.17 $\omega_1^{G_\mathcal{F}} = \omega_1^{ck}$. Thus Z is not $\Delta_1^1(G_\mathcal{F}^{(\alpha)})$. ∎

Chapter 4

Mathias forcing to create non-cohesive sets

The goal of this chapter is to prepare the next one : Chapter 5. In order to separate SRT^2_2 from RT^2_2 in ω-models, we have to show the following : for any set A, there is a set $G \in [A]^\omega \cup [\omega - A]^\omega$ which computes no p-cohesive set [1].

The combination of the two following facts illustrates the difficulty of proving the theorem to come.

1. Effective Mathias forcing is the only known technic to find for any set A an element $G \in [A]^\omega \cup [\omega - A]^\omega$ which has some lowness property (cone avoiding, non PA, etc...).

2. Sets which are sufficiently generic for Mathias forcing are cohesive.

We give in this chapter the beginning of a solution in order to overcome the difficulty raised by (1) and (2) : we show how to use Mathias forcing so that sufficiently generic sets are not cohesive. We do so while proving Liu's theorem :

Theorem 4.0.1 (Liu [23]):
For any set A an element $G \in [A]^\omega \cup [\omega - A]^\omega$ is not of PA degree.

We show that such an element G in the previous theorem can always be picked non-cohesive.

4.1 Partition genericity

We start by enriching our toolbox with a new notion that will make our life simpler in the proofs to come. Recall that all the partition regular class we consider are non-trivial. We use here the notation $\mathcal{U}_C$ for the intersection of Σ^0_1 class : $\bigcap_{e \in C} \mathcal{U}_e$.

Definition 4.1.1 : Let $\mathcal{U}_C$ be a largeness class. We say that X is *partition generic below* $\mathcal{U}_C$ if for every Σ^0_1 set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{U}_C$ is a largeness class, we have $X \in \mathcal{U} \cap \mathcal{U}_C$.

If X is partition generic below 2^ω we simply say that X is *partition generic*. We say that X is *bi-partition generic below* $\mathcal{U}_C$ is both X and $\overline{X}$ are partition generic below $\mathcal{U}_C$. ◊

[1] This is anyway the spirit. In practice we will not be able to show that for *any* set A, but for sufficiently many of them.

We have that ω is partition-generic. From Proposition 1.4.5 every non-trivial partition regular class if of measure 1. It follows that any Kurtz-random belongs to every Π^0_2 partition regular classes and thus that any Kurtz-random is *bi-partition generic*.

The class of partition generic elements is not itself partition regular. But the class of elements which are partition generic *somewhere* is partition regular. This is made clear through the two following lemmas.

Lemma 4.1.2 : Let $\mathcal{U}_C$ be a largeness class. Suppose X is partition generic below $\mathcal{U}_C$. Let $Y_0 \cup Y_1 \supseteq X$. Suppose $Y_i \notin \mathcal{L}(\mathcal{U}_C)$. Then Y_{1-i} is partition generic below $\mathcal{U}_C$. ⋆

PROOF: We assume that X is partition generic below $\mathcal{U}_C$ and $Y_i \notin \mathcal{L}(\mathcal{U}_C)$.

Suppose for contradiction that there is a Σ^0_1 class $\mathcal{V}$ such that $\mathcal{V} \cap \mathcal{U}_C$ is a largeness class, and such that $Y_{1-i} \notin \mathcal{V} \cap \mathcal{U}_C$. In particular there is a Σ^0_1 class $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{U}_C$ is a largeness class and such that $Y_{i-1} \notin \mathcal{U}$ and $Y_i \notin \mathcal{U}$.

Note that as X is partition generic below $\mathcal{U}_C$, we have $X \in \mathcal{L}(\mathcal{U} \cap \mathcal{U}_C)$. However we have $Y_i \notin \mathcal{U} \cap \mathcal{U}_C$ and $Y_{i-1} \notin \mathcal{U} \cap \mathcal{U}_C$. This contradicts that $X \in \mathcal{L}(\mathcal{U} \cap \mathcal{U}_C)$. Thus Y_{i-1} must be partition generic below $\mathcal{U}_C$. ■

Lemma 4.1.3 : Let $\mathcal{U}_C$ be a largeness class. Suppose X is partition generic below $\mathcal{U}_C$. Let $Y_0 \cup \dots \cup Y_k \supseteq X$. Then there is a Σ^0_1 class $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{U}_C$ is a largeness class, together with some $i \leqslant k$ such that Y_i is partition generic below $\mathcal{U} \cap \mathcal{U}_C$. ⋆

PROOF: We assume that X is partition generic below $\mathcal{U}_C$. We proceed by induction on $i \leqslant k$. Suppose Y_0 is not partition generic below $\mathcal{U}_C$. Thus there must be an open set $\mathcal{V}_0$ such that $\mathcal{V}_0 \cap \mathcal{U}_C$ is a largeness class but with $Y_0 \notin \mathcal{V}_0$. By Lemma 4.1.2 it follows that $Y_1 \cup \dots \cup Y_k$ is partition generic below $\mathcal{V}_0 \cap \mathcal{U}_C$.

We continue inductively : if Y_1 is not partition generic below $\mathcal{V}_0 \cap \mathcal{U}_C$, there must be an open set $\mathcal{V}_1$ such that $\mathcal{V}_1 \cap \mathcal{V}_0 \cap \mathcal{U}_C$ is a largeness class but with $Y_1 \notin \mathcal{V}_1$. It follows that $Y_2 \cup \dots \cup Y_k$ is partition generic below $\mathcal{V}_1 \cap \mathcal{V}_0 \cap \mathcal{U}_C$.

If we can continue like this for every $i < k$. Then we have Σ^0_1 classes $\mathcal{V}_0, \dots, \mathcal{V}_{k-1}$ such that $\mathcal{V}_{k-1} \cap \dots \cap \mathcal{V}_0 \cap \mathcal{U}_C$ is a largeness class with Y_k partition generic below $\mathcal{V}_{k-1} \cap \dots \cap \mathcal{V}_0 \cap \mathcal{U}_C$.

Otherwise we stop for some $j < k - 1$ and we have Σ^0_1 classes $\mathcal{V}_0, \dots, \mathcal{V}_j$ such that $\mathcal{V}_j \cap \dots \cap \mathcal{V}_0 \cap \mathcal{U}_C$ is a largeness class and such that Y_{j+1} is partition generic below $\mathcal{V}_j \cap \dots \cap \mathcal{V}_0 \cap \mathcal{U}_C$. ■

Partition genericity clearly relativizes to some oracle : Given $\mathcal{U}_C = \bigcap_{e \in C} \mathcal{U}_e$ with each $\mathcal{U}_e$ a $\Sigma^0_1(B)$ class, a set X is B-partition generic below $\mathcal{U}_C$ if for every $\Sigma^0_1(B)$ open set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{U}_C$ is a largeness class, we have $X \in \mathcal{U}$.

It is also clear how Lemma 4.1.3 relativizes : Suppose X is B-partition generic below $\mathcal{U}_C$. Let $Y_0 \cup \dots \cup Y_k \supseteq X$. Then there is a $\Sigma^0_1(B)$ class $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{U}_C$ is a largeness class, together with some $i \leqslant k$ such that Y_i is B-partition generic below $\mathcal{U} \cap \mathcal{U}_C$.

4.2 Liu's theorem

We give in this section a simplification on the original proof of Liu's theorem, using the machinery developed so far : partition regularity, largeness and partition genericity. We actually show slightly more than Liu's theorem : not only $G \in [A]^\omega \cup [\omega - A]^\omega$ can be picked non-PA, but we can make sure that it belongs to any Π^0_2 largeness class fixed in advance. We also show a relative version of this:

Theorem 4.2.1 (Improvement of Liu's):
Let A be any set. Let B be non-PA. Let $\bigcap_n \mathcal{U}_n$ be a $\Pi^0_2(B)$ largeness class. There is an element $G \in [A]^\omega \cup [\omega - A]^\omega$ with $G \in \bigcap_n \mathcal{U}_n$ and such that $G \oplus B$ is non-PA.

The forcing conditions will depend on the oracle B and on the $\Pi^0_2(B)$ largeness class $\bigcap_n \mathcal{U}_n$, that we then both fix now.

Lemma 4.2.2 : Let $A_0 \cup A_1 \supseteq \omega$ be an arbitrary cover of ω. For some $i < 2$, there exists a $\Sigma^0_1(B)$ class $\mathcal{U}$ such that $\mathcal{U} \cap \bigcap_n \mathcal{U}_n$ is a largeness class, and such that A_i is B-partition generic below $\mathcal{U} \cap \bigcap_n \mathcal{U}_n$. ⋆

Proof: We have that ω is B-partition generic below $\bigcap_n \mathcal{U}_n$. The current lemma then follows from Lemma 4.1.3. ∎

Let now $A_0 \cup A_1 \supseteq \omega$ be an arbitrary cover of ω. From the previous lemma, there exists $i < 2$ and a $\Sigma^0_1(B)$ class $\mathcal{U}$ be such that A_i is B-partition generic below $\mathcal{U} \cap \bigcap_n \mathcal{U}_n$. From now on we fix $A = A_i$. Let $\mathbb{P}$ be the set of conditions $(\sigma, X, \mathcal{U})$ where:

1. (σ, X) is a Mathias condition
2. $\sigma \subseteq A$
3. $\mathcal{U}$ is a $\Sigma^0_1(B)$ class such that $\mathcal{U} \cap \bigcap_n \mathcal{U}_n$ is a largeness class
4. $X \subseteq A$ is partition generic below $\mathcal{U} \cap \bigcap_n \mathcal{U}_n$

The order is defined by $(\sigma, X, \mathcal{U}) \leq (\tau, Y, \mathcal{V})$ if:

1. $(\sigma, X) \leq (\tau, Y)$ as Mathias conditions
2. $\mathcal{U} \subseteq \mathcal{V}$

Note that here again we know in advance in which side A_0 or A_1 our generic will be : whichever that can be B-partition generic somewhere below $\bigcap_n \mathcal{U}_n$.

Notation

Let $\mathcal{F} \subseteq \mathbb{P}$ be a sufficiently generic filter. We write $G_{\mathcal{F}} \in 2^\omega$ for the unique set such that $\sigma < G_{\mathcal{F}}$ for every $(\sigma, X, \mathcal{U}) \in \mathcal{F}$.

Definition 4.2.3 : Let $p = (\sigma, X, \mathcal{U})$ be a forcing condition. Let $\Phi_e(B \oplus G, n)$ be a functional. We define:

$$\begin{array}{lcl} p \Vdash \exists n\ \Phi_e(B \oplus G, n) \downarrow = \Phi_n(n) & \leftrightarrow & \exists n\ \Phi_e(B \oplus \sigma, n) \downarrow = \Phi_n(n) \\ p \Vdash \exists n\ \Phi_e(B \oplus G, n) \uparrow & \leftrightarrow & \exists n\ \forall \sigma \subseteq X\ \Phi_e(B \oplus \sigma_i \cup \sigma, n) \uparrow \end{array}$$

Note that if $p \Vdash \exists n\ \Phi_e(B \oplus G, n) \downarrow = \Phi_n(n)$ it is clear that for any generic filter $\mathcal{F} \subseteq \mathbb{P}$ with $p \in \mathcal{F}$ we have $\exists n\ \Phi_e(B \oplus G_{\mathcal{F}}, n) \downarrow = \Phi_n(n)$. Similarly if $p \Vdash \exists n\ \Phi_e(B \oplus G, n) \uparrow$ then for any generic filter $\mathcal{F} \subseteq \mathbb{P}$ with $p \in \mathcal{F}$, we have $\exists n\ \Phi_e(B \oplus G_{\mathcal{F}}, n) \uparrow$.

Lemma 4.2.4 : Let $p = (\sigma, X, \mathcal{U})$ be a forcing condition. Let $\Phi_e(B \oplus G, n)$ be a functional. There exists a condition $q \leqslant p$ such that:

$$\begin{array}{rcl} & q & \Vdash & \exists n\ \Phi_e(B \oplus G, n) \downarrow = \Phi_n(n) \\ \text{or} & q & \Vdash & \exists n\ \Phi_e(B \oplus G, n) \uparrow \end{array}$$

PROOF: Recall that $\bigcap_{k\in\omega} \mathcal{U}_k$ is the fixed $\Pi^0_2(B)$ largeness class such that X is B-partition generic below $\mathcal{U} \cap \bigcap_{k\in\omega} \mathcal{U}_k$. Let $P(n,k,i)$ be the predicate:

$$\forall X_0 \cup \dots \cup X_k \supseteq \omega\ \exists j \leqslant k\ X_j \in \mathcal{U} \cap \mathcal{U}_k \text{ and } \exists \tau \subseteq X_j - \{0, \dots, |\sigma|\}\ \Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = i$$

Suppose first that the following is true for every $k \in \omega$:

$$\forall n\ \exists i < 2\ P(n,k,i)$$

Note that it is uniformly B-c.e. to know whether $P(n,k,i)$ is true. Let us fix $k \in \omega$. Let us define the computable functional $\Psi_k : \omega \to \{0,1\}$ defined by $\Psi_k(n)$ to be the first value $i < 2$ that is found such that $P(n,k,i)$ is true. By hypothesis the functional Ψ_k is a total B-computable function. In particular as B is non-P.A. there exists n such that $\Psi_k(n) = \Phi_n(n)$. We thus have :

$$\forall X_0 \cup \dots \cup X_k \supseteq \omega\ \exists j \leqslant k\ X_j \in \mathcal{U} \cap \mathcal{U}_k \text{ and } \exists \tau \subseteq X_j - \{0, \dots, |\sigma|\}\ \exists n\ \Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = \Phi_n(n)$$

As $\mathcal{U}_{k+1} \subseteq \mathcal{U}_k$ we also have for every k, k' with $k' \geqslant k$ that

$$\forall X_0 \cup \dots \cup X_{k'} \supseteq \omega\ \exists j \leqslant k'\ X_j \in \mathcal{U} \cap \mathcal{U}_k \text{ and } \exists \tau \subseteq X_j - \{0, \dots, |\sigma|\}\ \exists n\ \Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = \Phi_n(n)$$

As this is true for every k, k' with $k' \geqslant k$, it implies by Lemma 1.4.8 that for every k the set ω belongs to $\mathcal{L}(\mathcal{U}_k \cap \mathcal{V})$ where:

$$\mathcal{V} = \{Y \in \mathcal{U}\ :\ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\}\ \exists n\ \Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = \Phi_n(n)\}$$

and in particular $\mathcal{V} \cap \bigcap_{k\in\omega} \mathcal{U}_k$ is a largeness class.

As X is B-partition generic below $\mathcal{U} \cap \bigcap_{k\in\omega} \mathcal{U}_k$, it belongs to$\mathcal{V} \cap \bigcap_{k\in\omega} \mathcal{U}_k$. We thus must have some $\tau \subseteq X - \{0, \dots, |\sigma|\}$ such that $\exists n\ \Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = \Phi_n(n)$. Also as X is B-partition generic below $\mathcal{U} \cap \bigcap_{k\in\omega} \mathcal{U}_k$ and as $\mathcal{L}(\mathcal{V} \cap \bigcap_{k\in\omega} \mathcal{U}_k)$ contains only infinite sets, we must have by Lemma 4.1.2 that $X - \{0, \dots, |\sigma \cup \tau|\}$ is B-partition generic below $\mathcal{U} \cap \bigcap_{k\in\omega} \mathcal{U}_k$. Let $q = (\sigma \cup \tau, X - \{0, \dots, |\sigma \cup \tau|\}, \mathcal{U})$. We have that $q \Vdash \exists n\ \Phi_e(B \oplus G, n) \downarrow = \Phi_n(n)$.

Suppose now that there exists $k \in \omega$ such that

$$\exists n\ \forall i < 2\ \neg P(n,k,i)$$

In particular we have for some k and some n we have a cover $X^0_0 \cup \dots \cup X^0_k \supseteq \omega$ and a cover $X^1_0 \cup \dots \cup X^1_k \supseteq \omega$ such that for $i < 2$ we have:

$$\forall j < k\ X^i_j \notin \mathcal{U} \cap \mathcal{U}_k \text{ or } \forall \tau \subseteq X^i_j - \{0, \dots, |\sigma|\}\ \Phi_e(B \oplus \sigma \cup \tau, n) \uparrow \text{ or } \Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = 1 - i$$

We have that $\{X^0_a \cap X^1_b\}_{a<k,b<k}$ is a cover of ω. As X is B-partition generic below $\mathcal{U} \cap \bigcap_{k\in\omega} \mathcal{U}_k$ and as $\mathcal{U} \cap \bigcap_{k<\omega} \mathcal{U}_k$ is a largeness class, there must be by Lemma 4.1.3 some $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \cap \bigcap_{k<\omega} \mathcal{U}_k$ is a largeness class, together with $a, b < k$ such that $X^0_a \cap X^1_b \cap X$ is B-partition generic below $\mathcal{V} \cap \bigcap_{k<\omega} \mathcal{U}_k$. Note that as $X^0_a \cap X^1_b \cap X \in \mathcal{L}(\mathcal{V} \cap \bigcap_{k<\omega} \mathcal{U}_k)$, we must have also that both X_a^0 and X_b^0 belong to $\mathcal{L}(\mathcal{V} \cap \bigcap_{k<\omega} \mathcal{U}_k)$ (as largeness classes are closed upwards). In particular $X_a^0, X_b^1 \in \mathcal{V} \cap \mathcal{U}_k \subseteq \mathcal{U} \cap \mathcal{U}_k$. It follows that:

$$\begin{array}{l} \forall \tau \subseteq X_a^0 - \{0,\dots,|\sigma|\}\ \Phi_e(B \oplus \sigma \cup \tau, n) \uparrow \text{ or } \Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = 1 \\ \forall \tau \subseteq X_b^1 - \{0,\dots,|\sigma|\}\ \Phi_e(B \oplus \sigma \cup \tau, n) \uparrow \text{ or } \Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = 0 \end{array}$$

Let $\tau \subseteq (X_a^0 \cap X_b^1) - \{0,\dots,|\sigma|\}$. As we cannot have both $\Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = 1$ and $\Phi_e(B \oplus \sigma \cup \tau, n) \downarrow = 0$, it must be that $\Phi_e(B \oplus \sigma \cup \tau, n) \uparrow$. Thus $\forall \tau \subseteq (X_a^0 \cap X_b^1) - \{0,\dots,|\sigma|\}\ \Phi_e(B \oplus \sigma \cup \tau, n) \uparrow$. Let $q = (\sigma, X_a^0 \cap X_b^1 \cap X, \mathcal{V})$. We have $q \Vdash \exists n\ \Phi_e(B \oplus G, n) \uparrow$.■

Lemma 4.2.5 : Let $p = (\sigma, X, \mathcal{U})$ be a forcing condition. For any $n \in \omega$ there is a forcing condition $q = (\tau, Y, \mathcal{U}) \leqslant (\sigma, X, \mathcal{U})$ such that $\tau \subseteq \mathcal{U}_n$ (where $\bigcap_n \mathcal{U}_n$ is the largeness class we start with). ⋆

Proof: We must have $X \in \mathcal{L}(\mathcal{U} \cap \bigcap_{n\in\omega} \mathcal{U}_n)$. In particular as $\{0,\dots,|\sigma|\} \cap X = \varnothing$ and as largeness classes are closed upwards, we have $\sigma \cup X \in \mathcal{L}(\mathcal{U} \cap \bigcap_{n\in\omega} \mathcal{U}_n)$ and thus $\sigma \cup X \in \mathcal{U}_n$. It follows that there exists a prefix $\tau \prec X$ such that $[\sigma \cup \tau] \subseteq \mathcal{U}_n$. As $\mathcal{U} \cap \bigcap_{n\in\omega} \mathcal{U}_n$ contains only infinite sets, and as X is B-partition generic below $\mathcal{U} \cap \bigcap_{n\in\omega} \mathcal{U}_n$, we have by Lemma 4.1.2 that $X - \{0,\dots,|\sigma \cup \tau|\}$ is B-partition generic below $\mathcal{U} \cap \bigcap_{n\in\omega} \mathcal{U}_n$. The condition q is given by $(\sigma \cup \tau, X, \mathcal{U})$. ■

Theorem (4.2.1): *Let A be any set. Let B be non-PA. Let $\bigcap_n \mathcal{U}_n$ be a $\Pi_2^0(B)$ largeness class. There is an element $G \in [A]^\omega \cup [\omega - A]^\omega$ with $G \in \bigcap_n \mathcal{U}_n$ and such that $G \oplus B$ is non-PA.*

Proof: We use the framework developed with Lemma 4.2.5 and Lemma 4.2.4 to build $G \subseteq A$ generic enough. ■

4.3 Mathias forcing to build non-cohesive set

We now show how enhance the previous proof in order to obtain non-cohesive solution, still with Mathias forcing. We use Theorem 4.2.1 as a blackbox. The idea is rather simple : We iterate the forcing several time. Given a computable set X, we find $G_0 \subseteq A$ with $G_0 \in \mathcal{L}_X$ not PA. Then we find $G_1 \subseteq A$ with $G_1 \in \mathcal{L}_{\overline{X}}$ such that $G_0 \oplus G_1$ is not PA. As $G_0 \oplus G_1 \geqslant_T G_0 \cup G_1$ it is clear that $G_0 \cup G_1$ is not PA. Furthermore as $G_0 \cup G_1$ intersects both X and $\overline{X}$ infinitely often, it is not cohesive.

There is however a catch in this construction : recall that we work within whichever among A or $\overline{A}$ is partition generic somewhere. Suppose for instance both A and $\overline{A}$ are partition generic. Once we have built $G_0 \subseteq A$ with $G_0 \in \mathcal{L}_X$ not PA, it might be that A is nowhere G_0-partition generic below $\mathcal{L}_{\overline{X}}$. We then have no choice but to build G_1 inside $\overline{A}$. Let us illustrate this with an example :

Example 4.3.1 : We build a set A such that:

- Both A and $\overline{A}$ are partition generic
- Any infinite subset of A computes an infinite subset of $\overline{A}$ and any infinite subset of $\overline{A}$ computes an infinite subset of A.

We build A as $\sigma_0 \preccurlyeq \sigma_1 \preccurlyeq \dots$ such that $[\sigma_i]$ is included in the i-th Σ^0_1 largeness class. We also make sure that each σ_i is of even length and that it contains $2n$ for $2n < |\sigma_i|$ iff it does not contain $2n+1$. It is then clear how an infinite subset of one side computes an infinite subset of the other side. The construction goes as follow : suppose σ_i is defined. Consider the $i+1$-th Σ^0_1 (non-trivial) largeness class $\mathcal{U}$. Let X be the even numbers. It must be that $\sigma_i X$ or $\sigma_i \overline{X}$ belongs to $\mathcal{U}$. We take σ_{i+1} to be of even length and equal to $\sigma_i \tau$ for $\tau \preccurlyeq X$ or $\tau \preccurlyeq \overline{X}$ such that $[\sigma_i \tau] \subseteq \mathcal{U}$.

Now given such a set A, once we have built $G_0 \subseteq A$, A is not anymore G_0-partition generic, as it does not belong to $\mathcal{L}_{\Phi(G_0)}$ where $\Phi(G_0)$ is an infinite subset of $\overline{A}$. $\diamond$

Of course the previous example does not say anything about A being G_0-partition generic *somewhere*, but it is possible to show there is no way out like this. For instance one can build a Δ^0_2 set A such that every set $G \in [A]^\omega \cup [\overline{A}]^\omega$ which contains infinitely many even and odd numbers, compute the halting problem (and in particular is not PA). Given a set A, there is then no hope to find a generic G witnessed no to be cohesive using any computable set and its complement.

In order to overcome this difficulty, we simply iterate a third time. This is done in the next proposition.

Proposition 4.3.2 : Let A be any set. There is an element $G \in [A]^\omega \cup [\omega - A]^\omega$ which is not PA and not cohesive. $\star$

Proof: Let $X_0 \cup X_1 \cup X_2 \supseteq \omega$ be three computable infinite sets. Let $A_0 = A$ and $A_1 = \overline{A}$. Using Theorem 4.2.1 we build $G_0 \subseteq A_{i_0}$ (for $i_0 \in \{0,1\}$) not PA with $G_0 \in \mathcal{L}_{X_0}$. We then build $G_1 \subseteq A_{i_1}$ such that $G_0 \oplus G_1$ is not PA and with $G_1 \in \mathcal{L}_{X_1}$. We finally build $G_2 \subseteq A_{i_2}$ such that $G_0 \oplus G_1 \oplus G_2$ is not PA and with $G_2 \in \mathcal{L}_{X_2}$.

As $i_0, i_1, i_2 \in \{0,1\}$, we must have $i_a = i_b = i$ for some $a \neq b$ and then $G_a \cup G_b \subseteq A_i$ both not PA and not cohesive. ∎

A direct construction would of course be possible, and even necessary if one wants to push this simple idea to find a generic $G \in [A]^\omega \cup [\omega - A]^\omega$ which not only is no p-cohesive, but compute no p-cohesive set. For that we need to extend our notions of largeness classes in product spaces.

Chapter 5

Separation of RT^2_2 from SRT^2_2 in ω-models

5.1 Overview

The goal of this chapter is to show the following theorem:

Theorem 5.1.1 (M., Patey [28]):
For every set Z whose jump is not of PA degree over $\emptyset'$ and every $\Delta^{0,Z}_2$ set A, there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that $(G \oplus Z)'$ is not of PA degree over $\emptyset'$.

This theorem can then be iterated to construct an ω-model of $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ containing no set whose jump is of PA degree over $\emptyset'$ and thus no p-cohesive set.

Theorem 5.1.2 (M., Patey [28]):
There is an ω-model of $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ which is not a model of COH.

Proof: By Theorem 5.1.1, there is a countable sequence of sets $Z_0, Z_1, \dots$ such that for every $s \in \omega$, the jump of $Z_0 \oplus \dots \oplus Z_s$ is not of PA degree over $\emptyset'$, and for every $\Delta^0_2(Z_0 \oplus \dots \oplus Z_s)$ set A, there is some $t \in \omega$ such that $Z_t \subseteq A$ or $Z_t \subseteq \overline{A}$. Let $\mathcal{I} = \{X \in 2^\omega : \exists s\ X \leqslant_T Z_0 \oplus \dots \oplus Z_s\}$. The collection $\mathcal{I}$ is a Turing ideal. Let $\mathcal{M}$ be the ω-structure whose second-order part is $\mathcal{I}$. Every instance of SRT^2_2 in $\mathcal{I}$ has a solution in $\mathcal{I}$, so $\mathcal{M}$ is an ω-model of SRT^2_2. Moreover, $\mathcal{I}$ does not contain any set whose jump is of PA degree over $\emptyset'$. By Theorem 1.3.4, $\mathcal{I}$ does not contain any p-cohesive set, so $\mathcal{M}$ is not a model of COH. ■

The rest of this chapter is dedicated to the proof of Theorem 5.1.1. Note this theorem can be seen as a one jump iteration of Theorem 4.0.1, with the difference is that we do not start from any set A, but only from a Δ^0_2 sets A. It is still an open question to know whether Theorem 5.1.1 holds for any set A.

5.2 Sketch of the proof

The full proof of Theorem 5.1.1 is somewhat a bit of a complicated construction. However like in most complicated constructions, the underlying intuition behind it is not that hard to get, and having this intuition in mind while reading the proof helps a lot the understanding. The goal of this section is to provide such an intuition for the reader.

We adopt for that an iterative reasoning : We start from the ideas behind the proof of Theorem 4.0.1 given in Chapter 4 and try to apply them in the case of Theorem 5.1.1 (with $Z = \varnothing$). We then identify what goes wrong and how we are naturally "forced" to work with product spaces and to go through the ideas exposed in Section 4.3.

Note that the full proof can be read independently from the rest of the document, but this section suppose the reader went through Chapter 4 (and in particular understood the basics of largeness and partition regular classes).

Assume we have a Δ^0_2 set A with $A^0 = A$ and $A^1 = \omega - A$.

5.2.1 A first step : defeat one functional

Let Φ_e be a functional. We want to build $G \in [A^0]^\omega \cup [A^1]^\omega$ such that $\Phi_e(G', n) \downarrow\neq \Phi_n(\emptyset', n)$ or such that $\Phi_e(G', n)\uparrow$ for some n. Note that we have a computable function $\eta : \omega \times \omega \times \omega \to \omega$ such that $\exists n\ \Phi_e(G', n) \downarrow= i$ iff $\exists x\ \forall y\ \Phi_{\eta(e,n,i)}(G, x, y)$. We then want to decide the truth of Σ^0_2 statements.

To continue let us define the computable function $\zeta : \omega \times 2^{<\omega} \times \omega \to$ to be such that:

$$\mathcal{U}_{\zeta(e,\sigma,x)} = \{X \ : \ \exists \tau \subseteq X - \{0, \dots, |\sigma|\}\ \exists y\ \neg\Phi_e(\sigma \cup \tau, x, y)\}$$

Assume first for this example that A^0 is partition generic and consider forcing conditions $(\sigma, X, \mathcal{U})$ where (σ, X) is a Mathias condition with $\sigma \subseteq A^0$, with X a low set and such that $A^0 \cap X$ is partition generic below $\mathcal{U}$ for some Σ^0_1 largeness class $\mathcal{U}$. Let us define the forcing question

$$(\sigma, X, \mathcal{U})\ ?\vdash \exists x\ \forall y\ \Phi_e(G, x, y)$$

if

$$\mathcal{U} \cap \bigcap_{\tau \subseteq X \cap A^0, x \in \omega} \mathcal{U}_{\zeta(e,\sigma\cup\tau,x)} \text{ is not a largeness class.}$$

Note that the forcing question is $\Sigma^0_1(\emptyset')$ (using that $A^0 \cap X$ is $\emptyset'$-computable).

Forcing the truth of Σ^0_2 statements

Let us suppose the following is true:

$$\forall n\ \exists i \in \{0, 1\}\ p\ ?\vdash \exists x\ \forall y\ \Phi_{\eta(n,i)}(G, x, y)$$

Then the function $f : \omega \to \{0, 1\}$ which on n finds the first $i \in \{0, 1\}$ such that $p\ ?\vdash \exists x\ \forall y\ \Phi_{\eta(e,n,i)}(G, x, y)$ is $\emptyset'$-computable and total. Thus there must be some n such that $\Phi_n(\emptyset', n) \downarrow= f(n)$ and then some n such that $\mathcal{U} \cap \bigcap_{\tau\subseteq X\cap A^0, x\in\omega} \mathcal{U}_{\zeta(\eta(e,n,i),\sigma\cup\tau,x)}$ is not a largeness class with $i = \Phi_n(\emptyset', n)$. There is then a cover $Y_0 \cup \dots \cup Y_k \supseteq \omega$ such that $Y_j \notin \mathcal{U} \cap \bigcap_{\tau\subseteq X\cap A^0, x\in\omega} \mathcal{U}_{\zeta(\eta(e,n,i),\sigma\cup\tau,x)}$ for $j \leqslant k$. As $X \cap A^0$ is partition generic below $\mathcal{U}$ there is $j \leqslant k$ such that $A^0 \cap X \cap Y_j$ is partition generic below $\mathcal{V} \subseteq \mathcal{U}$ where $\mathcal{V}$ is a Σ^0_1 largeness class. In particular $A^0 \cap X \cap Y_j \in \mathcal{V} \subseteq \mathcal{U}$ and then $A^0 \cap X \cap Y_j \notin \bigcap_{\tau\subseteq X\cap A^0, x\in\omega} \mathcal{U}_{\zeta(\eta(e,n,i),\sigma\cup\tau,x)}$. Let $\tau \subseteq X \cap A^0$ and $x \in \omega$ be such that $A^0 \cap X \cap Y_j \notin \mathcal{U}_{\zeta(\eta(e,n,i),\sigma\cup\tau,x)}$. It means that for any $\rho \subseteq A^0 \cap X \cap Y_j - \{0, \dots, |\sigma \cup \tau|\}$ and any y we have $\Phi(\sigma \cup \tau \cup \rho, x, y)$. Thus $(\sigma \cup \tau, X \cap Y_j - \{0, \dots, |\sigma \cup \tau|\}, \mathcal{V})$ is a condition ensuring that any generic G extending it will satisfy $\exists x\ \forall y\ \Phi_{\eta(e,n,i)}(G, x, y)$ and thus $\Phi_e(G', n) \downarrow= i = \Phi_n(\emptyset', n)$.

Forcing the truth of Π^0_2 statements

Suppose now the following is true :

$$\exists n\ \forall i \in \{0,1\}\ p\,?\nVdash \exists x\ \forall y\ \Phi_{\eta(e,n,i)}(G,x,y)$$

Let n be such that $p\,?\nVdash \exists x\ \forall y\ \Phi_{\eta(e,n,0)}(G,x,y)$ and $p\,?\nVdash \exists x\ \forall y\ \Phi_{\eta(e,n,1)}(G,x,y)$. For $i \in \{0,1\}$ we then have that both

(a) $\mathcal{L}_0 = \mathcal{U} \cap \bigcap_{\tau \subseteq X \cap A^0, x \in \omega} \mathcal{U}_{\zeta(\eta(e,n,0),\sigma\cup\tau,x)}$

(b) $\mathcal{L}_1 = \mathcal{U} \cap \bigcap_{\tau \subseteq X \cap A^0, x \in \omega} \mathcal{U}_{\zeta(\eta(e,n,1),\sigma\cup\tau,x)}$

are such that $\mathcal{U} \cap \mathcal{L}_0$ and $\mathcal{U} \cap \mathcal{L}_1$ are largeness classes. Note that as $A^0 \cap X$ is partition generic below $\mathcal{U}$, we have $A^0 \cap X \in \mathcal{U} \cap \mathcal{L}_0$ and $A^0 \cap X \in \mathcal{U} \cap \mathcal{L}_1$. It means that:

(1) For any $\tau \subseteq A^0 \cap X$ and any x there exists $\rho_0 \subseteq A^0 \cap X - \{0,\dots,|\sigma\cup\tau|\}$ such that $\exists y\ \neg\Phi_{\eta(e,n,0)}(\sigma\cup\tau\cup\rho_0,x,y))$ holds.

(2) For any $\tau \subseteq A^0 \cap X$ and any x there exists $\rho_1 \subseteq A^0 \cap X - \{0,\dots,|\sigma\cup\tau|\}$ such that $\exists y\ \neg\Phi_{\eta(e,n,1)}(\sigma\cup\tau\cup\rho_1,x,y))$ holds.

One then easily see how to use (1) and (2) to iteratively build $G \subseteq A^0$ such that we ultimately have $\forall x\ \exists y\ \neg\Phi_{\eta(e,n,i)}(G,x,y)$ for every $i \in \{0,1\}$. We then have $\Phi_e(G',n) \neq 0$ and $\Phi_e(G',n) \neq 1$ implying $\Phi_e(G',n)\uparrow$.

5.2.2 Defeating more functionals (part 1)

The issue now comes from iterating to more functionals in the Π^0_2 case. Let $\mathcal{L}_0, \mathcal{L}_1$ be like in (a) and (b) above. The first thing we want is that for any forcing condition $(\tau, Y, \mathcal{V}) \leqslant (\sigma, X, \mathcal{U})$ we still have $A^0 \cap Y \in \mathcal{L}_0$ and $A^0 \cap Y \in \mathcal{L}_1$. For that the largeness class $\mathcal{V}$ should be compatible with $\mathcal{L}_0$ and $\mathcal{L}_1$, that is, $\mathcal{V} \cap \mathcal{L}_0$ and $\mathcal{V} \cap \mathcal{L}_1$ must be largeness classes. It follows that we want to include both $\mathcal{L}_0$ and $\mathcal{L}_1$ in our forcing condition.

The easy case

Suppose first that $\mathcal{U} \cap \mathcal{L}_0 \cap \mathcal{L}_1$ is a largeness class. Then we can simply enrich the forcing by considering the condition (σ, X, C) where C is a $\emptyset'$ set of indices for Σ^0_1 classes such that $\mathcal{U} \cap \mathcal{L}_0 \cap \mathcal{L}_1 = \bigcap_{e \in C} \mathcal{U}_e$. Future extensions $(\tau, Y, D) \leqslant (\sigma, X, C)$ shall be such that $D \supseteq C$ is a Δ^0_2 set of indices such that $\bigcap_{e \in D} \mathcal{U}_e$ is a largeness class and such that $A^0 \cap Y$ is partition generic below $\bigcap_{e \in D} \mathcal{U}_e$. For a condition (σ, X, C) the forcing question becomes

$$(\sigma, X, C)\ ?\vdash \exists x\ \forall y\ \Phi_e(G,x,y)$$

if

$$\bigcap_{e \in C} \mathcal{U}_e \cap \bigcap_{\tau \subseteq X \cap A^0, x \in \omega} \mathcal{U}_{\zeta(e,\sigma\cup\tau,x)} \text{ is not a largeness class.}$$

Note that in the Σ^0_2 case or in the Π^0_2 case, nothing changes and we can then defeat as many functionals as we want, as long as when the Π^0_2 outcome occurs with largeness classes $\mathcal{L}_0$ and $\mathcal{L}_1$, we have that $\bigcap_{e \in C} \mathcal{U}_e \cap \mathcal{L}_0 \cap \mathcal{L}_1$ is a largeness class.

The hard case

The real issue comes when $\mathcal{U}\cap\mathcal{L}_0\cap\mathcal{L}_1$ is not a largeness class. This is what naturally forces us to work within product spaces. A largeness class $\mathcal{V}\subseteq 2^\omega\times 2^\omega$ is simply a class which is upward closed on each of its component and such that $\forall X_0\cup\cdots\cup X_k\supseteq\omega\ \forall Y_0\cup\cdots\cup Y_k\supseteq\omega\ \exists i_0,i_1\leqslant k$ s.t. $(X_{i_0},Y_{i_1})\in\mathcal{V}$. One defines the notion of partition genericity accordingly.

Still with $\mathcal{L}_0$ and $\mathcal{L}_1$ as in (a) and (b) above, we now want to consider the largeness class $(\mathcal{U}\cap\mathcal{L}_0)\times(\mathcal{U}\cap\mathcal{L}_1)$. Suppose for now that $(A^0\cap X, A^0\cap X)$ is partition generic below $(\mathcal{U}\cap\mathcal{L}_0)\times(\mathcal{U}\cap\mathcal{L}_1)$: this is some extra simplification we will get rid of in the next section. Note that we still build one generic $G\subseteq A^0$, but from this point on the construction of G will need to take finite extension sometimes from $\mathcal{U}\cap\mathcal{L}_0$ and sometimes from $\mathcal{U}\cap\mathcal{L}_1$. For this reason we now modify the definition of ζ for further question so that $\zeta(e,\sigma,x)$ is such that

$$\mathcal{U}_{\zeta(e,\sigma,x)}=\{(X,Y)\ :\ \exists\tau\subseteq X\cup Y-\{0,\dots,|\sigma|\}\ \exists y\ \neg\Phi_e(\sigma\cup\tau,x,y)\}$$

We also modify our condition (σ,X) into $(\sigma,\langle X_0,X_1\rangle,C)$ with $X_0=X_1=X$, so that now C is the set of indices for elements of $2^\omega\times 2^\omega$ such that $\bigcap_{e\in C}\mathcal{U}_e=(\mathcal{U}\cap\mathcal{L}_0)\times(\mathcal{U}\cap\mathcal{L}_1)$. The forcing question itself becomes:

$$(\sigma,\langle X_0,X_1\rangle,C)\mathrel{?\vdash}\exists x\ \forall y\ \Phi_e(G,x,y)$$

if

$$\bigcap_{e\in C}\mathcal{U}_e\cap\bigcap_{\tau\subseteq A^0\cap(X_0\cup X_1),x\in\omega}\mathcal{U}_{\zeta(e,\sigma\cup\tau,x)}\text{ is not a largeness class.}$$

With the same reasoning as before, we now have in the Σ^0_2 case an extension $(\sigma\cup\tau,\langle X_0\cap Y_0,X_1\cap Y_1\rangle,C\cup\{a\})$ with $\tau\subseteq A^0\cap(X_0\cup X_1)$ such that $\langle A^0\cap X_0\cap Y_0,A^0\cap X_1\cap Y_1\rangle$ is partition generic below $\mathcal{U}_a\cap\bigcup_{e\in C}\mathcal{U}_e$ and such that for some x we have $\forall\rho\subseteq A^0\cap((X_0\cap Y_0)\cup(X_1\cap Y_1))$ and all y that $\Phi_{\eta(e,n,i)}(\sigma\cup\tau\cup\rho,x,y)$ holds for some i and some n such that $\Phi_n(\emptyset',n)\downarrow=i$.

In the Π^0_2 case we have two largeness classes $\mathcal{L}_{*0}\subseteq\mathcal{L}_0\times\mathcal{L}_1$ and $\mathcal{L}_{*1}\subseteq\mathcal{L}_0\times\mathcal{L}_1$ such that working now in the product space $\mathcal{L}_{*0}\times\mathcal{L}_{*1}$ will now allow us to make the jump of our generic diverge, building it as before iteratively by finite extension. Here again we can iterate in growing product spaces, defeating as many functional as we want, but with the extra-assumption that $\langle A^0\cap X_0,A^0\cap X_1,\dots,A^0\cap X_m\rangle$ is always partition generic in our current largeness class.

5.2.3 Defeating more functionals (part 2)

In the hard case of the previous section we assumed $(A^0\cap X,A^0\cap X)$ was partition generic below $\mathcal{U}\cap\mathcal{L}_0\times\mathcal{U}\cap\mathcal{L}_1$. But similarly to what was discussed in Section 4.3, we cannot necessarily ensure that : It may be that A^0 is partition generic but such that (A^0,A^0) is nowhere partition generic below some largeness class $\mathcal{L}_0\times\mathcal{L}_1$. Worse than that, it may be that neither (A^0,A^0) nor (A^1,A^1) are partition generic somewhere below some largeness class $\mathcal{L}_0\times\mathcal{L}_1$. But having (A^0,A^1) or (A^1,A^0) partition generic is not of any help for us as this would only lead to the construction of a generic $G\subseteq A^0\cup A^1$, achieving then nothing.

Here is how we proceed. For a reservoir X, we first make (X,X) partition generic below $\mathcal{U}\cap\mathcal{L}_0\times\mathcal{U}\cap\mathcal{L}_1$ and then find a trick so that $(X\cap A^i,X\cap A^i)$ belongs to the large classes we are interested in, for some $i\in\{0,1\}$. The first step is in fact rather simple : we can always make sure that our reservoirs are low. Recall that $\mathcal{L}_X$ is the class of elements intersecting X infinitely often. If X is partition generic below $\mathcal{U}\cap\mathcal{L}_0$ and below $\mathcal{U}\cap\mathcal{L}_1$, we always have that $\mathcal{U}\cap\mathcal{L}_0\cap\mathcal{L}_X$ and $\mathcal{U}\cap\mathcal{L}_1\cap\mathcal{L}_X$ are largeness classes such that (X,X) is partition generic below $(\mathcal{U}\cap\mathcal{L}_0\cap\mathcal{L}_X)\times(\mathcal{U}\cap\mathcal{L}_1\cap\mathcal{L}_X)$ in a very strong way : not only $(X,X)\in\mathcal{V}$ for any Σ^0_1 class $\mathcal{V}\subseteq 2^\omega\times 2^\omega$ such that $\mathcal{V}\cap(\mathcal{U}\cap\mathcal{L}_0\cap\mathcal{L}_X\times\mathcal{U}\cap\mathcal{L}_1\cap\mathcal{L}_X)$ is a largeness class, but $(X,X)\in\mathcal{V}$ for a class $\mathcal{V}$ of any complexity such that $\mathcal{V}\cap(\mathcal{U}\cap\mathcal{L}_0\cap\mathcal{L}_X\times\mathcal{U}\cap\mathcal{L}_1\cap\mathcal{L}_X)$ is a largeness class (because $(X,\omega - X)$, $(\omega - X, X)$ or $(\omega - X, \omega - X)$ obviously don't belong to $\mathcal{V} \cap (\mathcal{U} \cap \mathcal{L}_0 \cap \mathcal{L}_X \times \mathcal{U} \cap \mathcal{L}_1 \cap \mathcal{L}_X))$. Note that this require us to now work with $\Sigma^0_1(Z)$ classes for low sets Z. The fact that Z is low makes the question "is $\mathcal{U}$ a largeness class?" for a $\Sigma^0_1(Z)$ open set a $\Pi^0_2(Z)$ question and then $\Pi^0_1(\emptyset')$. So the complexity of future forcing questions does not change.

Now that (X, X) is partition generic below $(\mathcal{U} \cap \mathcal{L}_0 \cap \mathcal{L}_X) \times (\mathcal{U} \cap \mathcal{L}_1 \cap \mathcal{L}_X)$ the second step — making $(X \cap A^i, X \cap A^i)$ belongs to our largeness classes for $i \in \{0,1\}$ — is more difficult. In particular we cannot do the same with A^0 or A^1 which are presumably not low. We are then forced to use a trick similar to the one discussed in Section 4.3 : we need three largeness classes $\mathcal{L}_0, \mathcal{L}_1$ and $\mathcal{L}_2$ such that being in two of them is enough to ensure $\Phi_e(G', n) \uparrow$ for some n. In order to do so, we shall use a lemma from Liu, stating that given any $\emptyset'$-c.e. set W of finite $\{0,1\}$-valued partial function, either W contains v such that $v(n) \downarrow = \Phi_n(\emptyset', n) \downarrow$ for every $n \in \operatorname{dom} v$, or $\omega - W$ contains as many pairwise incompatible finite $\{0,1\}$-valued partial functions as we want.

We change our function η which now takes a code e and a finite $\{0,1\}$-valued partial function v, in such a way that:

$$\exists n \in \operatorname{dom} v \ \Phi_e(G', n) \downarrow \text{ iff } \exists x \ \forall y \ \Phi_{\eta(e,v)}(G, x, y)$$

We also change our forcing conditions. They are first of the form $p = (\sigma_0, \sigma_1, X, C)$ where

- $\sigma_i \subseteq A^i$, for $i \in \{0,1\}$
- $X \in 2^\omega$ is a low reservoir such that (σ_i, X) are Mathias conditions.
- C is a Δ^0_2 set of indices such that $\bigcap_{e \in C} \mathcal{U}_e \subseteq \mathcal{L}_X$ is a largeness class

We finally change the forcing question so that it integrates a side $i \in \{0,1\}$, as we don't know anymore if we will succeed the construction in A^0 or A^1.

$$(\sigma_0, \sigma_1, X, C) \mathrel{?\!\vdash^i} \exists x \ \forall y \ \Phi_e(G, x, y)$$
$$\text{if}$$
$$\bigcap_{e \in C} \mathcal{U}_e \cap \bigcap_{\tau \subseteq A^i \cap X, x \in \omega} \mathcal{U}_{\zeta(e, \sigma_i \cup \tau, x)} \text{ is not a largeness class.}$$

Now given a functional Φ_e, a forcing condition $p = (\sigma_0, \sigma_1, X, C)$ and a side $i \in \{0,1\}$, we consider the $\emptyset'$-c.e. set W of finite $\{0,1\}$-valued partial functions defined by:

$$W = \{v \ : \ p \mathrel{?\!\vdash^i} \exists x \ \forall y \ \Phi_{\eta(e,v)}(G, x, y)\}$$

If W contains a valuation v such that $v(n) \downarrow = \Phi_n(\emptyset', n) \downarrow$ for every $n \in \operatorname{dom} v$ then we force as before with the Σ^0_2 case that $\Phi_e(G'_i, n) \downarrow = \Phi_n(\emptyset', n)$ for some n — where $G_i \subseteq A^i$ is the generic built on side i. Otherwise we can find using Liu's lemma three pairwise incompatible finite $\{0,1\}$-valued partial functions v_0, v_1, v_2 such that $p \mathrel{?\!\nvdash^i} \exists x \ \forall y \ \Phi_{\eta(e,v_j)}(G, x, y)$ for $j \in \{0,1,2\}$. Just as before this leads to three largeness classes $\mathcal{L}_0, \mathcal{L}_1$ and $\mathcal{L}_2$, corresponding respectively to the three valuations v_0, v_1, v_2 such that each $\mathcal{L}_j$ can help us make the jump of our generic always disagree with v_j on $\operatorname{dom} v_j$. Note that for any largeness class $\mathcal{V} \subseteq \mathcal{L}_0 \times \mathcal{L}_1 \times \mathcal{L}_2$ we must have $(A^{i_0}, A^{i_1}, A^{i_2}) \in \mathcal{V}$ for $i_0, i_1, i_2 \in \{0,1\}$ with $i_a = i_b$ for some $a \neq b$. Given such a largeness class $\mathcal{V}$, suppose for instance that $(A^1, A^0, A^0) \in \mathcal{V}$. Suppose also the largeness classes $\mathcal{L}_0, \mathcal{L}_1$ and $\mathcal{L}_2$ where created for side $i = 0$ (that is, with $p \mathrel{?\!\nvdash^0} \exists x \ \forall y \ \Phi_{\eta(e,v_j)}(G, x, y)$). Then we can achieve one more step in satisfying the Π^0_2 outcome with the help of $\mathcal{L}_1 \times \mathcal{L}_2$: as $(A^1, A^0, A^0) \in \mathcal{V} \subseteq \mathcal{L}_0 \times \mathcal{L}_1 \times \mathcal{L}_2$ then $A^0 \in \mathcal{L}_1$ and $A^0 \in \mathcal{L}_2$. We would of course have to repeat the question for the same functional e and the side $i = 1$.

5.2.4 Defeating more functionals (part 3)

After forcing our first Π^0_2 outcome on some side $i \in \{0,1\}$ we then have a forcing condition of the form $(\sigma_0, \sigma_1, \langle X_0, X_1, X_2\rangle, C)$ such that $\bigcap_{e\in C} \mathcal{U}_e \subseteq \mathcal{L}_{X_0} \times \mathcal{L}_{X_1} \times \mathcal{L}_{X_2}$. We now have three "branches" within which the Π^0_2 outcome may be satisfied : $\mathcal{L}_{X_0} \times \mathcal{L}_{X_1}$, $\mathcal{L}_{X_0} \times \mathcal{L}_{X_2}$ and $\mathcal{L}_{X_1} \times \mathcal{L}_{X_2}$. We do not know which one will end up being the "correct" one, that is the one containing (A^i, A^i) all the time as the construction goes through. Each of these branch will now be treated as separately as possible, given that we still want a unique largeness class. For this reason we now duplicated the finite part of the forcing condition which now becomes:

$$p = (\sigma_0^{0,1}, \sigma_1^{0,1}, \sigma_0^{0,2}, \sigma_1^{0,2}, \sigma_0^{1,2}, \sigma_1^{1,2}, \langle X_0, X_1, X_2\rangle, C)$$

where $\sigma_i^{a,b} = \sigma_i$. We then need to consider each "branch" for the next forcing question, which now becomes :

$$p \,?\!\vdash^i \exists x\ \forall y\ \Phi_e(G, x, y)$$

iff

$$\bigcap_{e\in C} \mathcal{U}_e \cap \begin{array}{l} \displaystyle\bigcap_{\tau\subseteq A^i\cap(X_0\cup X_1), x\in\omega} \left\{\langle Y_0, Y_1, Y_2\rangle \ : \ \begin{array}{l}\exists \rho \subseteq Y_0 \cup Y_1 - \{0, \dots, |\sigma_0^{0,1} \cup \tau|\}\ \exists y \in \omega \\ \text{s.t. } \neg\Phi_e(\sigma_0^{0,1} \cup \tau \cup \rho, x, y)\end{array}\right\} \\ \displaystyle\bigcap_{\tau\subseteq A^i\cap(X_0\cup X_2), x\in\omega} \left\{\langle Y_0, Y_1, Y_2\rangle \ : \ \begin{array}{l}\exists \rho \subseteq Y_0 \cup Y_2 - \{0, \dots, |\sigma_i^{0,2} \cup \tau|\}\ \exists y \in \omega \\ \text{s.t. } \neg\Phi_e(\sigma_0^{0,2} \cup \tau \cup \rho, x, y)\end{array}\right\} \\ \displaystyle\bigcap_{\tau\subseteq A^i\cap(X_1\cup X_2), x\in\omega} \left\{\langle Y_0, Y_1, Y_2\rangle \ : \ \begin{array}{l}\exists \rho \subseteq Y_1 \cup Y_2 - \{0, \dots, |\sigma_i^{1,2} \cup \tau|\}\ \exists y \in \omega \\ \text{s.t. } \neg\Phi_e(\sigma_i^{1,2} \cup \tau \cup \rho, x, y)\end{array}\right\} \end{array}$$

is not a largeness class.

In case of the Σ^0_2 outcome, we can ensure the Σ^0_2 formula for one of the branch (on side i), but without increasing the number of branches we consider (we still are with largeness subclasses of $2^\omega \times 2^\omega \times 2^\omega$). In case of the Π^0_2 outcome, we have to act similarly to what was done to handle the first Π^0_2 outcome : we find sufficiently many incomparable finite $\{0,1\}$-valued partial functions (say m many) each of them giving a largeness class $\mathcal{L}_j$ for $j < m$. We then duplicate everything to work with the largeness class $\mathcal{L}_0 \times \dots \times \mathcal{L}_{m-1}$. Note that each $\mathcal{L}_j$ is itself a largeness subclass of $2^\omega \times 2^\omega \times 2^\omega$.

What is the correct number m ? we need m to be such that if

$$(A_0^0, A_0^1, A_0^2, \dots, A_{m-1}^0, A_{m-1}^1, A_{m-1}^2) \in \mathcal{L}_0 \times \dots \times \mathcal{L}_{m-1}$$

where each A_b^a is either A^0 or A^1, then we must have $A_{j_1}^{i_0} = A_{j_1}^{i_1} = A_{j_2}^{i_0} = A_{j_2}^{i_1}$ for $j_1 \neq j_2 < m$ and $i_0 \neq i_1 \in \{0,1,2\}$. This way we can keep defeating the first functional with the Π^0_2 outcome on one branch, while also defeating the new one. It turns out that m must be at least 7. This is what will be formalized with *symmetric sets* in the next sections.

The idea behind "branches" will be formalized with the $\mathbb{Q}$-forcing and the $\mathbb{P}$-forcing. A $\mathbb{P}$-forcing condition will be a "full" forcing condition as in the example given above. A $\mathbb{Q}$-forcing condition will correspond to one branch of the $\mathbb{P}$-forcing condition.

5.3 Preliminaries

5.3.1 Liu's lemma

The proof requires a key lemma that Liu used for his separation of WKL_0 from RT^2_2.

Notation

Given a partial function defined on ω, we write $v(n)\downarrow$ to mean that v is defined on n and $v(n)\uparrow$ otherwise.

We say that two partial functions v_1, v_2 are *compatible* if $v_1(n)\downarrow$ and $v_2(n)\downarrow$ implies $v_1(n) = v_2(n)$. They are otherwise *incompatible.*

Definition 5.3.1 : A *valuation* is a finite partial function $v \subseteq \omega \to \{0,1\}$. ◊

In what follows we always consider that valuations are presented in a strongly finite way, that is, as a finite object by oppose to an infinite object which happens to be finite, the important point being to uniformly know the last value on which a valuation is defined.

Notation

We write J_X for the partial function $n \mapsto \Phi_n(X,n)$

Lemma 5.3.2 (Liu, [23]) : Let X be a set and $Y \geq_T X$ be non PA relative to X. Let W be a Y-c.e. set of valuations. Either W contains a valuation v compatible with J_X and such that $\operatorname{dom} v \subseteq \operatorname{dom} J_X$, or for any k there are k pairwise incompatible valuations outside of W. ⋆

Proof: Suppose W contains no valuation v compatible with J_X and such that $\operatorname{dom} v \subseteq \operatorname{dom} J_X$. It means that for any valuation $v \in W$, there exists n such that $v(n)\downarrow$ and either $\Phi_n(X,n)\uparrow$ or $v(n)\downarrow\neq J_X(n)\downarrow$.

Let $\mathcal{A}$ be the set of finite sets of integers F such that for all $v \in W$ which is defined on F and such that $\operatorname{dom} v - F \subseteq \operatorname{dom} J_X$ we have $v(m)\downarrow\neq J_X(m)\downarrow$ for some $m \in \operatorname{dom} v - F$.

Let $F \in \mathcal{A}$. Suppose for contradiction that for all integers $n \notin F$ we have $F \cup \{n\} \notin A$, that is, there exists $v \in W$ which is defined on $F \cup \{n\}$, such that $\operatorname{dom} v - F \cup \{n\} \subseteq \operatorname{dom} J_X$ and such that $v(m)\downarrow= J_X(m)\downarrow$ for all $m \in \operatorname{dom} v - F \cup \{n\}$.

We can use that to define a total Y-computable $\{0,1\}$-valued function f such that $f(n) \neq J_X(n)$ for all n, contradicting that Y is not PA relative to X. For $n \notin F$ we search for $v \in W$ which is defined on $F \cup \{n\}$, such that $\operatorname{dom} v - F \cup \{n\} \subseteq \operatorname{dom} J_X$ and such that $v(m)\downarrow= J_X(m)\downarrow$ for all $m \in \operatorname{dom} v - F \cup \{n\}$. We then set $f(n) = v(n)$. Note that we have $\operatorname{dom} v - F \cup \{n\} \subseteq \operatorname{dom} J_X$ together with $v(m)\downarrow= J_X(m)\downarrow$ for all $m \in \operatorname{dom} v - F \cup \{n\}$. Suppose in addition that $v(n)\downarrow= J_X(n)\downarrow$ then we obtain $\operatorname{dom} v - F \subseteq \operatorname{dom} J_X$ together with $v(m)\downarrow= J_X(m)\downarrow$ for all $m \in \operatorname{dom} v - F$ which contradicts our hypothesis on F. Thus it must be that either $J_X(n)\uparrow$ or $v(n)\downarrow\neq J_X(n)\downarrow$. In any case we have $f(n) \neq J_X(n)$ for all n outside of F. We can easily add values for f on F so that $f(n) \neq J_X(n)$ for all n which contradicts that Y is not PA relative to X. Thus for any $F \in \mathcal{A}$ there exists n such that $F \cup \{n\} \in \mathcal{A}$.

Note that $F = \emptyset$ belongs to $\mathcal{A}$: otherwise we would have a valuation $v \in W$ such that $\operatorname{dom} v \subseteq \operatorname{dom} J_X$ with $v(m)\downarrow= J_X(m)\downarrow$ for all $m \in \operatorname{dom} v$, contradicting our hypothesis on W. Let n_0 be the smallest such that $\{n_0\} \in \mathcal{A}$. Define inductively n_{k+1} as the smallest not in $\{n_0, \dots, n_k\}$ such that $\{n_0, \dots, n_{k+1}\} \in \mathcal{A}$.

Fix k and suppose now that a valuation v is defined exactly on $\{n_0, \dots, n_k\}$. Then $\operatorname{dom} v - \{n_0, \dots, n_k\} \subseteq \operatorname{dom} J_X$ and $v(m)\downarrow= J_X(m)\downarrow$ for all $m \in \operatorname{dom} v - \{n_0, \dots, n_k\}$. As $\{n_0, \dots, n_k\} \in \mathcal{A}$ it follows that $v \notin W$.

Then for any k there are 2^k pairwise incompatible valuations which are not in W. ■

5.3.2 Largeness classes in product spaces

In order to find an element $G \in [A]^\omega \cup [\omega - A]^\omega$ which computes no p-cohesive set, we would need to iterate countably many times an argument like the one of Proposition 4.3.2. In order to be able to do so, we need to "include" the iterations inside the forcing. This calls for an extension of Largeness classes to product spaces.

For technical reasons, yet to come, we will need a tight control on the indices of our products. To illustrate this, suppose we have a subclass of $2^\omega \times 2^\omega \times 2^\omega \times 2^\omega \times 2^\omega$ and want its projection along the first and third component. We will need to do such operations and for that we consider subclasses of $I \to 2^\omega$ for a finite set I.

The definitions, propositions and lemmas given here are straightforward extension to product spaces of those given in Section 1.4.1.

Definition 5.3.3 : Let I be a finite set. A *largeness subclass* of $I \to 2^\omega$ is a subset $\mathcal{A} \subseteq I \to 2^\omega$ such that:

1. $\mathcal{A}$ is not empty
2. $\mathcal{A}$ is *upward closed* on each component : If $\langle X_\rho : \rho \in I\rangle \in \mathcal{A}$ and $X_\rho \subseteq Y_\rho$ for every $\rho \in I$, then $\langle Y_\rho : \rho \in I\rangle \in \mathcal{A}$
3. For every k, for every $\rho \in I$, for every k-cover $Y^0_\rho \cup \dots \cup Y^k_\rho \supseteq \omega$, there is a function $f : I \to \{0,\dots,k\}$ such that $\langle Y^{f(\rho)}_\rho : \rho \in I\rangle \in \mathcal{A}$. ♢

Definition 5.3.4 : Let I be a finite set. A *partition regular subclass* of $I \to 2^\omega$ is a subset $\mathcal{L} \subseteq I \to 2^\omega$ such that:

1. $\mathcal{L}$ is a largeness class
2. If $\langle X_\rho : \rho \in I\rangle \in \mathcal{L}$ and $Y^0_\rho \cup \dots \cup Y^k_\rho \supseteq X_\rho$ for every $\rho \in I$, then there is $f : I \to \{0,\dots,k\}$ such that $\langle Y^{f(\rho)}_\rho : \rho \in I\rangle \in \mathcal{L}$

A partition regular class $\mathcal{L}$ is *non-trivial* if it contains only sets which are infinite on each component. ♢

All the partition regular classes we manipulate in this document will be non-trivial.

Proposition 5.3.5 : Let I be a finite set. Suppose $\{\mathcal{L}_i\}_{i\in I}$ is an arbitrary non-empty collection of partition regular subclass of $I \to 2^\omega$. Then $\bigcup_{i\in I} \mathcal{L}_i$ is a partition regular subclass of $I \to 2^\omega$. ⋆

Proof: Like the proof of Proposition 1.4.6. ∎

In particular for every class $\mathcal{A} \subseteq I \to 2^\omega$ such that $\mathcal{A}$ contains a partition regular class, there is a largest partition regular class included in $\mathcal{A}$. It leads to an extension of Definition 1.4.7 : $\mathcal{L}(\mathcal{A})$ is the largest partition regular class of $\mathcal{A}$, and the empty set if no such class exists.

Notation

Given a set $A \subseteq \omega$ and a finite set I we write $\bigotimes_I A$ for the element $\langle X_\rho : \rho \in I\rangle \in I \to 2^\omega$ such that $X_\rho = A$ for $\rho \in I$.
Given a set $\mathcal{A} \subseteq 2^\omega$ and a finite set I we write $\bigotimes_I \mathcal{A}$ for the class of elements $\bigotimes_I X$ for $X \in \mathcal{A}$.

We now connect largeness classes to partition regular classes, by showing a lemma analogous to Lemma 1.4.8 with the difference that $\mathcal{A}$ needs to be upward closed.

Lemma 5.3.6 : Let I be a finite set. For any upward closed class $\mathcal{A} \subseteq I \to 2^\omega$ the class $\mathcal{L}(\mathcal{A})$ equals:

$$\left\{ \begin{array}{lll} \langle X_\rho \ : \ \rho \in I \rangle \in I \to 2^\omega & : & \forall k \ \forall \rho \in I \ \forall X^0_\rho \cup \cdots \cup X^k_\rho \supseteq X_\rho \\ & & \exists f : I \to \{0, \ldots, k\} \ \text{ s.t. } \langle X^{f(\rho)}_\rho \ : \ \rho \in I \rangle \in \mathcal{A} \end{array} \right\}$$

Proof: For this proof we refer to $\mathcal{L}(\mathcal{A})$ as defined by this proposition, in order to show that it matches Definition 1.4.7. Note that by definition we must have $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$, as if $\langle X_\rho \ : \ \rho \in I \rangle \notin \mathcal{A}$ then itself as a 1-cover is not in $\mathcal{A}$.

Let us show that $\mathcal{L}(\mathcal{A})$ contains every partition regular class included in $\mathcal{A}$. Suppose $\mathcal{L} \subseteq \mathcal{A}$ is partition regular. Then given $\langle X_\rho \ : \ \rho \in I \rangle \in \mathcal{L}$, for every k, every ρ and every $X^0_\rho \cup \cdots \cup X^k_\rho \supseteq X_\rho$ we have $\langle X^{f(\rho)}_\rho \ : \ \rho \in I \rangle \in \mathcal{L} \subseteq \mathcal{A}$ for some $f : I \to \{0, \ldots, k\}$. It follows that $\langle X_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$ and thus that $\mathcal{L} \subseteq \mathcal{L}(\mathcal{A})$.

Let us show that if $\mathcal{L}(\mathcal{A})$ is non-empty, it is a partition regular class. Suppose $\langle X_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$. Let $Y_\rho \supseteq X_\rho$ for every $\rho \in I$. Then for every k, for $\rho \in I$, every k-cover $Z^0_\rho \cup \cdots \cup Z^k_\rho$ of Y_ρ is also a k-cover of X_ρ. As $\langle X_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$, there must be a function $f : I \to \{0, \ldots, k\}$ such that $\langle Z^{f(\rho)}_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$. It follows that $\langle Y_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$. Thus $\mathcal{L}(\mathcal{A})$ is upward closed.

Let $\langle X_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$ and let $Y^0_\rho \cup \cdots \cup Y^k_\rho \supseteq X_\rho$ for every ρ. Let us show there is some $f : I \to \{0, \ldots, k\}$ such that $\langle Y^{f(\rho)}_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$. Suppose for contradiction that this is not the case. In particular for every $f : I \to \{0, \ldots, k\}$ there are sets $Y^0_{f,\rho} \cup \cdots \cup Y^{k_f}_{f,\rho} \supseteq Y^{f(\rho)}_\rho$ such that for all g we have $\langle Y^{g(\rho)}_{f,\rho} \ : \ \rho \in I \rangle \notin \mathcal{A}$.

Also each sets $\{Y^i_{f,\rho}\}_{f : I \to \{0,\ldots,k\}, i \leqslant k_f}$ is a finite cover of X_ρ. Let $Z^0_\rho, \ldots, Z^{k_\rho}_\rho$ be a finite cover of X_ρ such that each $Z^{k_\rho}_\rho$ is either included or disjoint from each set $Y^i_{f,\rho}$ for $f : I \to \{0, \ldots, k\}$ and $i \leqslant k_f$. As $\langle X_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$ there must be a function h such that $\langle Z^{h(\rho)}_\rho \ : \ \rho \in I \rangle \in \mathcal{A}$. Let f be defined by $f(\rho)$ to be i for the smallest i such that $Z^{h(\rho)}_\rho \subseteq Y^i_\rho$. Then $Z^{h(\rho)}_\rho \subseteq Y^{f(\rho)}_\rho$. Let now g be the function which to ρ associates the smallest i such that $Z^{h(\rho)}_\rho \subseteq Y^i_{f,\rho}$. We then have $Z^{h(\rho)}_\rho \subseteq Y^{g(\rho)}_{f,\rho}$. As $\mathcal{A}$ is upward closed we have $\langle Y^{g(\rho)}_{f,\rho} \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$ which is a contradiction. So there is some $f : I \to \{0, \ldots, k\}$ such that $\langle Y^{f(\rho)}_\rho \ : \ \rho \in I \rangle \in \mathcal{L}(\mathcal{A})$. Thus if $\mathcal{L}(\mathcal{A})$ is non-empty it is a partition regular class.

It follows that $\mathcal{L}(\mathcal{A})$ is empty if $\mathcal{A}$ contains no partition regular class and $\mathcal{L}(\mathcal{A})$ is the largest partition regular subclass of $\mathcal{A}$ otherwise. ∎

Corollary 5.3.7 : Let I be a finite set. An upward closed subclass of $I \to 2^\omega$ is a largeness class iff it contains a partition regular class.

Proof: Suppose a class $\mathcal{A}$ is a largeness class. Then by definition we must have $\bigoplus_I \omega \in \mathcal{L}(\mathcal{A})$. It follows that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$ is a partition regular class.

Suppose now $\mathcal{A}$ contains a partition regular class. Then $\mathcal{L}(\mathcal{A})$is not empty and then $\bigoplus_I \omega \in \mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$ and then $\mathcal{A}$ is a largeness class. ∎

Proposition 5.3.8 : Let I be a finite set. Suppose $\{\mathcal{L}_n\}_{n\in\omega}$ is a collection of partition regular (resp. largeness) subclass of $I \to 2^\omega$ with $\mathcal{L}_{n+1} \subseteq \mathcal{L}_n$. Thus $\bigcap_{n\in\omega} \mathcal{L}_n$ is a partition regular (resp. largeness) class. ⋆

Proof: Like the proof of Proposition 1.4.10. ∎

Proposition 5.3.9 : Let I be a finite set. Let $\mathcal{U}$ be an upward closed Σ^0_1 subclass of $I \to 2^\omega$ for the product topology. Then $\mathcal{L}(\mathcal{U})$ is a Π^0_2 class subclass of $I \to 2^\omega$. ⋆

Proof: Like the proof of Proposition 1.4.14. ∎

Corollary 5.3.10 : Let I be a finite set. Let $\mathcal{U}$ be an upward closed Σ^0_1 subclass of $I \to 2^\omega$ for the product topology. The sentence "$\mathcal{U}$ is a largeness class" is Π^0_2.

Proof: By Proposition 5.3.8 we have that $\mathcal{U}$ is a largeness class iff $\oplus_I \omega \in \mathcal{L}(\mathcal{U})$, which is a Π^0_2 sentence. ∎

5.3.3 A strong version of partition genericity

One of the main issue, making the construction complicated, is that if X is partition generic below some class $\mathcal{U}$, it is not necessarily the case that (X, X) will be partition generic below $\mathcal{U} \times \mathcal{U}$. This will be dealt with for the set A itself (the set inside which we want to build our generic), by using basically 'a lot' of components. This will be done in the next section with the use of *symmetric sets*.

We however still have to deal with the reservoirs of our Mathias conditions. To do so, we will cheat and use the fact that our reservoir have simple complexity : they will all be low sets. There is then a radical way to make any set X partition generic below some class $\mathcal{U}$: take $\mathcal{U}$ to be $\mathcal{L}_X$. Doing so, (X, X) will clearly remains partition generic below $\mathcal{L}_X \times \mathcal{L}_X$. The fact that our reservoir are all low sets makes it possible without making the forcing question two complex.

We state here the few tools related to the use of this trick.

Definition 5.3.11 : Let I be a finite set. For any set $\langle X_\rho \ : \ \rho \in I\rangle$ where each X_ρ is infinite, we define $\mathcal{L}_{\langle X_\rho \ : \ \rho\in I\rangle}$ as the partition regular class of all the sets $\langle Y_\rho \ : \ \rho \in I\rangle$ such that $|X_\rho \cap Y_\rho| = \infty$ for every $\rho \in I$. ◊

Proposition 5.3.12 : Let $\langle X_\rho \ : \ \rho \in I\rangle, \langle Y_\rho \ : \ \rho \in I\rangle \in I \to 2^\omega$ be such that $X_\rho =^* Y_\rho$. Then

$$\mathcal{L}_{\langle X_\rho \ : \ \rho\in I\rangle} = \mathcal{L}_{\langle Y_\rho \ : \ \rho\in I\rangle}$$

Proof: For any ρ, a set intersects X_ρ infinitely often iff it intersects Y_ρ infinitely often. The proposition follows. ∎

Lemma 5.3.13 : Let $\langle X_\rho\ :\ \rho\in I\rangle, \langle Y_\rho\ :\ \rho\in I\rangle \in I \to 2^\omega$. We have

$$\mathcal{L}\left(\mathcal{L}_{\langle X_\rho\ :\ \rho\in I\rangle} \cap \mathcal{L}_{\langle Y_\rho\ :\ \rho\in I\rangle}\right) = \mathcal{L}_{\langle X_\rho \cap Y_\rho\ :\ \rho\in I\rangle}$$

PROOF: Let us show the left to right inclusion. Let $\langle Z_\rho\ :\ \rho\in I\rangle$ be in some partition regular subclass of $\mathcal{L}_{\langle X_\rho\ :\ \rho\in I\rangle} \cap \mathcal{L}_{\langle Y_\rho\ :\ \rho\in I\rangle}$. Suppose there is $\tau\in I$ such that $Z_\tau \cap X_\tau \cap Y_\tau$ is finite. Thus $Z_\tau \subseteq^* (\omega - X_\tau) \cup (\omega - Y_\tau)$.

Let $Z^0_\rho = Z_\rho$ for $\rho\neq\tau$ and $Z^0_\tau = Z_\tau \cap (\omega - X_\tau)$. Let $Z^1_\rho = Z_\rho$ for $\rho\neq\tau$ and $Z^1_\tau = Z_\tau \cap (\omega - Y_\tau)$. As $\langle Z_\rho\ :\ \rho\in I\rangle$ is in a non-trivial partition regular class, we must have $\langle Z^i_\rho\ :\ \rho\in I\rangle$ in this class for some $i\in\{0,1\}$. But $\langle Z^0_\rho\ :\ \rho\in I\rangle \notin \mathcal{L}_{\langle X_\rho\ :\ \rho\in I\rangle}$ and $\langle Z^1_\rho\ :\ \rho\in I\rangle \notin \mathcal{L}_{\langle Y_\rho\ :\ \rho\in I\rangle}$, which is a contradiction. Thus $Z_\tau \cap X_\tau \cap Y_\tau$ is infinite for every $\tau\in I$. Thus $\langle Z_\rho\ :\ \rho\in I\rangle \in \mathcal{L}_{\langle X_\rho\cap Y_\rho\ :\ \rho\in I\rangle}$.

Let us show the right to left inclusion. First if $X_\tau\cap Y_\tau$ is finite for some $\tau\in I$ then $\mathcal{L}_{\langle X_\rho\cap Y_\rho\ :\ \rho\in I\rangle}$ equals the empty set and is then included everywhere. Suppose now $X_\tau\cap Y_\tau$ is infinite for every $\tau\in I$. The set $\mathcal{L}_{\langle X_\rho\cap Y_\rho\ :\ \rho\in I\rangle}$ is a partition regular subclass of $\mathcal{L}_{\langle X_\rho\ :\ \rho\in I\rangle} \cap \mathcal{L}_{\langle Y_\rho\ :\ \rho\in I\rangle}$. By definition of $\mathcal{L}(\mathcal{A})$ for a class $\mathcal{A}$ the left to right inclusion holds. ∎

Lemma 5.3.14 : Let $\mathcal{A}\subseteq I\to 2^\omega$ be a largeness class. Let $X^0_\rho \cup\cdots\cup X^k_\rho \supseteq \omega$ for every $\rho\in I$. Then there must be some $f:\{0,\dots,k\}\to I$ such that $\mathcal{A}\cap\mathcal{L}_{\langle X^{f(\rho)}_\rho\ :\ \rho\in I\rangle}$ is a largeness class. ⋆

PROOF: Suppose otherwise. There for every $f:\{0,\dots,k\}\to I$ there are covers $X^0_{f,\rho}\cup\cdots\cup X^{k_f}_{f,\rho}\supseteq\omega$ for every ρ such that $\langle X^{g(\rho)}_{f,\rho}\ :\ \rho\in I\rangle \notin \mathcal{A}\cap\mathcal{L}_{\langle X^{f(\rho)}_\rho\ :\ \rho\in I\rangle}$ for every $g: I\to\{0,\dots,k_f\}$.

For every $\rho\in I$ let $Y^0_\rho\cup\cdots\cup Y^m_\rho\supseteq\omega$ be such that Y^i_ρ is included or disjoint from $X^j_{f,\rho}$ for every f and every $j\leqslant k_f$. There must be some $h: I\to\{0,\dots,m\}$ such that $\langle Y^{h(\rho)}_\rho\ :\ \rho\in I\rangle\in\mathcal{A}$. Now let f be the function which to ρ associates i such that $Y^{h(\rho)}_\rho$ intersects X^i_ρ infinitely often. Then we have $\langle Y^{h(\rho)}_\rho\ :\ \rho\in I\rangle \in \mathcal{A}\cap\mathcal{L}_{\langle X^{f(\rho)}_\rho\ :\ \rho\in I\rangle}$.

Let now g which to ρ assigns i such that $Y^{h(\rho)}_\rho \subseteq X^{g(\rho)}_{f,\rho}$. We then have $\langle X^{g(\rho)}_{f,\rho}\ :\ \rho\in I\rangle \in \mathcal{A}\cap\mathcal{L}_{\langle X^{f(\rho)}_\rho\ :\ \rho\in I\rangle}$ which is a contradiction. ∎

Proposition 5.3.15 : Let $\mathcal{A}\subseteq\mathcal{L}_{\langle X_\rho\ :\ \rho\in I\rangle}$ be a largeness class. For every $\rho\in I$ let $Y^0_\rho\cup\cdots\cup Y^k_\rho\supseteq\omega$. Then there must be some $f:\{0,\dots,k\}\to I$ such that $\mathcal{A}\cap\mathcal{L}_{\langle X_\rho\cap Y^{f(\rho)}_\rho\ :\ \rho\in I\rangle}$ is a largeness class. ⋆

PROOF: By Lemma 5.3.14 there must be a function $f:\{0,\dots,k\}\to I$ such that $\mathcal{A}\cap\mathcal{L}_{\langle Y^{f(\rho)}_\rho\ :\ \rho\in I\rangle}$ is a largeness class. Also we have

$$\mathcal{L}\left(\mathcal{A}\cap\mathcal{L}_{\langle X_\rho\ :\ \rho\in I\rangle}\cap\mathcal{L}_{\langle Y^{f(\rho)}_\rho\ :\ \rho\in I\rangle}\right)\subseteq\mathcal{A}\cap\mathcal{L}\left(\mathcal{L}_{\langle X_\rho\ :\ \rho\in I\rangle}\cap\mathcal{L}_{\langle Y^{f(\rho)}_\rho\ :\ \rho\in I\rangle}\right)$$

Thus by Lemma 5.3.13 we have

$$\mathcal{L}\left(\mathcal{A}\cap\mathcal{L}_{\langle X_\rho\ :\ \rho\in I\rangle}\cap\mathcal{L}_{\langle Y^{f(\rho)}_\rho\ :\ \rho\in I\rangle}\right)\subseteq\mathcal{A}\cap\mathcal{L}_{\langle X_\rho\cap Y^{f(\rho)}_\rho\ :\ \rho\in I\rangle}$$

Thus $\mathcal{A}\cap\mathcal{L}_{\langle X_\rho\cap Y^{f(\rho)}_\rho\ :\ \rho\in I\rangle}$ contains a partition regular class and is then a largeness class. ∎

5.3.4 Symmetric set of strings

We now present a key tool for the combinatorics of the construction. The goal is to be able to iterate the idea behind Proposition 4.3.2 : with $A_0 \cup A_i \supseteq \omega$, if we have a largeness class $\mathcal{L} \subseteq 2^\omega \times 2^\omega \times 2^\omega$, we must have $(A_{i_0}, A_{i_1}, A_{i_2}) \in \mathcal{L}$ for $i_0, i_1, i_2 \in \{0,1\}$ and there must be $a \neq b$ such that $A_{i_a} = A_{i_b}$. As long as it is enough for A_0 or A_1 to be in two components of $\mathcal{L}$, we are good.

If now we want A_0 or A_1 to be in two components among the first three, and also in two other distinct components (to defeat more functionals), how should we iterate our product ? This is the goal of this section.

Definition 5.3.16 : Given a finite function $\sigma : \{0, \dots, n\} \to \omega$ we write $(\sigma)^n$ for the set of finite functions $\tau : \{0, \dots, n\} \to \omega$ with $\tau(i) < \sigma(i)$ for $i \leqslant n$. ◊

Note that a finite function σ defined on $\{0, \dots, n\}$ for some n can be seen as a string of $\omega^{<\omega}$. We will therefore often manipulate them as strings, using for instance $|\sigma|$ for the sie of σ or $\sigma\tau$ for the concatenation of σ and τ.

Notation

Given a set I of finite functions defined on $\{0, \dots, n\}$ for some n, we write $I^\prec$ for the tree corresponding to the closure of I by prefixes of its elements.

Definition 5.3.17 : We say that a tree $T \subseteq \omega^{<\omega}$ is *exactly* 2*-bushy* if every node of T which is not a leaf has exactly 2 immediate extensions in T. ◊

Definition 5.3.18 : Given a finite function $\sigma : \{0, \dots, n\} \to \omega$, we say that a subset $I \subseteq (\sigma)^n$ is *symmetric* if $I^\prec$ is an exactly 2-bushy tree and if given $\tau \in I^\prec$ which is not a leaf and its two immediate extensions n_1, n_2 in $I^\prec$ we have for every string σ that $\tau n_1 \sigma \in I^\prec$ iff $\tau n_2 \sigma \in I^\prec$. ◊

In other words, below every node σ of $I^\prec$, the left subtree of σ must be identical to the right subtree of σ, except for the roots of both subtrees. Here is a example of a tree presentation of the symmetric set generated by finite functions defined on $\{0, 1, 2, 3\}$:

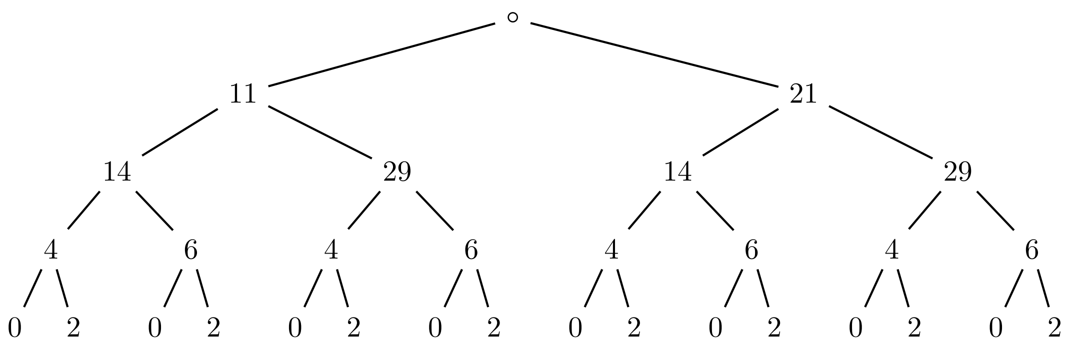

Figure 5.1: The tree presentation of a symmetric set

Recall that given a finite function $\sigma : \{0, \dots, n\} \to \omega$ we write $(\sigma)^n$ for the set of finite functions $\tau : \{0, \dots, n\} \to \omega$ with $\tau(i) < \sigma(i)$ for $i \leqslant n$.

Definition 5.3.19 : Let $u_0 = 1$, $\sigma_0 = u_0$ and $I_0 = (\sigma_0)^0$. Let u_{n+1} be the smallest number such that for any partition $P_1 \bigsqcup P_2 = (u_{n+1}\sigma_n)^n$, one of the part contains a symmetric set. Let $I_{n+1} = (u_{n+1}\sigma_n)^n$ ◇

The sets I_n and their symmetric subsets will play an important role in the forcing. We introduce for that some notation.

Notation

We write $I \lhd I_n$ to mean that I is a symmetric subset of I_n. We write $I \lhd \bigcup_n I_n$ to mean that $I \lhd I_n$ for some n.

We explain here how to compute the exact values of u_n and then the sets I_n. To do so we let v_n be the number of symmetric subsets of I_n. Note that the symmetric subset of I_{n+1} are exactly the sets $\{m_0\tau : \tau \in I\} \cup \{m_1\tau : \tau \in I\}$ for any $m_1 \neq m_2$ with $m_1, m_2 < u_{n+1}$ and any $I \lhd I_n$. It follows that $v_{n+1} = \binom{u_{n+1}}{2} v_n$: the number of possibility to pick two values smaller than u_{n+1} times the number of symmetric subset of I_n.

We then have that $u_{n+1} = 2v_n + 1$: each partition of the elements of I_{n+1} induces u_{n+1} partitions of the elements of I_n. For each of these partition of I_n, one of the part contains a symmetric subset. As they are v_n possible symmetric subsets of I_n, if we have $2v_n + 1$ partitions, at least two of them will share the same symmetric set on the same side. By induction on can show that this number is optimal by forcing every possibilities.

Now from $v_{n+1} = \binom{u_{n+1}}{2} v_n$ and $v_0 = 1$ we obtain $v_{n+1} = \Pi_{1 \leqslant i \leqslant n+1} \binom{u_i}{2}$. Together with $u_0 = 1$, $v_0 = 1$ and $u_{n+1} = 2v_n + 1$ we then have:

$$
\begin{array}{rcll}
u_0 & = & 1 & \\
u_1 & = & 3 & \\
u_n & = & 2\Pi_{1 \leqslant i < n} \binom{u_i}{2} + 1 & \text{for } n > 1
\end{array}
$$

The first values are $u_0 = 1$, $u_1 = 3$, $u_2 = 7$, $u_3 = 127$, $u_4 = 1008127$. We now introduce another notation that will be helpful to manipulate symmetric sets.

Definition 5.3.20 : Let $I \lhd I_n$. A set J is a 1-extension of I, in which case we write $J \preccurlyeq_1 I$, if J is of the form

$$J = \{m_0\tau : \tau \in I\} \cup \{m_1\tau : \tau \in I\}$$

For $m_0, m_1 \leqslant u_{n+1}$. Note that we have $J \lhd I_{n+1}$.

For $I, J \lhd \bigcup_n I_n$ we write $J \preccurlyeq I$ and we say that J is an *extension* of I if we have $J = J_k \preccurlyeq_1 J_{k-1} \preccurlyeq_1 \ldots \preccurlyeq_1 J_1 \preccurlyeq I$ for some sets $J_1, \ldots, J_k \lhd \bigcup_n I_n$. ◇

Note that extensions are done “backward”. We give here a graphical example :

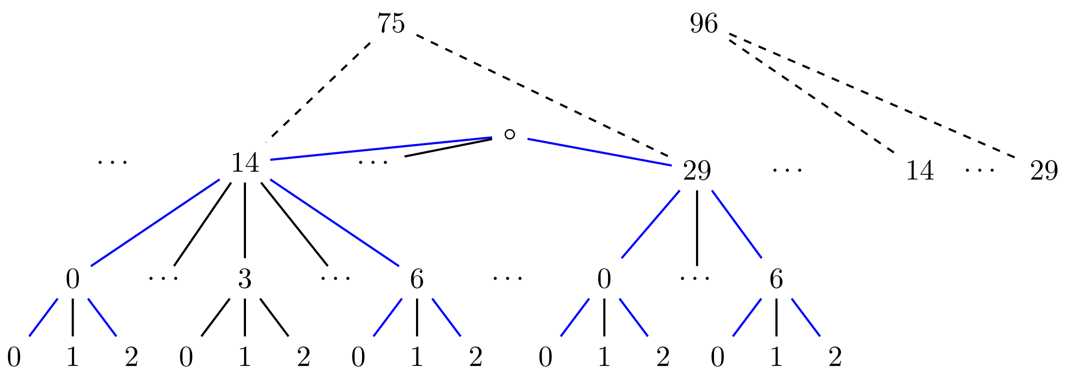

Figure 5.2: The tree with plain lines (partially represented) is a subset of I_3. The blue part represent some symmetric set $I \lhd I_3$. The dashed part represent a possible extension $J \preccurlyeq I$ with $J \lhd I_4$.

Notation

Given an $I \lhd \bigcup_n I_n$ and given a string τ we write τI for the set $\{\tau\rho \ : \ \rho \in I\}$.

Note that given $J \preccurlyeq I$ the set $\{\tau \ : \ \tau I \subseteq J\}$ is itself a symmetric set.

Definition 5.3.21 : For $I, J \lhd \bigcup_n I_n$ such that $J \preccurlyeq I$, the *complement* of I in J, denoted by $J - I$, is the symmetric set of elements τ such that $J = \{\tau\rho \ : \ \tau \in J - I, \rho \in I\}$. ◊

Definition 5.3.22 : Let $I \subseteq J \subseteq I_n$ for some n.

1. For $\mathcal{L} \subseteq J \to 2^\omega$, the *projection* of $\mathcal{L}$ to I, written $\mathcal{L}\upharpoonright_I$, is given by the class of all $\langle X_\tau \ : \ \tau \in I\rangle$ such that there exists X_τ for every $\tau \in J - I$ such that $\langle X_\tau \ : \ \tau \in J\rangle \in \mathcal{L}$.

2. For $\mathcal{L} \subseteq I \to 2^\omega$, the *completion* of $\mathcal{L}$ in J, written $\bigotimes_J \mathcal{L}$, is given by the class of all $\langle X_\tau \ : \ \tau \in J\rangle \in J \to 2^\omega$ such that $\langle X_\tau \ : \ \tau \in I\rangle \in I \to 2^\omega$. ◊

Definition 5.3.23 : Let $J, I \lhd \bigcup_n I_n$ with $J \preccurlyeq I$. For $\mathcal{L} \subseteq I \to 2^\omega$, the *completion* of $\mathcal{L}$ in J, written $\bigotimes_J \mathcal{L}$ is the class of elements $\langle Y_{\tau\rho} \ : \ \tau \in J - I, \rho \in I\rangle \in J \to 2^\omega$ such that $Y_{\tau\rho} = X_\rho$ for every $\tau \in J - I$ and every $\rho \in I$. ◊

Mind the fact that the notation $\bigotimes$ can mean four different things depending on the context : From the notation after Proposition 5.3.5 if $A \in 2^\omega$ then $\bigotimes_I A$ denote the element of $I \to 2^\omega$ for which we "duplicate" A on each component. Still from the notation after Proposition 5.3.5 if $\mathcal{A} \subseteq 2^\omega$ then $\bigotimes_I A$ denote the subclass of $I \to 2^\omega$ which is its cross product on each component.

From Definition 5.3.22 above, if $I \subseteq J \subseteq I_n$ and $\mathcal{A} \subseteq I \to 2^\omega$ then $\bigotimes_J A$ denote the subclass of $J \to 2^\omega$ for which we add the cross product with 2^ω on missing components. Finally from Definition 5.3.23 above, if $J, I \lhd \bigcup_n I_n$ with $J \preccurlyeq I$ and $\mathcal{A} \subseteq I \to 2^\omega$ then $\bigotimes_J A$ denote the class for which we "duplicate" the elements indexed by $\rho \in I$ on every index $\tau\rho \in J$ for $\tau \in J - I$.

5.4 The forcing machinery

From now on we fix a set Z which is not PA over $\emptyset'$ and a $\Delta^0_2(Z)$ set $A \subseteq \omega$. We write A^0 for A and A^1 for $\omega - A$. We are going to build generics $G^0 \in [A^0]^\infty$ and $G^1 \in [A^1]^\infty$ such that one of them will not be PA($\emptyset'$).

We suppose $\{\Phi_e(G, x, y)\}_{e \in \omega}$ is a list of all the *functional* Δ_0 formula, that is, if $\Phi_e(\sigma, a, b)$ is true for some a, b and $\sigma \in 2^{<\omega}$, then $\Phi_e(\sigma\tau, a, b)$ is true for every $\tau \in 2^{<\omega}$.

Notation

We will write $\Phi_e(Z \oplus G, x, y)$ to mean that the formula is meant to be used with set parameter $Z \oplus G$ for the set Z relative to which the construction is done, and for the generic set G that we built. Similarly we write $\Phi_e(Z \oplus \sigma, x, y)$ to mean $\Phi_e(\tau \oplus \sigma, x, y)$ for $\tau = Z{\upharpoonright}_{|\sigma|}$.

5.4.1 The $\mathbb{Q}$-forcing

The full forcing — the $\mathbb{P}$-forcing — can be seen as a tree of simpler forcing conditions, the $\mathbb{Q}$-forcing, that we define now.

Definition 5.4.1 : For $n \in \omega$, a $\mathbb{Q}_n$-condition is a tuple $(\sigma_0, \sigma_1, \langle X_\rho \ : \ \rho \in I\rangle, \mathcal{H})$ where

1. $(\sigma_i, \bigcup_{\rho \in I} X_\rho)$ is a Mathias condition for every $i \in \{0, 1\}$
2. $\sigma_i \subseteq A^i$ for $i \in \{0, 1\}$
3. $\mathcal{H}$ is a large subclass of $I \to 2^\omega$ for some $I \lhd I_n$
4. $\mathcal{H} \subseteq \mathcal{L}_{\langle X_\rho \ : \ \rho \in I\rangle}$

Let $\mathbb{Q} = \bigcup_n \mathbb{Q}_n$. A $\mathbb{Q}_n$-condition is *valid for side* $i \in \{0, 1\}$ if $\langle A^i \cap X_\rho \ : \ \rho \in I\rangle \in \mathcal{H}$ ◊

We now define the forcing extension.

Definition 5.4.2 : Given two conditions $p, q \in \mathbb{Q}$ with $p = (\sigma_0, \sigma_1, \langle X_\rho \ : \ \rho \in I\rangle, \mathcal{H})$ and $q = (\tau_0, \tau_1, \langle Y_\rho \ : \ \rho \in J\rangle, \mathcal{K})$. We define $q \preccurlyeq p$ by:

(1) $(\tau_i, \bigcup_{\rho \in J} Y_\rho) \preccurlyeq (\sigma_i, \bigcup_{\rho \in I} X_\rho)$ as Mathias conditions for every $i \in \{0, 1\}$

(2) $J \preccurlyeq I$

(3) $Y_{\tau\rho} \subseteq X_\rho$ for every $\rho \in I$ and any $\tau \in J - I$.

(4) $\mathcal{K} \subseteq \bigotimes_J \mathcal{H}$ ◊

Note that (4) in the above definition is equivalent to $\mathcal{K}{\upharpoonright}_{\tau I} \subseteq \mathcal{H}$ for every $\tau \in J - I$.

Lemma 5.4.3 : Let $p, q \in \mathbb{Q}$ with $q \preccurlyeq p$. If q is valid for side i then p is valid for side i. ⋆

Proof: Let $p = (\sigma_0, \sigma_1, \langle X_\rho \ : \ \rho \in I\rangle, \mathcal{H})$. Let $q = (\tau_0, \tau_1, \langle Y_\rho \ : \ \rho \in J\rangle, \mathcal{K})$. Suppose $\langle A^i \cap Y_\rho \ : \ \rho \in J\rangle \in \mathcal{K}$. Note that $Y_{\tau\rho} \subseteq X_\rho$ for every $\tau \in J - I$. Thus as $\mathcal{K}$ is upward closed on each of its component we have $\langle A^i \cap X_\rho \ : \ \rho \in I\rangle \in \mathcal{K}{\upharpoonright}_{\tau I}$ for every $\tau \in J - I$. As $\mathcal{K}{\upharpoonright}_{\tau I} \subseteq \mathcal{H}$ for every $\tau \in J - I$ we have $\langle A^i \cap X_\rho \ : \ \rho \in I\rangle \in \mathcal{H}$. Thus p is valid for side i. ■

Recall that Z is the set relative to which we do the construction. In order to define the forcing relations, we need the following functions:

Definition 5.4.4 : For every $I \lhd \bigcup_n I_n$ we define $\zeta_I : \omega \times 2^\omega \times \omega \to \omega$ to be the function which on e, σ and x associates the code for the open subset $\mathcal{U}_{\zeta_I(e,\sigma,x)}$ of $I \to 2^\omega$ defined by:

$$\left\{ \langle X_\rho \ : \ \rho \in I \rangle \ : \ \exists \tau \subseteq \bigcup_{\rho \in I} X_\rho - \{0, \dots, |\sigma|\} \ \exists y \ \neg \Phi_e(Z \oplus (\sigma \cup \tau), x, y) \right\}$$

Proposition 5.4.5 : Let $I, J \lhd \bigcup_n I_n$. Suppose $J \preccurlyeq I$. Then for any e, σ, x we have

$$\bigotimes_J \mathcal{U}_{\zeta_I(e,\sigma,x)} \subseteq \mathcal{U}_{\zeta_J(e,\sigma,x)}$$

Proof: Let $\langle Y_{\tau\rho} \ : \ \tau \in J - I, \rho \in I \rangle$ be such that $Y_{\tau\rho} = X_\rho$ for $\langle X_\rho \ : \ \rho \in I \rangle \in \mathcal{U}_{\zeta_I(e,\sigma,x)}$. Let $B_I = \bigcup_{\rho \in I} X_\rho$ and $B_J = \bigcup_{\tau \in J - I, \rho \in I} Y_{\tau\rho}$.

Note that $B_I = B_J$. By hypothesis there exists $\tau \subseteq B_I - \{0, \dots, |\sigma|\}$ and y such that $\neg\Phi_e(Z \oplus (\sigma \cup \tau), x, y)$, then also there exists $\tau \subseteq B_J - \{0, \dots, |\sigma|\}$ and y such that $\neg\Phi_e(Z \oplus (\sigma \cup \tau), x, y)$. It follows that $\langle Y_{\tau\rho} \ : \ \tau \in J - I, \rho \in I \rangle \in \mathcal{U}_{\zeta_J(e,\sigma,x)}$. ■

We now define the forcing relations.

Definition 5.4.6 : Let $p = (\sigma_0, \sigma_1, \langle X_\rho \ : \ \rho \in I \rangle, \mathcal{H})$ be a $\mathbb{Q}$ condition. Let $\Phi_e(Z \oplus G, x, y)$ be a Δ^0_1 functional formula. Let $i \in \{0, 1\}$. We define:

1. $p \Vdash^i \exists x \ \forall y \ \Phi_e(Z \oplus G, x, y)$ if there exists x such that for all $\tau \subseteq \bigcup_{\rho \in I} X_\rho$ and for all y we have $\Phi_e(Z \oplus (\sigma_i \cup \tau), x, y)$.
2. $p \Vdash^i \forall x \ \exists y \ \neg\Phi_e(Z \oplus G, x, y)$ if for $X^i = A^i \cap \bigcup_{\rho \in I} X_\rho$ we have

$$\mathcal{H} \subseteq \bigcap_{\tau \subseteq X^i, x \in \omega} \mathcal{U}_{\zeta_I(e, \sigma_i \cup \tau, x)}$$

Proposition 5.4.7 : Let $p = (\sigma_0, \sigma_1, \langle X_\rho \ : \ \rho \in I \rangle, \mathcal{H})$ be a $\mathbb{Q}$ condition. Let $\Phi_e(Z \oplus G, x, y)$ be a Δ^0_1 functional formula. Let $i \in \{0, 1\}$. Let $q \preccurlyeq p$.

(1) If $p \Vdash^i \exists x \ \forall y \ \Phi_e(Z \oplus G, x, y)$ then $q \Vdash^i \exists x \ \forall y \ \Phi_e(Z \oplus G, x, y)$.

(2) If $p \Vdash^i \forall x \ \exists y \ \neg\Phi_e(Z \oplus G, x, y)$ then $q \Vdash^i \forall x \ \exists y \ \neg\Phi_e(Z \oplus G, x, y)$. ⋆

Proof: Let $q = (\tau_0, \tau_1, \langle Y_\rho \ : \ \rho \in J \rangle, \mathcal{K})$. For (1) let x be such that for all $\tau \subseteq \bigcup_{\rho \in I} X_\rho$ and for all y we have $\Phi_e(Z \oplus (\sigma_i \cup \tau), x, y)$. Let now $\tau \subseteq \bigcup_{\rho \in J} Y_\rho$. We have $\bigcup_{\rho \in I} Y_\rho \subseteq \bigcup_{\rho \in I} X_\rho$ and thus $\tau \subseteq \bigcup_{\rho \in I} X_\rho$. Note also that $\tau_i = \sigma_i \cup \rho$ for some $\rho \subseteq \bigcup_{\rho \in I} X_\rho$. It follows that $\tau_i \cup \tau = \sigma_i \cup \rho \cup \tau$. As $\rho \cup \tau \subseteq \bigcup_{\rho \in I} X_\rho$ then for all y we have $\Phi_e(Z \oplus (\sigma_i \cup \rho \cup \tau), x, y)$. Then for all y we have $\Phi_e(Z \oplus (\tau_i \cup \tau), x, y)$. Then $q \Vdash^i \exists x \ \forall y \ \Phi_e(Z \oplus G, x, y)$

For (2) note that we have $\mathcal{K} \subseteq \bigotimes_J \mathcal{H}$. Let $X^i = A^i \cap \bigcup_{\rho \in I} X_\rho$ and $Y^i = A^i \cap \bigcup_{\rho \in J} Y_\rho$. Fix $\tau \subseteq Y^i$ and $x \in \omega$. As $q \preccurlyeq p$ we have $Y^i \subseteq X^i$ and then $\tau \subseteq X^i$. Note that $\tau_i = \sigma_i \cup \rho$ for some

$\rho \subseteq X^i$. So $\tau_i \cup \tau = \sigma_i \cup \rho \cup \tau$. As $\rho \cup \tau \subseteq X^i$ by hypothesis we have $\mathcal{H} \subseteq \mathcal{U}_{\zeta_I(e,\sigma_i \cup (\rho \cup \tau),x)}$ and then using Proposition 5.4.5

$$\bigotimes_J \mathcal{H} \subseteq \bigotimes_J \mathcal{U}_{\zeta_I(e,\sigma_i \cup (\rho \cup \tau),x)} \subseteq \mathcal{U}_{\zeta_J(e,\sigma_i \cup (\rho \cup \tau),x)}$$

As $\mathcal{K} \subseteq \bigotimes_J \mathcal{H}$ we then have $\mathcal{K} \subseteq \mathcal{U}_{\zeta_J(e,\sigma_i \cup (\rho \cup \tau),x)} = \mathcal{U}_{\zeta_J(e,\tau_i \cup \tau,x)}$. As the choice of τ and x was arbitrary we then have

$$\mathcal{K} \subseteq \bigcap_{\tau \subseteq Y^i, x \in \omega} \mathcal{U}_{\zeta_J(e,\tau_i \cup \tau,x)}$$

Thus $q \Vdash^i \forall x\ \exists y\ \neg\Phi_e(Z \oplus G, x, y)$ ∎

Proposition 5.4.8 : Given a sufficiently generic $\mathbb{Q}$-filter $\mathcal{F}$, there are unique sets $G^0_{\mathcal{F}}, G^1_{\mathcal{F}}$ such that for every conditions $(\sigma_0, \sigma_1, \langle X_\rho\ :\ \rho \in I\rangle, \mathcal{H}) \in \mathcal{F}$ we have $\sigma_0 \prec G^0_{\mathcal{F}}$ and $\sigma_0 \prec G^1_{\mathcal{F}}$. ⋆

Proof: Trivial. ∎

Notation

Given a sufficiently generic $\mathbb{Q}$-filter $\mathcal{F} \subseteq \mathbb{Q}$ we write $G^i_{\mathcal{F}}$ for the associated generic on side i.

Definition 5.4.9 : A $\mathbb{Q}$-filter $\mathcal{F} \subseteq \mathbb{Q}$ is Π^0_2-complete on side i if for every Δ_0 functional formula $\Phi_e(Z \oplus G, x, y)$, whenever there is $p \in \mathcal{F}$ such that $p \Vdash^i \forall x\ \exists y\ \Phi_e(Z \oplus G, x, y)$ for $i \in \{0,1\}$ then for every x there exists $q \in \mathcal{F}$ with $q = (\sigma_0, \sigma_1, \langle X_\rho\ :\ \rho \in I\rangle, \mathcal{H})$ such that $\exists y\ \Phi_e(Z \oplus \sigma_i, x, y)$ holds. ◇

Lemma 5.4.10 : Let $\mathcal{F}$ be a $\mathbb{Q}$-filter Π^0_2-complete on side i. Let $p \in \mathcal{F}$. Let $\Phi_e(Z \oplus G, x, y)$ be a Δ_0 functional formula.

(1) If $p \Vdash^i \exists x\ \forall y\ \Phi_e(Z \oplus G, x, y)$ then $\exists x\ \forall y\ \Phi_e(Z \oplus G^i_{\mathcal{F}}, x, y)$ holds.

(2) If $p \Vdash^i \forall x\ \exists y\ \neg\Phi_e(Z \oplus G, x, y)$ then $\forall x\ \exists y\ \neg\Phi_e(Z \oplus G^i_{\mathcal{F}}, x, y)$ holds. ⋆

Proof: Let $p = (\sigma_0, \sigma_1, \langle X_\rho\ :\ \rho \in I\rangle, \mathcal{H})$. For (1) there is some x such that for all $\tau \subseteq \bigcup_{\rho \in I} X_\rho$ and all y we have $\Phi_e(Z \oplus (\sigma_i \cup \tau), x, y)$. Suppose for contradiction the there exists y such that $\neg\Phi_e(Z \oplus G^i_{\mathcal{F}}, x, y)$. Then $\neg\Phi_e(Z \oplus \sigma, x, y)$ holds for some $\sigma \preccurlyeq G^i_{\mathcal{F}}$. It follows that $\neg\Phi_e(Z \oplus \rho, x, y)$ holds for every extension of $\rho \succcurlyeq \sigma$. One of this extension must be of the form $\sigma_i \cup \tau$ for $\tau \subseteq \bigcup_{\rho \in I} X_\rho$, which is a contradiction. Thus $\exists x\ \forall y\ \Phi_e(Z \oplus G^i_{\mathcal{F}}, x, y)$ holds.

For (2) we need to use the fact that $\mathcal{F}$ is Π^0_2-complete on side i : for every x there exists $q \in \mathcal{F}$ with $q = (\tau_0, \tau_1, \langle Y_\rho\ :\ \rho \in J\rangle, \mathcal{K})$ such that $\exists y\ \neg\Phi_e(Z \oplus \tau_i, x, y)$ holds. Thus for every x there exists y such that $\neg\Phi_e(Z \oplus G^i_{\mathcal{F}}, x, y)$ holds. ∎

For the previous proposition to work, we need our filter to be Π^0_2-complete on side i. This is the difficulty in this proof, and the reason we need later the $\mathbb{P}$-forcing.

We now connect the forcing of Σ^0_2 statements, to valuations and making our generic not PA($\emptyset'$). For that we fix an enumeration $\{\Phi_e(X, n)\}_{e \in \omega}$ of the $\{0,1\}$-valued partial functional.

Definition 5.4.11 : Let $\eta : \omega \times 2^{<\omega} \to \omega$ be a function which takes an integer e and a valuation v in parameter and returns the code $\eta(e, v)$ such that for any $X \in 2^{\omega}$:

$$\exists n \in \operatorname{dom} v\ \Phi_e(X', n) = v(n) \text{ iff } \exists x\ \forall y\ \Phi_{\eta(e,v)}(X, x, y)$$

Definition 5.4.12 : Let $p \in \mathbb{Q}$. We define

1. $p \Vdash^i \exists x\ \Phi_e((Z \oplus G)', x) \downarrow = \Phi_x(\emptyset', x)$ if there exists a valuation v compatible with $J_{\emptyset'}$ and with $\operatorname{dom} v \subseteq \operatorname{dom} J_{\emptyset'}$ such that $p \Vdash^i \exists x\ \forall y\ \Phi_{\eta(e,v)}(Z \oplus G, x, y)$.
2. $p \Vdash^i \exists x\ \Phi_{e_i}((Z \oplus G)', x) \uparrow$ if there are two incompatible valuations v_0, v_1 such that $p \Vdash^i \forall x\ \exists y\ \neg\Phi_{\eta(e,v_0)}(Z \oplus G, x, y)$ and $p \Vdash^i \forall x\ \exists y\ \neg\Phi_{\eta(e,v_1)}(Z \oplus G, x, y)$. ◊

Definition 5.4.13 : A $\mathbb{Q}$-filter $\mathcal{F} \subseteq \mathbb{Q}$ is PA-generic on side i if for every functional $\Phi_e(G, n)$, there exists $p \in \mathcal{F}$ and n such that either

1. $p \Vdash^i \Phi_{e_i}((Z \oplus G)', n) \downarrow = \Phi_n(\emptyset', n)$
2. $p \Vdash^i \Phi_{e_i}((Z \oplus G)', n) \uparrow$ ◊

Lemma 5.4.14 : Let $\mathcal{F}$ be a $\mathbb{Q}$-filter Π^0_2-complete on side i for $i \in \{0, 1\}$. Let $p \in \mathbb{Q}$.

1. If $p \Vdash^i \exists n\ \Phi_e((Z \oplus G)', n) \downarrow = \Phi_n(\emptyset', n)$ then $\Phi_e((Z \oplus G^i_{\mathcal{F}})', n) \downarrow = \Phi_n(\emptyset', n)$ holds for some n.
2. If $p \Vdash^i \exists n\ \Phi_e((Z \oplus G)', n) \uparrow$ then $\Phi_e((Z \oplus G^i_{\mathcal{F}})', n) \uparrow$ holds for some n. ⋆

Proof: For (1) let v be a valuation compatible with $J_{\emptyset'}$ such that $\operatorname{dom} v \subseteq \operatorname{dom} J_{\emptyset'}$ for which we have $p \Vdash^i \exists x\ \forall y\ \Phi_{\eta(e,v)}(Z \oplus G, x, y)$. By Lemma 5.4.10 we must have some x such that $\forall y\ \Phi_{\eta(e,v)}(Z \oplus G^i_{\mathcal{F}}, x, y)$ holds. By definition of η we must have some $n \in \operatorname{dom} v$ such that $\Phi_e((Z \oplus G^i_{\mathcal{F}})', n) = v(n)$ holds. By assumptions on v we then have $\Phi_e((Z \oplus G^i_{\mathcal{F}})', n) \downarrow = \Phi_n(\emptyset', n)$.

For (2) let v_0, v_1 be two incompatible valuations such that $p \Vdash^i \forall x\ \exists y\ \neg\Phi_{\eta(e,v_0)}(Z \oplus G, x, y)$ and $p \Vdash^i \forall x\ \exists y\ \neg\Phi_{\eta(e,v_1)}(Z \oplus G, x, y)$. By Lemma 5.4.10 we must have that both $\forall x\ \exists y\ \neg\Phi_{\eta(e,v_0)}(Z \oplus G^i_{\mathcal{F}}, x, y)$ and $\forall x\ \exists y\ \neg\Phi_{\eta(e,v_1)}(Z \oplus G^i_{\mathcal{F}}, x, y)$ holds. By definition of η it must be that

$$\Phi_e((Z \oplus G^i_{\mathcal{F}})', n) \uparrow \text{ or } \Phi_e((Z \oplus G^i_{\mathcal{F}})', n) \downarrow \neq v_0(n) \text{ for every } n \in \operatorname{dom} v_0$$

and

$$\Phi_e((Z \oplus G^i_{\mathcal{F}})', n) \uparrow \text{ or } \Phi_e((Z \oplus G^i_{\mathcal{F}})', n) \downarrow \neq v_1(n) \text{ for every } n \in \operatorname{dom} v_1$$

As v_0 and v_1 are incompatible we must have some n such that $v_0(n) \downarrow \neq v_1(n) \downarrow$. Therefore we must have $\Phi_e((Z \oplus G^i_{\mathcal{F}})', n) \uparrow$ (using that Φ_e is $\{0, 1\}$-valued). ∎

5.4.2 The $\mathbb{P}$-forcing

The goal is now to create a Π^0_2-complete and PA-generic $\mathbb{Q}$-filter. We use for that the $\mathbb{P}$-forcing for which a condition is a combination of several $\mathbb{Q}$-conditions. Ultimately a filter for $\mathbb{P}$ can be seen as a finitely branching tree of $\mathbb{Q}$-conditions.

Recall that Z is a set not PA($\emptyset'$) and A a $\Delta^0_2(Z)$ set with $A^0 = A$ and $A^1 = \omega - A$. We fix in addition a countable Scott set $\mathcal{M}$ containing Z and low relative to Z, that is such that $M' \leqslant_T Z'$ for a presentation M of $\mathcal{M}$. Recall also that the notation $\mathcal{U}_C^{\mathcal{M}}$ for $C \subseteq \omega \times \omega$ means $\bigcap_{\langle a,b\rangle \in C} \mathcal{U}_a^{X_b}$ where $\{X_n\}_{n\in\omega}$ is an enumeration of the elements of $\mathcal{M}$.

Definition 5.4.15 : For $n \in \omega$, a $\mathbb{P}_n$-condition is a tuple

$$(\{\langle \sigma_0^I, \sigma_1^I\rangle \ : \ I \lhd I_n\}, \langle X_\rho \ : \ \rho \in I_n\rangle, C)$$

where

(1) $\mathcal{U}_C^{\mathcal{M}}$ is a large subclass of $I_n \to 2^\omega$

(2) $(\sigma_0^I, \sigma_1^I, \langle X_\rho \ : \ \rho \in I\rangle, \mathcal{U}_C^{\mathcal{M}}\upharpoonright_I)$ is a $\mathbb{Q}_n$ condition for every $I \lhd I_n$.

(3) C a $\Delta^0_2(Z)$ set.

(4) $X_\rho \in \mathcal{M}$ for every $\rho \in I_n$

(5) $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{\langle X_\rho \ : \ \rho \in I_n\rangle}$

For $I \lhd I_n$ we write p^I for the $\mathbb{Q}_n$-condition defined by

$$(\sigma_0^I, \sigma_1^I, \langle X_\rho \ : \ \rho \in I\rangle, \mathcal{U}_C^{\mathcal{M}}\upharpoonright_I)$$

Let $\mathbb{P} = \bigcup_n \mathbb{P}_n$ ◇

We now define forcing extension.

Definition 5.4.16 : Given two $\mathbb{P}$-conditions $p = (\{\langle \sigma_0^I, \sigma_1^I\rangle \ : \ I \lhd I_n\}, \langle X_\rho \ : \ \rho \in I_n\rangle, C)$ and $q = (\{\langle \tau_0^I, \tau_1^I\rangle \ : \ I \lhd I_m\}, \langle Y_\rho \ : \ \rho \in I_m\rangle, D)$ we define $q \leqslant p$ if

1. $m \geqslant n$
2. For every $J \lhd I_m$ and $I \lhd I_n$ such that $J \leqslant I$ we have $q^J \leqslant p^I$. ◇

We now define the forcing question, which is parametered by a set $S \subseteq \{I \lhd I_n\}$.

Definition 5.4.17 : Let $p = (\{\langle \sigma_0^I, \sigma_1^I\rangle \ : \ I \lhd I_n\}, \langle X_\rho \ : \ \rho \in I_n\rangle, C)$ be a $\mathbb{P}_n$ condition. Let $S \subseteq \{I \lhd I_n\}$. Let $\Phi_e(Z \oplus G, x, y)$ be a Δ^0_1 functional formula. For $I \lhd I_n$ and $i \in \{0,1\}$ let X_I^i be $A^i \cap \bigcup_{\rho \in I} X_\rho$. We define $p \mathrel{?{\vdash}}_S^i \exists x \ \forall y \ \Phi_e(Z \oplus G, x, y)$ to hold if

$$\mathcal{U}_C^{\mathcal{M}} \cap \bigcap_{I \in S} \bigcap_{\tau \subseteq X_I^i, x \in \omega} \bigotimes_{I_n} \mathcal{U}_{\zeta_I(e, \sigma_i^I \cup \tau, x)}$$

is not a largeness class ◇

At this point an example will probably help the understanding.

Example 5.4.18 : We have $I_1 = \{0,1,2\}$ with $\{0,1\} \lhd I_1$, $\{0,2\} \lhd I_1$ and $\{1,2\} \lhd I_1$. Let

$$(\{\langle\sigma_0^{0,1},\sigma_1^{0,1},\rangle,\langle\sigma_0^{0,2},\sigma_1^{0,2},\rangle,\langle\sigma_0^{1,2},\sigma_1^{1,2},\rangle\},\langle X_0,X_1,X_2\rangle,C)$$

be a $\mathbb{P}_1$-condition. Suppose $S = \{\{0,1\},\{0,2\},\{1,2\}\}$ and let $i = 0$. Then

$$p\,?{\vdash}_S^0\ \exists x\ \forall y\ \Phi_e(Z \oplus G, x, y)$$

holds iff

$$\mathcal{U}_C^{\mathcal{M}} \cap \begin{array}{l} \bigcap_{\tau \subseteq A^0 \cap (X_0 \cup X_1), x \in \omega} \left\{ \langle Y_0,Y_1,Y_2\rangle : \begin{array}{l} \exists \rho \subseteq Y_0 \cup Y_1 - \{0,\dots,|\sigma_0^{0,1} \cup \tau|\}\ \exists y \in \omega \\ \text{s.t. } \neg\Phi_e(Z \oplus (\sigma_0^{0,1} \cup \tau \cup \rho), x, y) \end{array} \right\} \\ \bigcap_{\tau \subseteq A^0 \cap (X_0 \cup X_2), x \in \omega} \left\{ \langle Y_0,Y_1,Y_2\rangle : \begin{array}{l} \exists \rho \subseteq Y_0 \cup Y_2 - \{0,\dots,|\sigma_0^{0,2} \cup \tau|\}\ \exists y \in \omega \\ \text{s.t. } \neg\Phi_e(Z \oplus (\sigma_0^{0,2} \cup \tau \cup \rho), x, y) \end{array} \right\} \\ \bigcap_{\tau \subseteq A^0 \cap (X_1 \cup X_2), x \in \omega} \left\{ \langle Y_0,Y_1,Y_2\rangle : \begin{array}{l} \exists \rho \subseteq Y_1 \cup Y_2 - \{0,\dots,|\sigma_0^{1,2} \cup \tau|\}\ \exists y \in \omega \\ \text{s.t. } \neg\Phi_e(Z \oplus (\sigma_0^{1,2} \cup \tau \cup \rho), x, y) \end{array} \right\} \end{array}$$

is not a largeness class. $\diamond$

We now show that the question has the right complexity.

Lemma 5.4.19 : Let $p = (\{\langle\sigma_0^I,\sigma_1^I\rangle : I \lhd I_n\}, \langle X_\rho : \rho \in I_n\rangle, C)$ be a $\mathbb{P}_n$ condition. Let $S \subseteq \{I \lhd I_n\}$. Let $\Phi_e(Z \oplus G, x, y)$ be a Δ_1^0 functional formula. The question $p\,?{\vdash}_S^i\ \exists x\ \forall y\ \Phi_e(Z \oplus G, x, y)$ is $\Sigma_1^0(Z')$ uniformly in e, i and S. $\star$

Proof: For $I \lhd I_n$ and $i \in \{0,1\}$ let X_I^i be $A^i \cap \bigcup_{\rho \in I} X_\rho$. Let D be a set of indices such that

$$\mathcal{U}_D^{\mathcal{M}} = \mathcal{U}_C^{\mathcal{M}} \cap \bigcap_{I \in S} \bigcap_{\tau \subseteq X_I^i, x \in \omega} \bigotimes_{I_n} \mathcal{U}_{\zeta_I(e, \sigma_i^I \cup \tau, x)}$$

Let M be the set coding for $\mathcal{M}$. Note that as A, C and M are all $\Delta_2^0(Z)$ then also D is $\Delta_2^0(Z)$. By Proposition 5.3.8 $\mathcal{U}_D^{\mathcal{M}}$ is a largeness class iff for every finite set $F \subseteq D$ the open set $\mathcal{U}_F^{\mathcal{M}}$ is a largeness class. By a relativized version of Corollary 5.3.10 the question "is $\mathcal{U}_F^{\mathcal{M}}$ is a largeness class ?" is $\Pi_2^0(M)$ uniformly in F, in e, i and S. It is then $\Pi_1^0(M')$ and then $\Pi_1^0(Z')$ uniformly in F, in e, i and S. Thus the question "is $\mathcal{U}_D^M$ is a largeness class ?" is $\Pi_1^0(Z')$ uniformly in e, i and S.

It follows that the question $p\,?{\vdash}_S^i\ \exists x\ \forall y\ \Phi_e(Z \oplus G, x, y)$ is $\Sigma_1^0(Z')$ uniformly in e, i and S. ■

We now turn to the proof of the two main lemmas, which can be seen as the core of the proof.

Lemma 5.4.20 : Let $p = (\{\langle\sigma_0^I,\sigma_1^I\rangle : I \lhd I_n\}, \langle X_\rho : \rho \in I_n\rangle, C)$ be a $\mathbb{P}_n$ condition. Let $S \subseteq \{I \lhd I_n\}$. Let $\Phi_e(Z \oplus G, x, y)$ be a Δ_1^0 functional formula. Suppose $p\,?{\vdash}_S^i\ \exists x\ \forall y\ \Phi_e(Z \oplus G, x, y)$ for some $i \in \{0,1\}$. Then there is an extension $q \preccurlyeq p$ with $q \in \mathbb{P}_n$ and some $I \in S$ such that $q^I \Vdash^i \exists x\ \forall y\ \Phi_e(Z \oplus G, x, y)$. $\star$

Proof: Let $X_I^i = A^i \cap \bigcup_{\rho \in I} X_\rho$ for $I \lhd I_n$. As $p\,?{\vdash}_S^i\ \exists x\ \forall y\ \Phi_e(Z \oplus G, x, y)$ we have

$$\mathcal{U}_C^{\mathcal{M}} \cap \bigcap_{I \in S} \bigcap_{\tau \subseteq X_I^i, x \in \omega} \bigotimes_{I_n} \mathcal{U}_{\zeta_I(e, \sigma_i^I \cup \tau, x)}$$

is not a largeness class. Thus by Proposition 5.3.8 there exists a $\Sigma_1^0(X)$ class $\mathcal{U} \subseteq I_n \to 2^\omega$ for some $X \in \mathcal{M}$ such that $\mathcal{U}_C^{\mathcal{M}} \cap \bigcap_{I \in S} \bigcap_{\tau \subseteq X_I^i, x \in \omega} \bigotimes_{I_n} \mathcal{U}_{\zeta_I(e, \sigma_i^I \cup \tau, x)} \subseteq \mathcal{U}$ and already $\mathcal{U}$ is not a largeness class. Then there exists covers $X^0_\rho \cup \cdots \cup X^k_\rho \supseteq \omega \in \mathcal{M}$ for every $\rho \in I_n$ such that for every function $f : I_n \to \{0, \dots, k\}$ we have $\langle X^{f(\rho)}_\rho \ : \ \rho \in I_n\rangle \notin \mathcal{U}$. As $\mathcal{U}^{\mathcal{M}}_C \subseteq \mathcal{L}_{\langle X_\rho \ : \ \rho \in I_n\rangle}$ there must be by Proposition 5.3.15 some $f : I_n \to \{0, \dots, k\}$ such that $\mathcal{U}^{\mathcal{M}}_C \cap \mathcal{L}_{\langle X_\rho \cap X^{f(\rho)}_\rho \ : \ \rho \in I_n\rangle}$ is a largeness class and in particular such that $\langle X_\rho \cap X^{f(\rho)}_\rho \ : \ \rho \in I_n\rangle \in \mathcal{U}^{\mathcal{M}}_C \subseteq \mathcal{U}$. Then we must have $\langle X_\rho \cap X^{f(\rho)}_\rho \ : \ \rho \in I_n\rangle \notin \bigcap_{I \in S} \bigcap_{\tau \subseteq X^i_I, x \in \omega} \bigotimes_{I_n} \mathcal{U}_{\zeta_I(e, \sigma^I_i \cup \tau, x)}$. Let $I_* \in S$, $\tau_* \subseteq A^i \cap \bigcup_{\rho \in I_*} X_\rho$ and $x_* \in \omega$ be such that

$$\langle X_\rho \cap X^{f(\rho)}_\rho \ : \ \rho \in I_n\rangle \notin \bigotimes_{I_n} \mathcal{U}_{\zeta_{I_*}(e, \sigma^{I_*}_i \cup \tau_*, x_*)}$$

It follows that

$$\langle X_\rho \cap X^{f(\rho)}_\rho \ : \ \rho \in I_*\rangle \notin \mathcal{U}_{\zeta_{I_*}(e, \sigma^{I_*}_i \cup \tau_*, x_*)}$$

Thus $\Phi_e(Z \oplus (\sigma^{I_*}_i \cup \tau_* \cup \rho), x_*, y)$ holds for every $\rho \subseteq \bigcup_{\rho \in I_*} X_\rho \cap X^{f(\rho)}_\rho$ and every $y \in \omega$.

Let $\tau^I_{i-1} = \sigma^I_{i-1}$ and $\tau^I_i = \sigma^I_i$ except for I_* for which we let $\tau^{I_*}_i = \sigma^{I_*}_i \cup \tau_*$. Let $m = \max_{i \in \{0,1\}, I \lhd I_n} |\tau^I_i|$. Let $Y_\rho = X_\rho \cap X^{f(\rho)}_\rho - \{0, \dots, m\}$ for $\rho \in I_n$. By Proposition 5.3.12 we have $\mathcal{L}_{\langle X_\rho \cap X^{f(\rho)}_\rho \ : \ \rho \in I_n\rangle} = \mathcal{L}_{\langle Y_\rho \ : \ \rho \in I_n\rangle}$ and thus $\mathcal{U}^{\mathcal{M}}_C \cap \mathcal{L}_{\langle Y_\rho \ : \ \rho \in I_n\rangle}$ is a largeness class. Using the fact that $Y_\rho \in \mathcal{M}$ for $\rho \in I_n$, let D be such that $\mathcal{U}^{\mathcal{M}}_D = \mathcal{U}^{\mathcal{M}}_C \cap \mathcal{L}_{\langle Y_\rho \ : \ \rho \in I_n\rangle}$. Let

$$q = (\{\langle \tau^I_0, \tau^I_1\rangle \ : \ I \lhd I_n\}, \langle Y_\rho \ : \ \rho \in I_n\rangle, D)$$

.

By design $q \preccurlyeq p$ is a $\mathbb{P}_n$-condition such that $q^{I_*} \Vdash^i \exists x \ \forall y \ \Phi_e(Z \oplus G, x, y)$. ■

Before showing the next lemma, we provide a picture which may be of help for the reader while reading the proof : an illustration of the relation between largeness classes of a condition $p \in \mathbb{P}_n$ and an extension $q \preccurlyeq p$ with $q \in \mathbb{P}_{n+1}$.

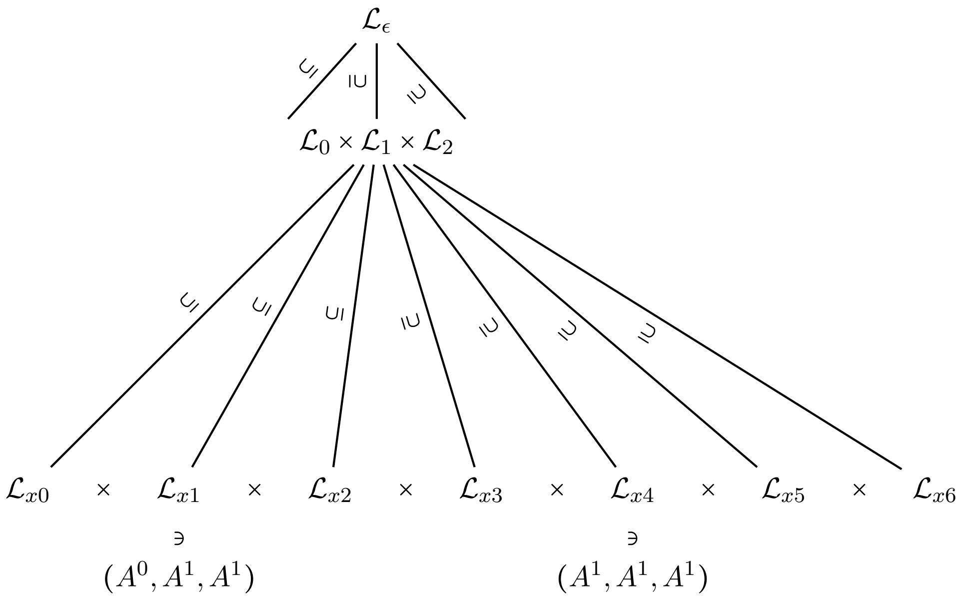

Figure 5.3: In this picture $\mathcal{L}_\epsilon$ is a largeness subclass of 2^ω. Then $\mathcal{L}_0, \mathcal{L}_1$ and $\mathcal{L}_2$ are the three largeness classes obtained when forcing our first Π^0_2 statement. Each of them is a subclass of $\mathcal{L}_\epsilon$. Then each $\mathcal{L}_{xa}$ for $a < 7$ are the seven largeness classes obtained when forcing our second Π^0_2 statement. Each of them is a largeness subclass of $\mathcal{L}_0 \times \mathcal{L}_1 \times \mathcal{L}_2$ and we are then at this point in the product space $I_2 \to 2^\omega$. We then have a partial illustration of a good combination in the belong relation $\langle A^{f(\rho)} : \rho \in I_2\rangle$ for a possible $f : I_n \to \{0,1\}$: Here $(A^1, A^1) \in \mathcal{L}_{x1}\restriction_{\{1,2\}}$ and $(A^1, A^1) \in \mathcal{L}_{x4}\restriction_{\{1,2\}}$. Having $(A, A) \in \mathcal{L}_1 \times \mathcal{L}_2$ helps us defeat one functional and having $(A^1, A^1) \in \mathcal{L}_{x1}\restriction_{\{1,2\}}$ and $(A^1, A^1) \in \mathcal{L}_{x4}\restriction_{\{1,2\}}$ helps us defeat another one.

Lemma 5.4.21 : Let $p = (\{\langle\sigma^I_0, \sigma^I_1\rangle : I \lhd I_n\}, \langle X_\rho : \rho \in I_n\rangle, C)$ be a $\mathbb{P}_n$ condition. Let $S \subseteq \{I \lhd I_n\}$. Let $\Phi_{e_0}(Z \oplus G, x, y), \dots, \Phi_{e_{u_{n+1}-1}}(Z \oplus G, x, y)$ be u_{n+1} many Δ^0_1 functional formulas. Let $i \in \{0,1\}$ and suppose $p\,?\!\nVdash^i_S \exists x\ \forall y\ \Phi_{e_j}(Z \oplus G, x, y)$ for every $j < u_{n+1}$. Then there is one extension $q \preccurlyeq p$ with $q \in \mathbb{P}_{n+1}$ such that for every $I \in S$ and every $J \in I_{n+1}$ extending I we have for a, b the two possible nodes of $J^{\preccurlyeq}$ of length 1 that:

$$q^J \Vdash^i \forall x\ \exists y\ \neg\Phi_{e_a}(Z \oplus G, x, y) \text{ and } q^J \Vdash^i \forall x\ \exists y\ \neg\Phi_{e_b}(Z \oplus G, x, y)$$

Proof: Let $X^i_I = A^i \cap \bigcup_{\rho \in I} X_\rho$ for $I \lhd I_n$. Let $\mathcal{C}$ be the class $\mathcal{U}^{\mathcal{M}}_C$. For every $j < u_{n+1}$, as $p\,?\!\nVdash^i_S \exists x\ \forall y\ \Phi_{e_j}(Z \oplus G, x, y)$ we have

$$\mathcal{C}_j = \mathcal{C} \cap \bigcap_{I \in S}\ \bigcap_{\tau \subseteq X^i_I, x \in \omega}\ \bigotimes_{I_n} \mathcal{U}_{\zeta_I(e_j, \sigma^I_i \cup \tau, x)}$$

is a largeness class.

Let $\mathcal{D} \subseteq I_{n+1} \to 2^\omega$ be the class $\mathcal{C}_0 \times \mathcal{C}_2 \times \dots \times \mathcal{C}_{u_{n+1}-1}$. Let D be a set of indices such that $\mathcal{U}^{\mathcal{M}}_D = \mathcal{D}$. For $J \lhd I_{n+1}$ with $J \preccurlyeq I$ let $\tau^J_0 = \sigma^I_0$ and $\tau^J_1 = \sigma^I_1$. Let $Y_{j\rho} = X_\rho$ for every $\rho \in I_n$ and every $j < u_{n+1}$. Finally let

$$q = (\{\langle\tau^J_0, \tau^J_1\rangle : J \lhd I_{n+1}\}, \langle Y_\rho : \rho \in I_{n+1}\rangle, D)$$

Let us show that q is a $\mathbb{P}_{n+1}$ condition. As a product of u_{n+1} large subclasses of $I_n \to 2^\omega$, $\mathcal{D}$ is a large subclass of $I_{n+1} \to 2^\omega$. As X^i_I is $\Delta^0_2(Z)$ for every $I \lhd I_n$ it follows that the set of indices D is $\Delta^0_2(Z)$. As $\mathcal{C} \subseteq \mathcal{L}_{\langle X_\rho \,:\, \rho\in I_n\rangle}$, then $\mathcal{C}_j \subseteq \mathcal{L}_{\langle X_\rho \,:\, \rho\in I_n\rangle}$ for $j < u_{n+1}$. Thus $\mathcal{C}_j \subseteq \mathcal{L}_{\langle Y_{j\rho} \,:\, \rho\in I_n\rangle}$ for $j < u_{n+1}$ and thus $\mathcal{D} \subseteq \mathcal{L}_{\langle Y_\rho \,:\, \rho\in I_{n+1}\rangle}$. It follows that q is a $\mathbb{P}_{n+1}$ condition.

Let us show that $q \preccurlyeq p$. Let $I \lhd I_n$ and $J \lhd I_{n+1}$ with $J \preccurlyeq I$ and let a, b be such that $J = \{a\rho \;:\; \rho \in I\} \cup \{b\rho \;:\; \rho \in I\}$. It is clear that $Y_{a\rho} \subseteq X_\rho$, $Y_{b\rho} \subseteq X_\rho$ and that $(\tau^I_0, \bigcup_{\rho\in I_{n+1}} Y_\rho) \preccurlyeq (\sigma^I_0, \bigcup_{\rho\in I_n} X_\rho)$ and $(\tau^I_1, \bigcup_{\rho\in I_{n+1}} Y_\rho) \preccurlyeq (\sigma^I_1, \bigcup_{\rho\in I_n} X_\rho)$ as Mathias conditions (the conditions are in fact equal). Note also that as $\mathcal{C}_a\!\upharpoonright_I \subseteq \mathcal{C}\!\upharpoonright_I$ and as $\mathcal{C}_b\!\upharpoonright_I \subseteq \mathcal{C}\!\upharpoonright_I$ we have

$$\mathcal{C}_a\!\upharpoonright_I \times \mathcal{C}_b\!\upharpoonright_I \subseteq \bigotimes_J \mathcal{C}\!\upharpoonright_I$$

As $\mathcal{D}\!\upharpoonright_J = \mathcal{C}_a\!\upharpoonright_I \times \mathcal{C}_b\!\upharpoonright_I$ we then have $\mathcal{D}\!\upharpoonright_J \subseteq \bigotimes_J \mathcal{C}\!\upharpoonright_I$. Thus $q^J \preccurlyeq p^I$. As this is the case for $I \lhd I_n$ and every $J \lhd I_{n+1}$ with $J \preccurlyeq I$ we then have $q \preccurlyeq p$.

Let us finally show that for every $I \in S$ and every $J \in I_{n+1}$ extending I we have for a, b the two possible nodes of $J^{\preccurlyeq}$ of length 1 that:

$$q^J \Vdash^i \forall x\ \exists y\ \neg\Phi_{e_a}(Z \oplus G, x, y) \text{ and } q^J \Vdash^i \forall x\ \exists y\ \neg\Phi_{e_b}(Z \oplus G, x, y)$$

We want to show for $Y^i_J = A^i \cap \bigcup_{\rho\in J} Y_\rho$ that we have:

$$\mathcal{U}^{\mathcal{M}}_D\!\upharpoonright_J \subseteq \bigcap_{\tau\subseteq Y^i_J, x\in\omega} \mathcal{U}_{\zeta_J(e_a, \tau^J_i \cup \tau, x)} \text{ and } \mathcal{U}^{\mathcal{M}}_D\!\upharpoonright_J \subseteq \bigcap_{\tau\subseteq Y^i_J, x\in\omega} \mathcal{U}_{\zeta_J(e_b, \tau^J_i \cup \tau, x)}$$

Let us show the first inclusion, the second one being shown symmetrically. We have

$$\mathcal{C}_a \subseteq \bigcap_{\tau\subseteq X^i_I, x\in\omega} \bigotimes_{I_n} \mathcal{U}_{\zeta_I(e_a, \sigma^I_i \cup \tau, x)} \text{ and then } \mathcal{C}_a\!\upharpoonright_I \subseteq \bigcap_{\tau\subseteq X^i_I, x\in\omega} \mathcal{U}_{\zeta_I(e_a, \sigma^I_i \cup \tau, x)}$$

Also $\mathcal{U}^{\mathcal{M}}_D\!\upharpoonright_J = \mathcal{C}_a\!\upharpoonright_I \times \mathcal{C}_b\!\upharpoonright_I$. Note that $Y^i_J = X^i_I$ and $\tau^J_i = \sigma^I_i$. Suppose $\langle X_\rho \;:\; \rho \in aI \cup bI\rangle \in \mathcal{U}^{\mathcal{M}}_D\!\upharpoonright_J$. Let $\tau \subseteq Y^i_J = X^i_I$ and $x \in \omega$ be such that there exists $\rho \subseteq \bigcup_{\rho\in aI} X_\rho$ and y such that $\neg\Phi_{e_a}(Z \oplus (\sigma^I_i \cup \tau \cup \rho), x, y)$. Then there exists $\rho \subseteq \bigcup_{\rho\in J} Y_\rho$ and y such that $\neg\Phi_{e_a}(Z \oplus (\tau^J_i \cup \tau \cup \rho), x, y)$. Thus

$$\mathcal{U}^{\mathcal{M}}_D\!\upharpoonright_J \subseteq \bigcap_{\tau\subseteq Y^i_J, x\in\omega} \mathcal{U}_{\zeta_J(e_a, \tau^J_i \cup \tau, x)} \text{ and then } q^J \Vdash^i \forall x\ \exists y\ \neg\Phi_{e_a}(Z \oplus G, x, y)$$

We show the same symmetrically for b. ∎

Before we continue, we give an illustration of a $\mathbb{P}$ filter, seen as a tree of $\mathbb{Q}$-forcing conditions

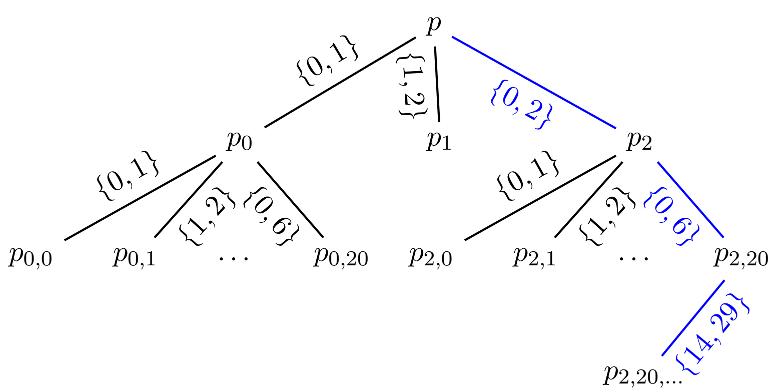

Figure 5.4: Here the blue part correspond to the beginning of a path of $\mathbb{Q}$-forcing conditions. At the first level there are three possibilities : $\{0,1\}, \{1,2\}$ and $\{0,2\}$. At the second level there are 21 possibilities : the number of possible extensions of some $I \lhd I_1$ by some $J \lhd I_2$, which is the number of possibility to pick 2 elements out of 7 : the two elements along each edge correspond to the two new element added in the symmetric set.

We now combine the two previous lemmas to show that given a $\mathbb{P}$-condition, we can find an extension deciding the truth of a Σ^0_2 formula for every $\mathbb{Q}$-condition composing it. We also directly combine it with the notation η for Definition 5.4.11. This will then be used to build a PA-generic $\mathbb{Q}$-filter.

Lemma 5.4.22 : Let $p \in \mathbb{P}_n$. Let $\Phi_e(G, x)$ be a functional. Let $i \in \{0,1\}$. There is a $\mathbb{P}_{n+1}$ or a $\mathbb{P}_n$ condition $q \leqslant p$ such that for every $I \lhd I_{n+1}$ we have

(1) Either $q^I \Vdash^i \exists x\ \Phi_e((Z \oplus G)', x) \downarrow = \Phi_x(\emptyset', x)$

(2) Or $q^I \Vdash^i \exists x\ \Phi_e((Z \oplus G)', x) \uparrow$ ⋆

Proof: Recall $\eta : \omega \times 2^\omega \to \omega$ be such that

$$\exists n \in \operatorname{dom} v\ \Phi_e(G', n) = v(n) \text{ iff } \exists x\ \forall y\ \Phi_{\eta(e,v)}(G, x, y)$$

Let $p_0 = p$. Suppose we have defined conditions $p_k \leqslant p_{k-1} \leqslant \ldots \leqslant p_0$ with $p_t \in \mathbb{P}_n$ for $t \leqslant k$, elements $I^t \lhd I_n$ for $t < k$ and valuations v_t for $t < k$ such that $p_{t+1}^{I^t} \Vdash^i \exists x\ \forall y\ \Phi_{\eta(e,v_t)}(Z \oplus G, x, y)$ for $t \leqslant k$.

Let $S_k = \{I\ :\ I \lhd I_n\} - \{I^t\ :\ t < k\}$. Let W_k be the set of valuations v such that $p_k\ ?\!\vdash^i_{S_k} \exists x\ \forall y\ \Phi_{\eta(e,v)}(Z \oplus G, x, y)$. By Lemma 5.4.19 the set W_k is Z'-c.e.

From Lemma 5.3.2 either W_k contains a valuation v compatible with $J_{\emptyset'}$ with $\operatorname{dom} v \subseteq \operatorname{dom} J_{\emptyset'}$, or there are u_{n+1} many pairwise incompatible valuations outside W_k.

Suppose we are in the first case and let $v_k \in W_k$ witness that. Then from Lemma 5.4.20 there exists $I^k \in S_k$ and $p_{k+1} \leqslant p_k$ with $p_{k+1} \in \mathbb{P}_n$ such that $p_{k+1}^{I^k} \Vdash^i \exists x\ \forall y\ \Phi_{\eta(e,v_k)}(Z \oplus G, x, y)$. Note that by definition this is the same as $p_{k+1}^{I^k} \Vdash^i \exists x\ \Phi_e((Z \oplus G)', x) \downarrow = \Phi_x(\emptyset', x)$. We can continue the induction.

If we always are in the first case until we exhausts all of $\{I\ :\ I \lhd I_n\}$, letting $m = |\{I\ :\ I \lhd I_n\}|$ we then have that $p_m \leqslant p$ is such that $p_m \in \mathbb{P}_n$ and $p_m^I \Vdash^i \exists x\ \Phi_e((Z \oplus G)', x) \downarrow = \Phi_x(\emptyset', x)$ for every $I \lhd I_n$ (using Lemma 5.4.3). Then $q = p_m$ satisfies the lemma.

Suppose now that for some k in the induction we are in the second case, by Lemma 5.3.2 fix u_{n+1} many pairwise incompatible valuations $w_0, \ldots, w_{u_{n+1}-1}$ outside of W_k such that $p_k\ ?\!\nvdash^i_{S_k} \exists x\ \forall y\ \Phi_{\eta(e,w_j)}(G, x, y)$ for every $j < u_{n+1}$.

From Lemma 5.4.21 we have an extension $q \leqslant p_k$ with $q \in \mathbb{P}_{n+1}$ such that for every $I \in S_k$ and every $J \lhd I_{n+1}$ with $J \leqslant I$ we have for a, b the two possible nodes of $J^{\leqslant}$ of length 1 that:

$$q^J \Vdash^i \forall x\ \exists y\ \neg\Phi_{\eta(e,w_a)}(Z \oplus G, x, y) \text{ and } q^J \Vdash^i \forall x\ \exists y\ \neg\Phi_{\eta(e,w_b)}(Z \oplus G, x, y)$$

Note that by definition this implies $q^J \Vdash^i \exists x\ \Phi_e((Z \oplus G)', x) \uparrow$. Let $J \lhd I_{n+1}$. Either $J \leqslant I$ for some $I \in S_k$ in which case $q^J \Vdash^i \exists x\ \Phi_e((Z \oplus G)', x) \uparrow$, or $J \leqslant I$ for some $I \notin S_k$. This is case we have $q \leqslant p_t$ for some t such that $p_t^I \Vdash^i \exists x\ \forall y\ \Phi_{\eta(e,v_{t-1})}(Z \oplus G, x, y)$. As $q^J \leqslant p_t^I$ we then have $q^J \Vdash^i \exists x\ \forall y\ \Phi_{\eta(e,v_{t-1})}(Z \oplus G, x, y)$ and then $q^J \Vdash^i \exists x\ \Phi_e((Z \oplus G)', x) \downarrow = \Phi_x(\emptyset', x)$. It follows that for every $J \lhd I_{n+1}$ we have (1) or (2). ∎

We now show the first step in the creation of a Π^0_2-complete $\mathbb{Q}$-filter.

Lemma 5.4.23 : Let $p \in \mathbb{P}_n$. Let $\Phi_e(Z \oplus G, x, y)$ be a Δ^0_1 functional formula. Let $J \lhd I_n$. Suppose $p^J \Vdash^i \forall x\ \exists y\ \neg\Phi_e(Z \oplus G, x, y)$ for $i \in \{0,1\}$. Suppose also that p^J is valid for side i. Let $x \in \omega$. Then there is an extension $q \leqslant p$ with $q \in \mathbb{P}_n$ such that for $q^J = (\sigma_0, \sigma_1, \langle X_\tau : \tau \in J\rangle, \mathcal{H})$ we have $\exists y\ \neg\Phi_e(Z \oplus \sigma_i, x, y)$ holds. Furthermore for every $I \lhd I_n$ and every

$j \in \{0,1\}$ we have that p^I is valid for side j iff q^I is valid for side j. ⋆

PROOF: Let $p = (\{\langle\sigma_0^I, \sigma_1^I\rangle : I \lhd I_n\}, \langle X_\rho : \rho \in I_n\rangle, C)$. We have

$$p^J = (\sigma_0^J, \sigma_1^J, \langle X_\rho : \rho \in J\rangle, \mathcal{U}_C^{\mathcal{M}}\upharpoonright_J)$$

Let $X_I^i = A^i \cap \bigcup_{\rho\in I} X_\rho$ for $I \lhd I_n$. As $p^J \Vdash^i \forall x\ \exists y\ \neg\Phi_e(Z \oplus G, x, y)$ we have

$$\mathcal{U}_C^{\mathcal{M}}\upharpoonright_J \subseteq \bigcap_{\tau \subseteq X_J^i, x \in \omega} \mathcal{U}_{\zeta_J(e, \sigma_i^J \cup \tau, x)}$$

As p^J is valid for side i we have $\langle A^i \cap X_\rho : \rho \in J\rangle \in \mathcal{U}_C^{\mathcal{M}}\upharpoonright_J$. For $\tau = \epsilon$ and $x \in \omega$. We then have

$$\langle A^i \cap X_\rho : \rho \in J\rangle \in \mathcal{U}_{\zeta_J(e, \sigma_i^J, x)}$$

Then there exists $\tau \subseteq \bigcup_{\rho\in J} A^i \cap X_\rho$ such that $\exists y\ \neg\Phi_e(Z \oplus (\sigma_i^J \cup \tau), x, y)$ holds. Let $\tau_i^J = \sigma_i^J \cup \tau$ and $\tau_{i-1}^J = \sigma_{i-1}^J$. For $I \neq J$ let $\tau_i^I = \sigma_i^I$ and $\tau_{i-1}^I = \sigma_{i-1}^I$. Let $m = \max_{i\in\{0,1\}, I \lhd I_n} |\tau_i^I|$. Let $Y_\rho = X_\rho - \{0, \dots, m\}$ for $\rho \in I_n$. Note that by Proposition 5.3.12 we have $\mathcal{L}_{\langle Y_\rho : \rho\in I_n\rangle} = \mathcal{L}_{\langle X_\rho : \rho\in I_n\rangle}$ and thus $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{\langle Y_\rho : \rho\in I_n\rangle}$.

It follows that $q = (\{\langle\tau_0^I, \tau_1^I\rangle : I \lhd I_n\}, \langle Y_\rho : \rho \in I_n\rangle, C)$ is a $\mathbb{P}_n$ condition such that $\exists y\ \neg\Phi_e(Z \oplus \tau_i^J, x, y)$.

Suppose now that $\langle A^j \cap X_\rho : \rho \in I\rangle \in \mathcal{U}_C^{\mathcal{M}}$ for some $j \in \{0,1\}$ and some $I \lhd I_n$. Then as $\mathcal{U}_C^{\mathcal{M}}\upharpoonright_I \subseteq \mathcal{L}_{\langle X_\rho : \rho\in I\rangle}$ we must have $\bigotimes_I A^j \in \mathcal{U}_C^{\mathcal{M}}\upharpoonright_I$ and then $\bigotimes_I A^j \in \mathcal{L}_{\langle Y_\rho : \rho\in I\rangle}$ and then $\langle A^j \cap Y_\rho : \rho \in I\rangle \in \mathcal{U}_C^{\mathcal{M}}\upharpoonright_I$. ∎

We shall now show that for every n, one branch in the tree of $\mathbb{P}_n$ conditions must be valid for some side $i \in \{0,1\}$.

Lemma 5.4.24 : Let $p \in \mathbb{P}_n$. There is $i \in \{0,1\}$ and $I \lhd I_n$ such that p^I is valid for side i. ⋆

PROOF: Let $p = (\{\langle\sigma_0^I, \sigma_1^I\rangle : I \lhd I_n\}, \langle X_\rho : \rho \in I_n\rangle, C)$. The lemma follows from the combinatoric used to define I_n. We have $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{\langle X_\rho : \rho\in I_n\rangle}$. It follows from Proposition 5.3.15 that there must be some $f : I_n \to \{0,1\}$ such that $\mathcal{U}_C^{\mathcal{M}} \cap \mathcal{L}_{\langle A^{f(\rho)} \cap X_\rho : \rho\in I_n\rangle}$ is a largeness class and thus such that $\langle A^{f(\rho)} \cap X_\rho : \rho \in I_n\rangle \in \mathcal{U}_C^{\mathcal{M}}$. By construction of I_n, there must exists some $I \lhd I_n$ such that $f(\rho) = 0$ for $\rho \in I$ or such that $f(\rho) = 1$ for $\rho \in I$. We then have $\langle A^i \cap X_\rho : \rho \in I\rangle \in \mathcal{U}_C^{\mathcal{M}}\upharpoonright_I$ for some $i \in \{0,1\}$. Thus p^I is valid for side i. ∎

Definition 5.4.25 : Let $\mathcal{F} \subseteq \mathbb{P}$ be a filter. A *path* of $\mathcal{F}$ is a function $f : \mathcal{F} \to \bigcup_n I_n$ such that:

(1) $p \in \mathbb{P}_n$ iff $f(p) \in I_n$

(2) $q \preccurlyeq p$ iff $q^{f(q)} \preccurlyeq p^{f(p)}$

A path is *valid* for side i if furthermore we have

(3) $p^{f(p)}$ is valid for side i ◇

Lemma 5.4.26 : Let $\mathcal{F} \subseteq \mathbb{P}$ be a filter. Then for some $i \in \{0,1\}$ such that this filter has a path valid for side i. ⋆

PROOF: We can assume without loss of generality that $\mathcal{F} = p_0 \geqslant p_1 \geqslant p_2 \geqslant \dots$ with $p_n \in \mathbb{P}_n$. Let us define the finitely branching tree $T = \{(p_n^I, i) : p_n \in \mathcal{F}, I \lhd I_n, i \in \{0,1\}\}$ with the partial order being $(p_n^I, i) \leqslant (q_m^J, j)$ iff $q_m^J \leqslant p_n^I$ and $i = j$. Let $S \subseteq T$ be the set of nodes (p_n^I, i) such that p_n^I is valid for side i. From Lemma 5.4.24 the set S is infinite. From Lemma 5.4.3 it is a subtree of T : if $(q_m^J, j) \in S$ and $(p_n^I, i) \leqslant (q_m^J, j)$ then $(p_n^I, i) \in S$.

From König's lemma it has an infinite path and there is then a function f satisfying (1) (2) and (3) of Definition 5.4.25. ■

Lemma 5.4.27 : There exists a $\mathbb{Q}$-filter which is both PA-generic and Π_2^0-complete on side i for some $i \in \{0,1\}$. ⋆

PROOF: Let $p_0 = (\langle \epsilon, \epsilon \rangle \omega, 2^\omega)$. For $n+1 = \langle e, x, j \rangle$ where $j \in \{0,1\}$ we define p_{n+1} the following way: If $j = 0$ we let using Lemma 5.4.22 $p_{n+1} \leqslant p$ be such that for every $i \in \{0,1\}$ and every $I \lhd I_m$ for m such that $p_{n+1} \in I_m$ we have

(a) Either $p^I \Vdash^i \exists x \; \Phi_e((Z \oplus G)', x) \downarrow = \Phi_x(\emptyset', x)$

(b) Or $p^I \Vdash^i \exists x \; \Phi_e((Z \oplus G)', x) \uparrow$

If $j \neq 0$ and using Lemma 5.4.23 we let p_{n+1} such that for every $I \lhd I_m$ for m such that $p_n \in \mathbb{P}_m$, we have for every $i \in \{0,1\}$ for which p_n^I is valid for side i that $p_n^I \Vdash^i \forall x \; \exists y \; \neg\Phi_e(G, x, y)$ implies $\exists y \; \neg\Phi_e(\sigma_i, x, y)$ where $p_n^I = (\sigma_0, \sigma_1, \langle X_\rho \; : \; \rho \in I \rangle, \mathcal{H})$.

Let $\mathcal{F} = \{p_n\}_{n \in \omega}$. By design every path of $\mathcal{F}$ is a PA-generic $\mathbb{Q}$-filter. By Lemma 5.4.26 let f be a valid path through $\mathcal{F}$. By construction $f(\mathcal{F})$ must be a PA-generic $\mathbb{Q}$-filter which is Π_2^0-complete on side i for some $i \in \{0,1\}$. ■

We can finally show the theorem:

Theorem (5.1.1): *For every set Z whose jump is not of PA degree over $\emptyset'$ and every $\Delta_2^{0,Z}$ set A, there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that $(G \oplus Z)'$ is not of PA degree over $\emptyset'$.*

PROOF: Using Lemma 5.4.27 let $\mathcal{F}$ be a PA-generic $\mathbb{Q}$-filter Π_2^0-complete on side i for $i \in \{0,1\}$. By design we must have $G_{\mathcal{F}}^i \in [A^i]^\omega$. From Lemma 5.4.14 it must be that $(Z \oplus G_{\mathcal{F}}^i)'$ is not PA$(\emptyset')$. ■

Chapter 6

Conclusion and future research

We conclude with some open questions regarding potential future research directions.

6.1 About cone avoidance

Consider the following theorems proved in Chapter 3.

Theorem (2.1.1 for $m = 0$): *Let Z be non-computable. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not G-computable.*

Theorem (2.1.1): *Let $m \geqslant 0$. Let Z be non $\emptyset^{(m)}$-computable. Let A be any set. Then there is a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that Z is not $G^{(m)}$-computable.*

Note the first one was first proved Dzhafarov and Jockusch [13]. A common way to state this theorem is to say "RT^1_2 admits strong cone avoidance". The meaning is the following : RT^1_2 is simply the pigeonhole principle for infinite sets : Given a set A, there exists an infinite subset in A or in its complement. "Cone avoidance" means solutions of instances of RT^1_2 can always avoid non-trivial cones in the Turing degrees : For any non-computable Z, we may find $G \in [A]^\omega \cup [\overline{A}]^\omega$ which is not in the cone of Turing degrees above Z. What interests us now is the difference between "cone avoidance" and "strong cone avoidance" : for the first one, we are only interested in computable instances, or at least instances which are not themselves in the cone of Turing degree we want to avoid. For the second one we are interested in *any* instance. Theorem 2.1.1 for $m = 0$ then says that "RT^1_2 admits strong cone avoidance".

Using the Cholak, Jockusch, and Slaman's [3] decomposition of RT^2_2 into $\mathrm{COH} + \mathrm{SRT}^2_2$, we can use strong cone avoidance of RT^1_2 to show that RT^2_2 admits cone avoidance (but not strong cone avoidance) : this is Seetapun's theorem.

Theorem 6.1.1 (Seetapun [37]):
Let $c : [\omega]^2 \to \{0,1\}$ be a color and let Z be non c-comutable. Then there is a set $G \in [\omega]^\omega$ such that c is monochromatic on $[G]^2$ and such that Z is not G-computable.

Note that using the development of Chapter 3 and Chapter 4 we are now able to show the following generalization, still using the Cholak, Jockusch, and Slaman's [3] decomposition of RT^2_2 into $\mathrm{COH} + \mathrm{SRT}^2_2$:

Theorem 6.1.2:
Let $c : [\omega]^2 \to \{0,1\}$ be a color and let Z be non $c^{(n)}$-computable for $n \in \omega$ (not $\Delta^1_1(c)$ respectively). Then there is a set $H \in [\omega]^\omega$ such that c is monochromatic on $[H]^2$ and such that Z is not $H^{(n)}$-computable (not $\Delta^1_1(H)$ respectively).

Proof: We briefly sketch the proof where Z is not $c^{(n)}$-computable for $n \in \omega$, the case not $\Delta^1_1(c)$ being similar. Let $c : [\omega]^2 \to \{0,1\}$ be a color and let Z be non $c^{(n)}$-computable for $n \in \omega$. Let $R_n = \{x \ : \ c(\{n,x\}) = 0\}$. Using a relativized version of Theorem 2.1.1 we find an infinite set X which is cohesive for the sequence of sets $\{R_n\}_{n\in\omega}$ and such that Z is not $(c \oplus X)^{(n)}$-computable. Note that a slightly different form of Theorem 2.1.1 is used : we do not have an arbitrary set A, but rather countably may sets $\{R_n\}_{n\in\omega}$. However it is clear that any set generic enough for $\mathbb{P}_\omega$ will be cohesive for $\{R_n\}_{n\in\omega}$.

Note that as X is $\{R_n\}_{n\in\omega}$-cohesive, then $\lim_{x\in X} c(n,x)$ exists for every n. Let $f : \omega \to X$ be an X-computable bijection. We basically use f to now work "as if ω were X".

Let $A \in 2^\omega$ be such that $A(n) = \lim_{x\in X} c(f(n),x)$. Using 2.1.1 again we find $G \in [A]^\omega \cup [\omega - A]^\omega$ such that Z is not $(c \oplus X \oplus G)^{(n)}$-computable. Suppose with loss of generality $G \in [A]^\omega$. Using X, G and c we compute $H \subseteq X$ infinite such that c is monochromatic on $[H]^2$ as follow : Let $G = \{n_0, n_1, n_2, \dots\}$. Put $f(n_0)$ in H. Suppose at stage k we have put $n_{a_1}, \dots, n_{a_k}$ in H. Then let a_{k+1} be the smallest such that $c(f(n_{a_i}), n_{a_{k+1}}) = 0$ for every $i \leqslant k$. By construction of A and as $G \subseteq A$ we must find such an element a_{k+1}.

Ultimately H is infinite and by construction c is monochromatic on $[H]^2$. As H is $c \oplus X \oplus G$-computable then $H^{(n)}$ does not compute Z. ■

It is easy to see that RT^2_2 does not admit strong cone avoidance : even SRT^2_2 does not. For any function f one can design an instance of SRT^2_2 every solution of which computes a function dominating f. For appropriate functions f one can then design instances of SRT^2_2 every solution of which computes the halting problem (in fact any Δ^1_1 set fixed in advance). However Wang was able to show that if we are allowed to keep 2 colors (among many) then we have the following:

Theorem 6.1.3 (Wang [42]):
Let $k \in \omega$ and $c : [\omega]^2 \to \{0, \dots, k\}$ be a any color and let Z be non computable. Then there is a set $H \in [\omega]^\omega$ such that $|c([H]^2)| \leqslant 2$ and such that Z is not H-computable.

This new principle, where we are allowed to keep 2 colors among k, is called $\mathrm{RT}^2_{\mathrm{k},2}$. Wang then proved that $\mathrm{RT}^2_{\mathrm{k},2}$ admits strong cone avoidance for any k. One can ask if the same can be done with jump-cone avoidance and above. This leads to the following question:

Question 6.1.1 : Fix $n \in \omega$. Let Z be non $\emptyset^{(n)}$-computable. Does there exists $m \in \omega$ such that for any $k \in \omega$ and $c : [\omega]^2 \to \{0, \dots, k\}$, there is a set $H \in [\omega]^\omega$ such that $|c([H]^2)| \leqslant m$ and such that Z is not $G^{(n)}$-computable ? ◊

Many similar questions can be ask for various type of combinatorial problems. We give here another example.

Theorem 6.1.4 (Dzhafarov, Patey [14]):
Let Z be non-computable. For any coloring $c : [2^{<\omega}]^2 \to \{0,1\}$, there exists a perfect tree $T \subseteq 2^{<\omega}$ such that c is monochromatic on the pairs of comparable nodes of T and such that Z is not T-computable.

Note that in the above theorem, a perfect tree is a non-empty set of strings such that $\sigma \in T$ implies $\sigma\tau_0 \in T$ and $\sigma\tau_1 \in T$ for some τ_0, τ_1 incomparable. In particular it is not required to be closed by prefixes. We can ask the following question:

Question 6.1.2 : Fix $n \in \omega$. Let Z be non $\emptyset^{(n)}$-computable. Does there exists $m \in \omega$ such that for any $k \in \omega$ and $c : [2^{<\omega}]^2 \to \{0,\dots,k\}$, there is a perfect tree T such that at most m colors are used by c on the pairs of comparable nodes of T and such that Z is not $T^{(n)}$-computable ? $\diamond$

6.2 RT^2_2 vs SRT^2_2

Before the separation between RT^2_2 and SRT^2_2, every theorem about the weakness of RT^1_2 worked for *any* set A, that is, theorems where of the form “for every set A, there exists $G \in [A]^\omega \cup [\omega - A]^\omega$ which is *computably weak*”.

However Theorem 5.1.1 only works for Δ^0_2 sets A, which is somewhat disappointing. The various attempts to obtain the same result for any set A have failed so far. We then ask the following question:

Question 6.2.1 : Does there exists a set A such that every $G \in [A]^\omega \cup [\omega - A]^\omega$ computes a p-cohesive set? $\diamond$

We connect this question to another more abstract one : remember the definition of *partition generic* : X is partition generic below $\mathcal{U}$ for a largeness class $\mathcal{U}$ if X belongs to every Π^0_2 partition regular class $\mathcal{C} \subseteq \mathcal{U}$. By considering the largeness classes $\mathcal{L}_X$ for computable infinite sets X, partition generics sets can be considered as the opposite of cohesive sets : they intersect every computable sets infinitely often.

Question 6.2.2 : Let A be any set. Does there exists a Σ^0_1 largeness class $\mathcal{U}$ and a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that G is not PA and partition generic below $\mathcal{U}$. $\diamond$

Note that we could ask the same question with cone avoidance instead of not being PA. The important point is to be able to control the truth of Σ^0_1 statements, while being partition generic somewhere. Following the ideas of Theorem 4.2.1 we can always make sure that our generic belongs to a Π^0_2 partition regular class that we fix in advance. The previous question ask if we can make it belong to as many Π^0_2 partition regular class as possible.

We end with a last question, which can be seen as a variation of the previous one:

Question 6.2.3 : Let C be a Σ^0_3 largeness class. Let A be any set. Does there exists a set $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that G is not PA and belongs to C. $\diamond$

Σ^0_3 partition regular classes which are not Π^0_2 includes for instance sets with positive upper density or sets of strings which are dense somewhere. It may be possible to build a counter example with one of these two classes.

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