Specification and Automatic Verification of Computational Reductions

Julien Grange[1], Fabian Velken[2], Nils Vortmeier[2], Thomas Zeume[2]

[1]LACL, Université Paris-Est Créteil, France
[2]Ruhr Universität Bochum, Germany

Complexity Days
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Teaching reductions

Goal : develop a platform to help students learn complexity theory

- ✓ Understand the classic reductions
- ☠ Design their own reductions, and get feedback

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- ☠ Design their own reductions, and get feedback
  - easy-to-grasp specification language for reductions
  - automatic tools to check the validity of such reductions
  - produce a counter-example if the reduction is incorrect

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- give an algorithmic procedure

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Definition (FO-interpretation between graphs)

$\rho = \left(\varphi_{\text{domain}}(\bar{x}), \varphi_{\sim}(\bar{x}, \bar{y}), \varphi_E(\bar{x}, \bar{y})\right)$ transforms

- a graph $\mathcal{G} = (V, E)$
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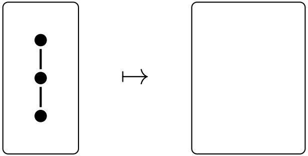
Best of both worlds :

- ✓ declarative
- ✓ left-to-right

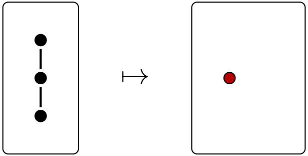
Specification language : *cookbook reductions*

Cookbook reduction



$$k\!-\!\textsc{Clique} \leq (k\!+\!1)\!-\!\textsc{Clique}$$

Cookbook reduction



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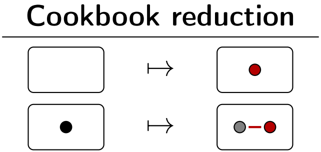

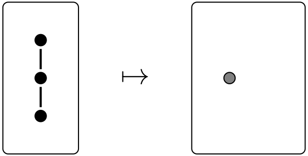

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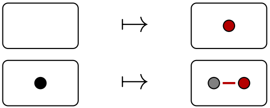

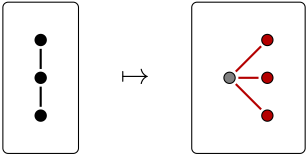

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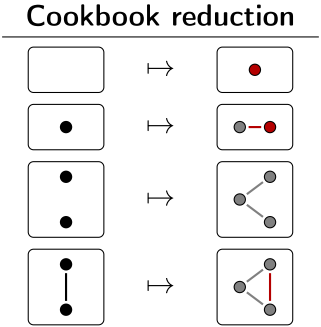

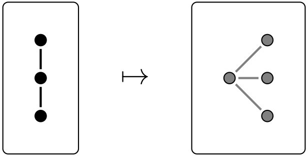

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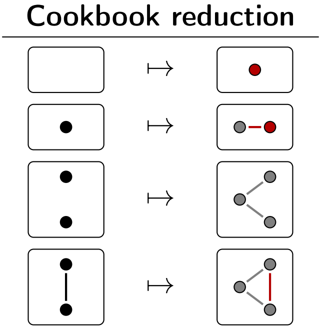

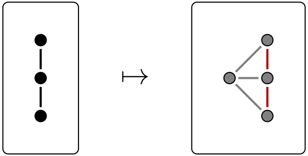

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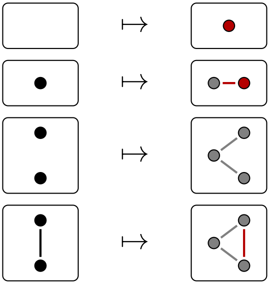

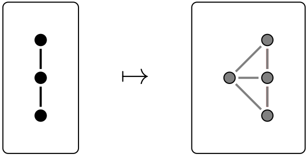

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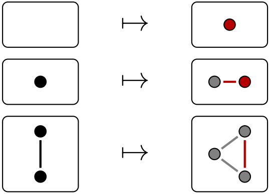

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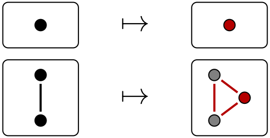
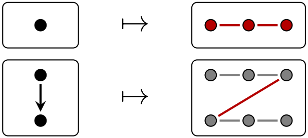
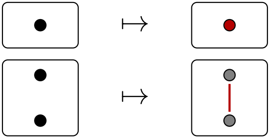


$k-\textsc{VertexCover} \leq k-\mathrm{FVS}$



$\textsc{HamCycle}_d \leq \textsc{HamCycle}_u$



$k-\textsc{Clique} \leq k-\textsc{IndepSet}$

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Theorem

Every cookbook reduction is equivalent to a quantifier-free interpretation

...but not all QF-interpretations are cookbook reductions

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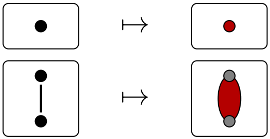

Edge-gadget reductions

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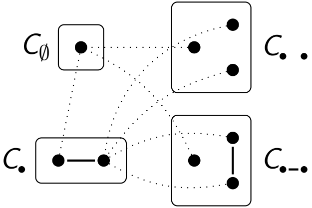

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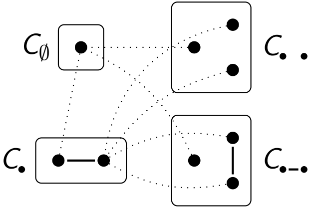

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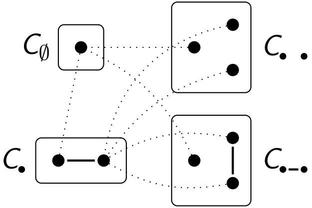

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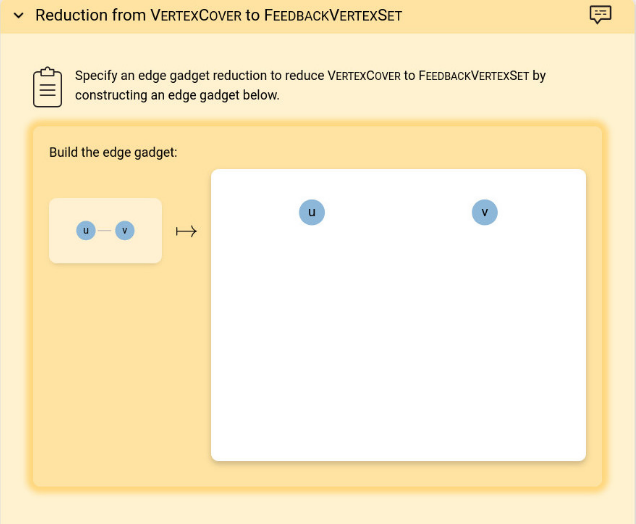
Hence, the correction of ρ only depends on the FO-type of its recipe at some depth.

Prototype on Iltis



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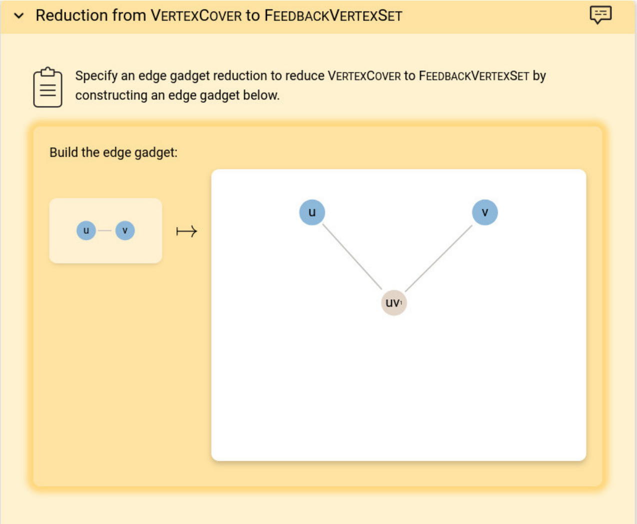

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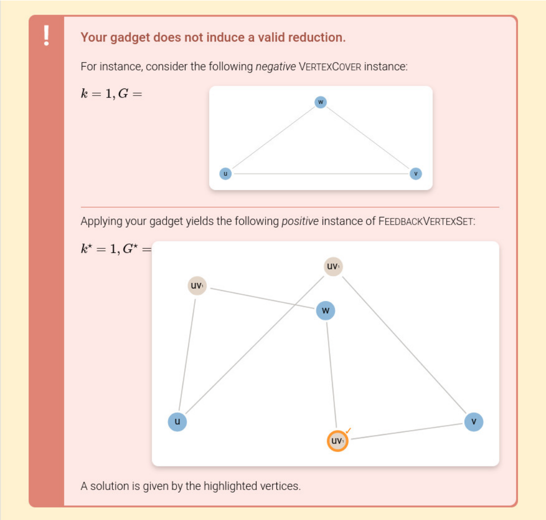

Wrong reduction: feedback via counter-example

Conclusion and perspectives

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~~Theorem~~ Conjecture

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