On Interpretations in Büchi Arithmetics

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Büchi arithmetics

Definition

A ***Büchi arithmetic*** BA_n, $n \geq 2$, *is the theory* $\mathrm{Th}(\mathbb{N}; =, +, V_n)$ *where* V_n *is an unary functional symbol such that* $V_n(x)$ *is the largest power of* n *that divides* x *(we set* $V_n(0) := 0$ *by definition).*

These theories were proposed by R. Büchi in order to describe the recognizability of sets of natural numbers by finite automata through definability in some arithmetic language.

The theories BA_n are complete and decidable.

Cobham-Semënov theorem states that for multiplicatively independent natural numbers n, m (two numbers n, m are called *multiplicatively independent* if the equation $n^k = m^l$ has no integer solutions beside $k = l = 0$), any set definable in BA_n and BA_m is definable in Presburger arithmetic $\mathrm{PrA} = \mathrm{Th}(\mathbb{N}; =, +)$.

Büchi-Bruyère theorem

Let $Digit_n(x, y)$ be the digit corresponding to n^y in the n-ary expansion of $x \in \mathbb{N}$. Consider an automaton over the alphabet $\{0, \ldots, n-1\}^m$ that, at step k, receives the input $(Digit_n(x_1, k), \ldots, Digit_n(x_m, k))$ of the digits corresponding to n^k in the n-ary expansion of $(x_1, \ldots, x_m)$.
We say the automaton accepts the tuple $(x_1, \ldots, x_m)$ if it accepts the sequence of tuples $(Digit_n(x_1, k), \ldots, Digit_n(x_m, k))$.

Proposition (Büchi 1960, Bruyère 1985, Haase, Różycki 2021)

Let $\varphi(x_1, \ldots, x_m)$ be a BA_n-formula. Then there is an effectively constructed automaton $\mathcal{A}$ such that $(a_1, \ldots, a_m)$ is accepted by $\mathcal{A}$ iff $\mathbb{N} \models \varphi(a_1, \ldots, a_m)$. Contrariwise, let $\mathcal{A}$ be a finite automaton working on m-tuples of n-ary natural numbers. Then there is an effectively constructed BA_n-formula (of quantifier complexity not surpassing Σ_2) $\varphi(x_1, \ldots, x_m)$ such that $\mathbb{N} \models \varphi(a_1, \ldots, a_m)$ iff $(a_1, \ldots, a_m)$ is accepted by $\mathcal{A}$.

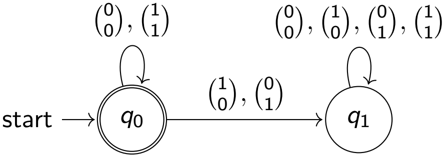

Figure: Automaton for $=_2 (x, y)$

Examples 2

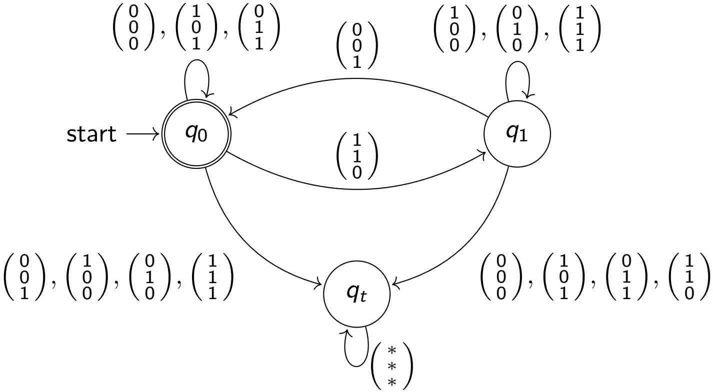

Figure: Automaton for $+_2(x, y, z)$ ($*$ represents any other digit)

Examples 3

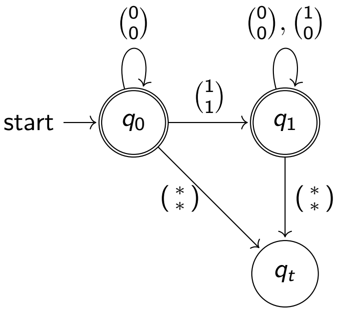

Figure: Automaton for $V_2(x, y)$ ($*$ represents any other digit)

Interpretations

Let $\mathcal{K}$, $\mathcal{L}$ be two first-order languages, $\mathcal{K}$ has no functional symbols [Tarski, Mostowski, Robinson 1953].

Definition

A ***non-parametric*** *m-**dimensional interpretation** ι of $\mathcal{K}$ in an $\mathcal{L}$-structure $\mathfrak{B}$ consists of the following $\mathcal{L}$-formulas:*

1. $D_\iota(\overline{y})$*(the domain formula);*
2. $P_\iota(\overline{x_1}, \dots, \overline{x_n})$*, for each predicate symbol* $P(x_1, \dots, x_n)$ *in* $\mathcal{K}$ *(including equality).*

Here $\overline{x_i}, \overline{y}$ are tuples of length m.

Translation of formulas under interpretation

Definition

*The **translation** $\varphi^\iota(\overline{x_1}, \ldots, \overline{x_n})$ of a $\mathcal{K}$-formula $\varphi(x_1, \ldots, x_n)$ into $\mathcal{L}$ under interpretation ι is now constructed by induction:*

- $(P(x_1, \ldots, x_n))^\iota := P_\iota(\overline{x}_1, \ldots, \overline{x}_n)$;
- $(\varphi \wedge \psi)^\iota = \varphi^\iota \wedge \psi^\iota$, $(\varphi \vee \psi)^\iota = \varphi^\iota \vee \psi^\iota$, $(\varphi \to \psi)^\iota = \varphi^\iota \to \psi^\iota$, $(\neg\varphi)^\iota = \neg(\varphi^\iota)$;
- $(\exists x \psi(x))^\iota := \exists \overline{x}(D(\overline{x}) \wedge \psi^\iota(\overline{x}))$, $(\forall x \psi(x))^\iota := \forall \overline{x}(D(\overline{x}) \to \psi^\iota(\overline{x}))$.

Internal models

As long as we fix some $\mathcal{L}$-structure $\mathfrak{B}$ (such that $\{\overline{y} \mid D_\iota(\overline{y})\} \neq \varnothing$ and the translation of $=^\iota$ is a congruence), a $\mathcal{K}$-structure $\mathfrak{A}$ emerges with the support $\{\overline{y} \in \mathfrak{B}^m \mid D_\iota(\overline{y})\}/{\sim_\iota}$ where $\sim_\iota$ is defined as $=_\iota (\overline{x}_1, \overline{x}_2)$.

Such a structure $\mathfrak{A}$ is called an **internal model**, and ι an **interpretation of $\mathfrak{A}$ in $\mathfrak{B}$**.

We say that an interpretation from is *unrelativized* if the domain formula is trivial; it has *absolute equality* if $=$ is interpreted as the identity of tuples.

Interpretations of theories

Given two theories, T in the language $\mathcal{K}$ and U in the language $\mathcal{L}$, an interpretation ι is called an **interpretation of** T **in** U if each theorem of T translated into a theorem of U.

Equivalently, for each model $\mathfrak{B}$ of U, the corresponding internal model $\mathfrak{A}$ is a model of T.

Definition

Interpretations ι_1 and ι_2 of T *in* U *are called* ***provably isomorphic*** *if there is a formula $F(\overline{x}, \overline{y})$ in the language of* U *expressing the isomorphism f between the corresponding internal models of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and the condition that f is an isomorphism is provable in* U.

Interpretations in elementary theories

Note that two interpretations in the theory $\mathrm{Th}(\mathfrak{B})$ are provably isomorphic iff there is an isomorphism between their corresponding internal models in $\mathfrak{B}$ expressible by an $\mathcal{L}$-formula.

As $\mathrm{BA}_n = \mathrm{Th}(\mathbb{N}; =, +, V_n)$ it is sufficient to consider interpretations in its standard model $\mathbb{N}$ when studying interpretations in BA_n itself.

Reflexive and sequential theories

A sufficiently strong first-order theory is called *reflexive* if it can prove the consistency of all its finitely axiomatizable subtheories. Well-known examples of reflexive theories include Peano arithmetic PA and Zermelo-Fraenkel set theory ZF.

Definition

Adjunctive set theory AS *[Visser 2012] is the theory in the language* $\{=, \in\}$ *containing the following two axioms:*

1. $\exists x \forall y \, (y \notin x)$ *(existence of the empty set);*
2. $\forall x \forall y \, \exists z \, \forall u \, (u \in z \leftrightarrow (u \in x \vee u = y))$ *(each set can be extended by any single object).*

A theory T is called *sequential* if if there is a one-dimensional, unrelativized interpretation with absolute equality of AS into T. Such theories are able to encode finite tuples of objects with a single object.

Visser's interpretation properties

All sequential theories that prove all instances of the induction scheme in their language are reflexive.

Each theory T that is both sequential and reflexive has the following property: T *cannot be interpreted in any of its finite subtheories*.

A. Visser has proposed to consider this interpretational-theoretic property as a generalization of reflexivity for weaker theories unable to formalize syntax.

Statement of the problem

In this context, Visser asked the question: *for which arithmetical theories T all their interpretations in themselves are provably isomorphic to the trivial one?* We note that, for theories without finite axiomatization, this also implies the absence of interpretations of T in any of its finitely axiomatizable subtheories. An example of a weak arithmetical theory for which this property does not hold is the theory $\mathrm{Th}(\mathbb{Z}; =, S(x))$ of integer numbers with successor $(y = S(x) \Leftrightarrow y = x + 1)$.

What had been done

The author had previously established:

Theorem (Pakhomov, Zapryagaev 2020)

1. *Let* ι *be a (one-dimensional or multi-dimensional) interpretation of* PrA *in* $(\mathbb{N}; =, +)$*. The the internal model induced by* ι *is always isomorphic to the standard one.*
2. *This isomorphism can always be expressed by a formula in the language of* PrA*.*

The result of point (1) was established by studying the linear orders interpretable in PrA, obtaining a necessary condition based on the notion of VD^*-**rank** [Khoussainov, Rubin, Stephan 2005].

Scattered linear orders and rank

Definition

Let $(L, <)$ be a linear order. By transfinite recursion, we introduce a family of equivalence relations $\simeq_\alpha$, $\alpha \in \mathrm{Ord}$ on L:

1. $\simeq_0$ *is equality;*
2. $a \simeq_{\alpha+1} b$, *if* $|\{c \in L \mid (a < c < b) \text{ or } (b < c < a)\}/\simeq_\alpha|$ *is finite;*
3. $\simeq_\lambda = \bigcup_{\beta<\lambda} \simeq_\alpha$ *when* λ *is a limit ordinal.*

A ***rank*** $\mathrm{rk}(L, <) \in \mathrm{Ord} \cup \{\infty\}$ *of the order* $(L, <)$ *is the smallest* α *such that* $L/\simeq_\alpha$ *is finite or* ∞ *if such does not exist.*

It is known [Rosenstein 1982] that the *scattered* linear orders, that is, not containing a suborder isomorphic to $\mathbb{Q}$, exactly coincide with the orders of rank below ∞.

Rank condition on the definable orders

The following condition has been established:

Theorem

All linear orders m-dimensionally interpretable in $(\mathbb{N}; =, +)$ *have rank* $\leq m$.

As $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$ is not even scattered, a non-standard model PrA cannot be interpreted in $(\mathbb{N}; =, +)$.
In fact, the following complete criterion was very recently reached:

Theorem (Pakhomov, Zapryagaev submitted)

A linear order $(L, <)$ *is m-dimensionally interpretable in* $(\mathbb{N}; =, +)$ *for some* $m \geq 1$ *iff there exists some* $k \in N$ *and a* PrA*-definable set* $D \in \mathbb{Z}^k$ *such that* L *is isomorphic to the restriction of the lexicographic ordering on* $\mathbb{Z}^k$ *onto* D.

Orders definable in BA_n

Yet, the same rank condition is not extended to BA_n. The statement holds:

Lemma

For each n, there is an order of rank n interpretable in BA_2.

Examples follow.

$n = 1$: $x \leq_1 y := x \leq y$

$n = 2$: $x \leq_2 y := V_2(x) < V_2(y) \vee V_2(x) = V_2(y) \wedge (x \leq y)$

$n = 3$: $x \leq_3 y := V_2(x) < V_2(y) \vee V_2(x) = V_2(y) \wedge V_2(x - V_2(x)) <$
$V_2(y - V_2(y)) \vee V_2(x) = V_2(y) \wedge V_2(x - V_2(x)) = V_2(y - V_2(y)) \wedge x \leq y$

What is done

The following result is achieved:

Theorem (Zapryagaev 2023)

Let ι be a (one-dimensional or multi-dimensional) interpretation of BA_n in $(\mathbb{N}; =, +, V_n)$. The the internal model induced by ι is always isomorphic to the standard one.

This gives a partial positive answer to Visser's question.

Bi-interpretability

First we find that the answer to the question does not depend on which particular theory BA_n is considered.

The following claim holds:

Theorem

For any $k, l \geq 2$, BA_k is interpretable in BA_l.

This can be shown by a combination of two claims:

Lemma

Each BA_{k^2} can be interpreted in BA_k.

Lemma

Each BA_k can be interpreted in BA_{k+1}, $k \geq 2$.

Automatic structures

Definition

A structure $\mathfrak{B}$ in the language containing equality and predicate symbols $P_1, \ldots, P_n$ is called automatic [Khoussainov, Nerode 2005] if there a language $\mathcal{L} \subseteq \Omega^*$ over a finite alphabet Ω and a surjective mapping $c \colon \mathcal{L} \to \mathfrak{B}$ such that the following sets are recognizable by some automaton over Ω ($\overline{x}_i \in \Omega^*$):

1. *The language $\mathcal{L}$;*
2. *The set of all pairs $(\overline{x}, \overline{y}) \in \mathcal{L}^2$ such that $c(\overline{x}) = c(\overline{y})$;*
3. *The set of all tuples $(\overline{x}_1, \ldots, \overline{x}_n) \in \mathcal{L}^n$ such that $\mathfrak{B} \models P_i(c(\overline{x}_1), \ldots, c(\overline{x}_n))$.*

As follows from the Büchi-Bruyère theorem, interpretability in the standard model of BA_n is an alternate description of automatic structures.

Non-standard models of BA_n

It is required to find whether for each interpretation ι of BA_n in $(\mathbb{N}; =, +, V_n)$ the internal model is isomorphic to the standard one. Hence, it is necessary to check whether some non-standard model of BA_n is interpretable in Büchi arithmetic. The order-types of the non-standard models of BA_n are described by the following classic result.

Proposition (folklore, analogous to Kemeny 1958)

Each non-standard model $\mathfrak{A}$ *of* BA_n *has the order type* $\mathbb{N} + \mathbb{Z} \cdot A$ *where* $\langle A, <_A \rangle$ *is a dense linear order without endpoints.*

In particular, each countable non-standard model of BA_n has the order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$.

Interpretations in BA_n

Let ι be an interpretation of BA_n or PrA with a non-standard internal model. As $\mathbb{N}$ is countable, its order type must be $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$. By defining the negative numbers, it is now possible to construct an interpretation ι' of an ordered abelian group $\mathcal{B}$, with the order type $\mathbb{Z} \cdot \mathbb{Q}$.
Consider the *galaxies*

$$[c] := \{d \in \mathcal{B} \mid |c - d| \text{ is a standard natural number}\}.$$

The standard integers form one of the galaxies, namely, the one containing zero. The addition $[c + d] := [c] + [d]$ is well defined. Furthermore:

Lemma

Let $\mathcal{Z}$ be the subgroup of the standard integers in $\mathcal{B}$. Then $\mathcal{B}/\mathcal{Z}$ contains a subgroup $\mathcal{Q}$ isomorphic to $(\mathbb{Q}, +)$.

Automatic abelian groups

One the other hand, as we have shown, each group interpretable in BA_n is automatic. The following condition is known to hold for automatic abelian groups.

Theorem (Braun, Strüngmann 2011)

Let $(A, +)$ be an automatic torsion-free abelian group. Then there exists a subgroup $B \subseteq A$ isomorphic to $\mathbb{Z}^m$ for some m such that the orders of the elements in $C = A/B$ are only divisible by a finite number of different primes $p_1, \ldots, p_s$.

It is shown this contradicts the existence of a subgroup Q isomorphic to $(\mathbb{Q}, +)$ in $\mathcal{B}/\mathbb{Z}$.

Plans for further research

- Establish whether each isomorphism between the internal model of BA_n and $(\mathbb{N}; =, +, V_n)$ is expressible by a BA_n-formula, obtaining the complete answer to Visser's question.
- Find an explicit axiomatization of BA_n for each n.
- Further elucidate the structure of non-standard models of BA_n.

Thank you!

Publications

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