Satisfaction classes with the full collection scheme

Bartosz Wcisło

Institute of Philosophy, University of Gdańsk

42ème Journées sur les Arithmétiques Faibles, Karlovassi
September 25, 2023

This talk concerns the general question which axioms can be added to the theory of compositional truth (or satisfaction) over PA to yield a nonconservative extension.

This talk concerns the general question which axioms can be added to the theory of compositional truth (or satisfaction) over PA to yield a nonconservative extension.

Our general framework: We add to PA a fresh unary predicate $T(x)$ together with axioms postulating that T behaves like a compositional truth predicate.

Uniwersytet Gdański

This talk concerns the general question which axioms can be added to the theory of compositional truth (or satisfaction) over PA to yield a nonconservative extension.

Our general framework: We add to PA a fresh unary predicate $T(x)$ together with axioms postulating that T behaves like a compositional truth predicate. We (usually) call the resulting theory CT^-.

This talk concerns the general question which axioms can be added to the theory of compositional truth (or satisfaction) over PA to yield a nonconservative extension.
Our general framework: We add to PA a fresh unary predicate $T(x)$ together with axioms postulating that T behaves like a compositional truth predicate. We (usually) call the resulting theory CT^-.
The definition comes on the next slide.

Definition

By CT^- (compositional truth), we mean a theory obtained by adding to PA the following axioms:

Definition

By CT^- (compositional truth), we mean a theory obtained by adding to PA the following axioms:

1. $\forall s, t \in \mathrm{ClTerm}_{\mathrm{PA}} \quad T(s = t) \equiv (\mathrm{val}(s) = \mathrm{val}(t))$.

Definition

By CT^- (compositional truth), we mean a theory obtained by adding to PA the following axioms:

1. $\forall s, t \in \mathrm{ClTerm}_{\mathrm{PA}} \quad T(s = t) \equiv (\mathrm{val}(s) = \mathrm{val}(t))$.
2. $\forall \phi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\neg\phi \equiv \neg T\phi$.

Definition

By CT^- (compositional truth), we mean a theory obtained by adding to PA the following axioms:

1. $\forall s, t \in \mathrm{ClTerm}_{\mathrm{PA}} \quad T(s = t) \equiv (\mathrm{val}(s) = \mathrm{val}(t))$.
2. $\forall \phi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\neg\phi \equiv \neg T\phi$.
3. $\forall \phi, \psi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\phi \vee \psi \equiv T\phi \vee T\psi$.

Uniwersytet Gdański

Definition

By CT^- (compositional truth), we mean a theory obtained by adding to PA the following axioms:

1. $\forall s, t \in \mathrm{ClTerm}_{\mathrm{PA}} \quad T(s = t) \equiv (\mathrm{val}(s) = \mathrm{val}(t))$.
2. $\forall \phi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\neg\phi \equiv \neg T\phi$.
3. $\forall \phi, \psi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\phi \vee \psi \equiv T\phi \vee T\psi$.
4. $\forall v \in \mathrm{Var} \forall \phi \in \mathrm{Form}^{\leq 1}_{\mathrm{PA}} \quad T\exists v\phi \equiv \exists x T\phi[\underline{x}/v]$.

Definition

By CT^- (compositional truth), we mean a theory obtained by adding to PA the following axioms:

1. $\forall s, t \in \mathrm{ClTerm}_{\mathrm{PA}} \quad T(s = t) \equiv (\mathrm{val}(s) = \mathrm{val}(t))$.
2. $\forall \phi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\neg\phi \equiv \neg T\phi$.
3. $\forall \phi, \psi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\phi \vee \psi \equiv T\phi \vee T\psi$.
4. $\forall v \in \mathrm{Var} \forall \phi \in \mathrm{Form}^{\leq 1}_{\mathrm{PA}} \quad T\exists v\phi \equiv \exists x T\phi[\underline{x}/v]$.
5. $\forall \bar{s}, \bar{t} \in \mathrm{ClTermSeq}_{\mathrm{PA}} \forall \phi \in \mathrm{Form}_{\mathrm{PA}} \ \mathrm{val}(\bar{s}) = \mathrm{val}(\bar{t}) \rightarrow T\phi(\bar{t}) \equiv T\phi(\bar{s})$.

Definition

By CT^- (compositional truth), we mean a theory obtained by adding to PA the following axioms:

1. $\forall s, t \in \mathrm{ClTerm}_{\mathrm{PA}} \quad T(s = t) \equiv (\mathrm{val}(s) = \mathrm{val}(t))$.
2. $\forall \phi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\neg\phi \equiv \neg T\phi$.
3. $\forall \phi, \psi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\phi \vee \psi \equiv T\phi \vee T\psi$.
4. $\forall v \in \mathrm{Var}\forall \phi \in \mathrm{Form}^{\leq 1}_{\mathrm{PA}} \quad T\exists v\phi \equiv \exists x T\phi[\underline{x}/v]$.
5. $\forall \bar{s}, \bar{t} \in \mathrm{ClTermSeq}_{\mathrm{PA}}\forall \phi \in \mathrm{Form}_{\mathrm{PA}} \ \mathrm{val}(\bar{s}) = \mathrm{val}(\bar{t}) \rightarrow T\phi(\bar{t}) \equiv T\phi(\bar{s})$.

Definition

By CT^- (compositional truth), we mean a theory obtained by adding to PA the following axioms:

1. $\forall s, t \in \mathrm{ClTerm}_{\mathrm{PA}} \quad T(s = t) \equiv (\mathrm{val}(s) = \mathrm{val}(t))$.
2. $\forall \phi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\neg\phi \equiv \neg T\phi$.
3. $\forall \phi, \psi \in \mathrm{Sent}_{\mathrm{PA}} \quad T\phi \vee \psi \equiv T\phi \vee T\psi$.
4. $\forall v \in \mathrm{Var} \forall \phi \in \mathrm{Form}^{\leq 1}_{\mathrm{PA}} \quad T\exists v\phi \equiv \exists x T\phi[\underline{x}/v]$.
5. $\forall \bar{s}, \bar{t} \in \mathrm{ClTermSeq}_{\mathrm{PA}} \forall \phi \in \mathrm{Form}_{\mathrm{PA}} \ \mathrm{val}(\bar{s}) = \mathrm{val}(\bar{t}) \rightarrow T\phi(\bar{t}) \equiv T\phi(\bar{s})$.

By CT we mean CT^- with full induction (in the extended language).

Uniwersytet Gdański

It turns out that the compositional axioms by themselves are insufficient to prove any new arithmetical theorems.

It turns out that the compositional axioms by themselves are insufficient to prove any new arithmetical theorems.

Theorem (Kotlarski–Krajewski–Lachlan)

CT^- *is conservative over* PA.

Uniwersytet Gdański

It turns out that the compositional axioms by themselves are insufficient to prove any new arithmetical theorems.

Theorem (Kotlarski–Krajewski–Lachlan)

CT^- *is conservative over* PA.

It turns out that the compositional axioms by themselves are insufficient to prove any new arithmetical theorems.

Theorem (Kotlarski–Krajewski–Lachlan)

CT^- *is conservative over* PA.

On the other hand, we have the following easy observation:

Proposition

CT *is not conservative over* PA.

It turns out that the compositional axioms by themselves are insufficient to prove any new arithmetical theorems.

Theorem (Kotlarski–Krajewski–Lachlan)

CT^- *is conservative over* PA.

On the other hand, we have the following easy observation:

Proposition

CT *is not conservative over* PA.

It turns out that the compositional axioms by themselves are insufficient to prove any new arithmetical theorems.

Theorem (Kotlarski–Krajewski–Lachlan)

CT^{-} *is conservative over* PA.

On the other hand, we have the following easy observation:

Proposition

CT *is not conservative over* PA.

We prove the above proposition by showing by induction on the number of steps in proofs that any formulae provable in PA is true under any assignment, thus showing that the uniform reflection holds in CT.

It turns out that the compositional axioms by themselves are insufficient to prove any new arithmetical theorems.

Theorem (Kotlarski–Krajewski–Lachlan)

CT^- *is conservative over* PA*.*

On the other hand, we have the following easy observation:

Proposition

CT *is not conservative over* PA*.*

We prove the above proposition by showing by induction on the number of steps in proofs that any formulae provable in PA is true under any assignment, thus showing that the uniform reflection holds in CT. The above argument overtly uses Π_1-induction, but we can do better.

Fact

Let CT_0 *be* CT^- *with induction for* Δ_0*-formulae containing the truth predicate. Then* CT_0 *is not conservative over* PA.

Fact

Let CT_0 *be* CT^- *with induction for* Δ_0*-formulae containing the truth predicate. Then* CT_0 *is not conservative over* PA*.*

Uniwersytet Gdański

Fact

Let CT_0 *be* CT^- *with induction for* Δ_0*-formulae containing the truth predicate. Then* CT_0 *is not conservative over* PA*. In fact, there is a number of natural truth-theoretic principles which are all equivalent to* CT_0*, for instance (Enayat – Pakhomov) "a disjunction over a finite set of arithmetical sentences is true iff one of the disjuncts is."*

How about pure collection?

Fact

Let CT_0 *be* CT^- *with induction for* Δ_0*-formulae containing the truth predicate. Then* CT_0 *is not conservative over* PA*. In fact, there is a number of natural truth-theoretic principles which are all equivalent to* CT_0*, for instance (Enayat – Pakhomov) "a disjunction over a finite set of arithmetical sentences is true iff one of the disjuncts is."*

How about pure collection? It is a classical result that the full induction scheme is equivalent to Δ_0-induction together with the instances of the following **collection scheme**:

$$\forall x < a \exists y \ \phi(x, y) \ \longrightarrow \ \exists b \forall x < a \exists y < b \ \phi(x, y).$$

Problem (Kaye)

Is CT^- with the full collection scheme (for the extended language) a conservative extension of PA?

Theorem

CT^- *with the full collection scheme is a conservative extension of* PA.

Theorem

CT^- *with the full collection scheme is a conservative extension of* PA.

Theorem

CT^- *with the full collection scheme is a conservative extension of* PA.

The proof strategy was suggested already by Kaye.

Theorem

CT^- *with the full collection scheme is a conservative extension of* PA.

The proof strategy was suggested already by Kaye. Recall that $M \models \mathrm{PA}$ is ω_1-**like** if $|M| = \aleph_1$, but for any $a \in M$, the initial segment $[0, a]$ is countable.

Theorem

CT^- *with the full collection scheme is a conservative extension of* PA.

The proof strategy was suggested already by Kaye. Recall that $M \models \mathrm{PA}$ is ω_1-**like** if $|M| = \aleph_1$, but for any $a \in M$, the initial segment $[0, a]$ is countable.

Remark

Let $M \models \mathrm{PA}$ be an ω_1-like model.

Theorem

CT^- *with the full collection scheme is a conservative extension of* PA.

The proof strategy was suggested already by Kaye. Recall that $M \models \mathrm{PA}$ is ω_1-**like** if $|M| = \aleph_1$, but for any $a \in M$, the initial segment $[0, a]$ is countable.

Remark

Let $M \models \mathrm{PA}$ be an ω_1-like model.

Theorem

CT^- *with the full collection scheme is a conservative extension of* PA.

The proof strategy was suggested already by Kaye. Recall that $M \models \mathrm{PA}$ is ω_1-**like** if $|M| = \aleph_1$, but for any $a \in M$, the initial segment $[0, a]$ is countable.

Remark

Let $M \models \mathrm{PA}$ be an ω_1-like model. Then for any $P \subseteq M$, the expansion (M, P) satisfies collection.

Theorem

CT^- *with the full collection scheme is a conservative extension of* PA.

The proof strategy was suggested already by Kaye. Recall that $M \models \mathrm{PA}$ is ω_1-**like** if $|M| = \aleph_1$, but for any $a \in M$, the initial segment $[0, a]$ is countable.

Remark

Let $M \models \mathrm{PA}$ be an ω_1-like model. Then for any $P \subseteq M$, the expansion (M, P) satisfies collection.

In order to prove the theorem, it is enough to prove the following result:

Theorem

Let $M \models \mathrm{PA}$ *be an arbitrary countable model. Then there exists an* ω_1*-like elementary extension* $M' \succ M$ *and* $T \subseteq M'$ *such that* $(M', T) \models \mathrm{CT}^-$ *and thus automatically* $(M', T) \models \mathrm{CT}^- + \mathrm{Coll}(L_{\mathrm{PAT}})$.

There is one obvious prove strategy which *does not* work. It would be enough to show that for any countable $(M, T) \models \mathrm{CT}^-$, we can find a proper end-extension $(M', T') \supset_e (M, T)$ to a model of CT^-.

Uniwersytet Gdański

There is one obvious prove strategy which *does not* work. It would be enough to show that for any countable $(M, T) \models \mathrm{CT}^-$, we can find a proper end-extension $(M', T') \supset_e (M, T)$ to a model of CT^-.

- Notice that such an extension is automatically elementary in the arithmetical part. If $M \models \phi(a)$, then $(M, T) \models T(\phi(\underline{a}))$, so $(M', T') \models T'(\phi(\underline{a}))$ and since it also a model of CT^-, $M' \models \phi(a)$.

There is one obvious prove strategy which *does not* work. It would be enough to show that for any countable $(M, T) \models \mathrm{CT}^-$, we can find a proper end-extension $(M', T') \supset_e (M, T)$ to a model of CT^-.

- Notice that such an extension is automatically elementary in the arithmetical part. If $M \models \phi(a)$, then $(M, T) \models T(\phi(\underline{a}))$, so $(M', T') \models T'(\phi(\underline{a}))$ and since it also a model of CT^-, $M' \models \phi(a)$.
- Notice that the axioms of CT^- are preserved in the unions of models. They are Π_2 modulo the arithmetical part and every arithmetical property is equivalent to an atomic property in a model with the truth predicate.

There is one obvious prove strategy which *does not* work. It would be enough to show that for any countable $(M, T) \models \mathrm{CT}^-$, we can find a proper end-extension $(M', T') \supset_e (M, T)$ to a model of CT^-.

- Notice that such an extension is automatically elementary in the arithmetical part. If $M \models \phi(a)$, then $(M, T) \models T(\phi(\underline{a}))$, so $(M', T') \models T'(\phi(\underline{a}))$ and since it also a model of CT^-, $M' \models \phi(a)$.
- Notice that the axioms of CT^- are preserved in the unions of models. They are Π_2 modulo the arithmetical part and every arithmetical property is equivalent to an atomic property in a model with the truth predicate.

There is one obvious prove strategy which *does not* work. It would be enough to show that for any countable $(M, T) \models \mathrm{CT}^-$, we can find a proper end-extension $(M', T') \supset_e (M, T)$ to a model of CT^-.

- Notice that such an extension is automatically elementary in the arithmetical part. If $M \models \phi(a)$, then $(M, T) \models T(\phi(\underline{a}))$, so $(M', T') \models T'(\phi(\underline{a}))$ and since it also a model of CT^-, $M' \models \phi(a)$.
- Notice that the axioms of CT^- are preserved in the unions of models. They are Π_2 modulo the arithmetical part and every arithmetical property is equivalent to an atomic property in a model with the truth predicate.

So this really *almost* works.

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ such that there is no proper end-extension $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

- $T(\forall x < a \exists ! y \phi(x, y))$

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

- $T(\forall x < a \exists! y \phi(x, y))$
- $T(\forall y \exists x < a \phi(x, y))$.

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

- $T(\forall x < a \exists ! y \phi(x, y))$
- $T(\forall y \exists x < a \phi(x, y))$.

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}_{\mathrm{PA}}^M$ and for some $a \in M$, we have:

- $T(\forall x < a \exists! y \phi(x, y))$
- $T(\forall y \exists x < a \phi(x, y))$.

Notice that those properties have to be preserved in an end-extension (M', T').

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

- $T(\forall x < a \exists! y \phi(x, y))$
- $T(\forall y \exists x < a \phi(x, y))$.

Notice that those properties have to be preserved in an end-extension (M', T'). Now take any $c \in M' \setminus M$.

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

- $T(\forall x < a \exists ! y \phi(x, y))$
- $T(\forall y \exists x < a \phi(x, y))$.

Notice that those properties have to be preserved in an end-extension (M', T'). Now take any $c \in M' \setminus M$. By assumption,

$$(M', T') \models \exists x < a T' \phi(x, c).$$

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

- $T(\forall x < a \exists ! y \phi(x, y))$
- $T(\forall y \exists x < a \phi(x, y))$.

Notice that those properties have to be preserved in an end-extension (M', T'). Now take any $c \in M' \setminus M$. By assumption,

$$(M', T') \models \exists x < a T' \phi(x, c).$$

Fix $a' < a$ such that $T' \phi(a', c)$. Since $M' \supset_e M$, $a' \in M$.

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

- $T(\forall x < a \exists! y \phi(x, y))$
- $T(\forall y \exists x < a \phi(x, y))$.

Notice that those properties have to be preserved in an end-extension (M', T'). Now take any $c \in M' \setminus M$. By assumption,

$$(M', T') \models \exists x < a T' \phi(x, c).$$

Fix $a' < a$ such that $T'\phi(a', c)$. Since $M' \supset_e M$, $a' \in M$. By the first clause, we know that there exists $d \in M$ such that $(M, T) \models T\phi(a', d)$.

Theorem (Smith)

There exists a countable model $(M, T) \models \mathrm{CT}^-$ *such that there is no proper end-extension* $(M, T) \subset_e (M', T')$.

Proof.

Let $(M, T) \models \mathrm{CT}^-$ be a model such that for some nonstandard formula $\phi(x, y) \in \mathrm{Form}^M_{\mathrm{PA}}$ and for some $a \in M$, we have:

- $T(\forall x < a \exists ! y \phi(x, y))$
- $T(\forall y \exists x < a \phi(x, y))$.

Notice that those properties have to be preserved in an end-extension (M', T'). Now take any $c \in M' \setminus M$. By assumption,

$$(M', T') \models \exists x < a T' \phi(x, c).$$

Fix $a' < a$ such that $T' \phi(a', c)$. Since $M' \supset_e M$, $a' \in M$. By the first clause, we know that there exists $d \in M$ such that $(M, T) \models T\phi(a', d)$. This contradicts the uniqueness of y such that $T'\phi(a', y)$. □

Now, it turns out that the counterexample found by Smith is essentially the only obstruction there is to the existence of the end-extensions.

Now, it turns out that the counterexample found by Smith is essentially the only obstruction there is to the existence of the end-extensions.

Definition

By the **internal induction** principle, INT, we mean the following sentence:

Now, it turns out that the counterexample found by Smith is essentially the only obstruction there is to the existence of the end-extensions.

Definition

By the **internal induction** principle, INT, we mean the following sentence:

Now, it turns out that the counterexample found by Smith is essentially the only obstruction there is to the existence of the end-extensions.

Definition

By the **internal induction** principle, INT, we mean the following sentence:

$$\forall \phi \in \mathrm{Form}_{\mathrm{PA}}\Big[T\phi(0) \wedge \forall x\Big(T\phi(\underline{x}) \rightarrow T\phi(\underline{S(x)})\Big) \rightarrow \forall x T\phi(\underline{x})\Big].$$

Now, it turns out that the counterexample found by Smith is essentially the only obstruction there is to the existence of the end-extensions.

Definition

By the **internal induction** principle, INT, we mean the following sentence:

$$\forall \phi \in \mathrm{Form}_{\mathrm{PA}}\Big[T\phi(0) \wedge \forall x\Big(T\phi(\underline{x}) \rightarrow T\phi(\underline{S(x)})\Big) \rightarrow \forall x T\phi(\underline{x})\Big].$$

Theorem

Let $(M, T) \models \mathrm{CT}^- + \mathrm{INT}$ *be a countable model. Then there exists a proper end-extension* $(M, T) \subset (M', T') \models \mathrm{CT}^- + \mathrm{INT}$.

In the rest of the proof, it will be more convenient to work with **satisfaction classes**.

In the rest of the proof, it will be more convenient to work with **satisfaction classes**. Instead of a unary predicate $T(x)$, we will have a binary relation $S(\phi, \alpha)$, where ϕ is an arithmetical formula in the sense of a model, and α is a ϕ-assignment,

In the rest of the proof, it will be more convenient to work with **satisfaction classes**. Instead of a unary predicate $T(x)$, we will have a binary relation $S(\phi, \alpha)$, where ϕ is an arithmetical formula in the sense of a model, and α is a ϕ-assignment, i.e., a function which assigns some elements of a model to the free variables of ϕ.

In the rest of the proof, it will be more convenient to work with **satisfaction classes**. Instead of a unary predicate $T(x)$, we will have a binary relation $S(\phi, \alpha)$, where ϕ is an arithmetical formula in the sense of a model, and α is a ϕ-assignment, i.e., a function which assigns some elements of a model to the free variables of ϕ.
We will assume that our satisfaction classes satisfy Tarski's compositional conditions on sets of formulae closed under direct subformulae.

In the rest of the proof, it will be more convenient to work with **satisfaction classes**. Instead of a unary predicate $T(x)$, we will have a binary relation $S(\phi, \alpha)$, where ϕ is an arithmetical formula in the sense of a model, and α is a ϕ-assignment, i.e., a function which assigns some elements of a model to the free variables of ϕ.

We will assume that our satisfaction classes satisfy Tarski's compositional conditions on sets of formulae closed under direct subformulae. We will also assume that they satisfy some strong regularity properties.

In the rest of the proof, it will be more convenient to work with **satisfaction classes**. Instead of a unary predicate $T(x)$, we will have a binary relation $S(\phi, \alpha)$, where ϕ is an arithmetical formula in the sense of a model, and α is a ϕ-assignment, i.e., a function which assigns some elements of a model to the free variables of ϕ.

We will assume that our satisfaction classes satisfy Tarski's compositional conditions on sets of formulae closed under direct subformulae. We will also assume that they satisfy some strong regularity properties. This is required both for the proof and to assure that they really correspond to models of CT^-.

We will denote the axioms for full satisfaction classes CS^-. If I is a cut, then $\mathrm{CS}^- \restriction I$ are axioms stating that S satisfies compositional clauses for all formulae ϕ with $\mathrm{dpt}(\phi) \in I$ (but with *arbitrary* assignments).

The proof of the theorem comes in two steps.

The proof of the theorem comes in two steps.

Lemma (Slicing)

Suppose that $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$. *Then there exists a model* $(M', S') \supseteq (M, S)$ *such that:*

The proof of the theorem comes in two steps.

Lemma (Slicing)

Suppose that $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$. *Then there exists a model* $(M', S') \supseteq (M, S)$ *such that:*

The proof of the theorem comes in two steps.

Lemma (Slicing)

Suppose that $(M, S) \models \mathsf{CS}^- + \mathsf{INT}$*. Then there exists a model* $(M', S') \supseteq (M, S)$ *such that:*

- $(M', S') \models \mathsf{CS}^- \restriction M + \mathsf{INT}$.

The proof of the theorem comes in two steps.

Lemma (Slicing)

Suppose that $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$. *Then there exists a model* $(M', S') \supseteq (M, S)$ *such that:*

- $(M', S') \models \mathrm{CS}^- \upharpoonright M + \mathrm{INT}$.
- *For any* $a \in M$ *and any function* $f : [0, a] \to M'$ *coded in* M', *the image* $f \cap M$ *is not cofinal in* M. *(*M *is* ***semiregular*** *in* M'*).*

The proof of the theorem comes in two steps.

Lemma (Slicing)

Suppose that $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$. *Then there exists a model* $(M', S') \supseteq (M, S)$ *such that:*

- $(M', S') \models \mathrm{CS}^- \restriction M + \mathrm{INT}$.
- *For any* $a \in M$ *and any function* $f : [0, a] \to M'$ *coded in* M'*, the image* $f \cap M$ *is not cofinal in* M*. (*M *is* ***semiregular*** *in* M'*).*

The proof of the theorem comes in two steps.

Lemma (Slicing)

Suppose that $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$. *Then there exists a model* $(M', S') \supseteq (M, S)$ *such that:*

- $(M', S') \models \mathrm{CS}^- \restriction M + \mathrm{INT}$.
- *For any* $a \in M$ *and any function* $f : [0, a] \to M'$ *coded in* M', *the image* $f \cap M$ *is not cofinal in* M. *(*M *is* ***semiregular*** *in* M'*).*

One more bit of classical models of PA.

The proof of the theorem comes in two steps.

Lemma (Slicing)

Suppose that $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$. *Then there exists a model* $(M', S') \supseteq (M, S)$ *such that:*

- $(M', S') \models \mathrm{CS}^- \restriction M + \mathrm{INT}$.
- *For any* $a \in M$ *and any function* $f : [0, a] \to M'$ *coded in* M'*, the image* $f \cap M$ *is not cofinal in* M*. (*M *is* ***semiregular*** *in* M'*).*

One more bit of classical models of PA. Recall that the extension $M \preceq M'$ is **conservative** if for any A definable in M' (with parametres), $A \cap M$ is definable in M.

Fix a model $(M, S) \models \mathsf{CS}^- + \mathsf{INT}$ and introduce a family of predicates S_ϕ for $\phi \in \mathsf{Form}_{\mathsf{PA}}(M)$ defined by:

Fix a model $(M, S) \models \mathsf{CS}^- + \mathsf{INT}$ and introduce a family of predicates S_ϕ for $\phi \in \mathrm{Form}_{\mathsf{PA}}(M)$ defined by:

$$S_\phi(\alpha) :\equiv S(\phi, \alpha).$$

Fix a model $(M, S) \models \mathsf{CS}^- + \mathsf{INT}$ and introduce a family of predicates S_ϕ for $\phi \in \mathrm{Form}_{\mathsf{PA}}(M)$ defined by:

$$S_\phi(\alpha) :\equiv S(\phi, \alpha).$$

They form a countable family of predicates such that the model $(M, S_\phi)_{\phi \in M}$ satisfy the full induction.

Uniwersytet Gdański

Fix a model $(M, S) \models \mathsf{CS}^- + \mathsf{INT}$ and introduce a family of predicates S_ϕ for $\phi \in \mathrm{Form}_{\mathrm{PA}}(M)$ defined by:

$$S_\phi(\alpha) :\equiv S(\phi, \alpha).$$

They form a countable family of predicates such that the model $(M, S_\phi)_{\phi \in M}$ satisfy the full induction. By MacDowell–Specker, there exists a conservative elementary end-extension $(M', S'_\phi)_{\phi \in M}$.

Fix a model $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$ and introduce a family of predicates S_ϕ for $\phi \in \mathrm{Form}_{\mathrm{PA}}(M)$ defined by:

$$S_\phi(\alpha) :\equiv S(\phi, \alpha).$$

They form a countable family of predicates such that the model $(M, S_\phi)_{\phi \in M}$ satisfy the full induction. By MacDowell–Specker, there exists a conservative elementary end-extension $(M', S'_\phi)_{\phi \in M}$. You glue S'_ϕ back together in order to obtain S' (you need regularity conditions to ensure that you can account for ϕ whose *depth* is in M).

Uniwersytet Gdański

Fix a model $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$ and introduce a family of predicates S_ϕ for $\phi \in \mathrm{Form}_{\mathrm{PA}}(M)$ defined by:

$$S_\phi(\alpha) :\equiv S(\phi, \alpha).$$

They form a countable family of predicates such that the model $(M, S_\phi)_{\phi \in M}$ satisfy the full induction. By MacDowell–Specker, there exists a conservative elementary end-extension $(M', S'_\phi)_{\phi \in M}$. You glue S'_ϕ back together in order to obtain S' (you need regularity conditions to ensure that you can account for ϕ whose *depth* is in M).
For semiregularity: let $a \in M$ and let $f \in M'$ be a function from a to M'.

Uniwersytet Gdański

Fix a model $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$ and introduce a family of predicates S_ϕ for $\phi \in \mathrm{Form}_{\mathrm{PA}}(M)$ defined by:

$$S_\phi(\alpha) :\equiv S(\phi, \alpha).$$

They form a countable family of predicates such that the model $(M, S_\phi)_{\phi \in M}$ satisfy the full induction. By MacDowell–Specker, there exists a conservative elementary end-extension $(M', S'_\phi)_{\phi \in M}$. You glue S'_ϕ back together in order to obtain S' (you need regularity conditions to ensure that you can account for ϕ whose *depth* is in M).
For semiregularity: let $a \in M$ and let $f \in M'$ be a function from a to M'. By conservativity, the set $f \cap M$ is definable in $(M, S_\phi)_{\phi \in M}$.

Fix a model $(M, S) \models \mathrm{CS}^- + \mathrm{INT}$ and introduce a family of predicates S_ϕ for $\phi \in \mathrm{Form}_{\mathrm{PA}}(M)$ defined by:

$$S_\phi(\alpha) :\equiv S(\phi, \alpha).$$

They form a countable family of predicates such that the model $(M, S_\phi)_{\phi \in M}$ satisfy the full induction. By MacDowell–Specker, there exists a conservative elementary end-extension $(M', S'_\phi)_{\phi \in M}$. You glue S'_ϕ back together in order to obtain S' (you need regularity conditions to ensure that you can account for ϕ whose *depth* is in M).
For semiregularity: let $a \in M$ and let $f \in M'$ be a function from a to M'. By conservativity, the set $f \cap M$ is definable in $(M, S_\phi)_{\phi \in M}$. Since the latter structure satisfies induction, the image of f is not cofinal in M. $\square$
Notice that since M' is recursively saturated, the extension $M \preceq_E M'$ is *not* conservative.

Lemma (Upwards Extension)

Let $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$, *where* I *is a semiregular nonstandard cut in* M. *Then there exists* $S' \supseteq S$ *such that* $(M, S') \models \mathrm{CS}^- + \mathrm{INT}$.

Lemma (Upwards Extension)

Let $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$*, where* I *is a semiregular nonstandard cut in* M*. Then there exists* $S' \supseteq S$ *such that* $(M, S') \models \mathrm{CS}^- + \mathrm{INT}$*.*

Lemma (Upwards Extension)

Let $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$*, where* I *is a semiregular nonstandard cut in* M*. Then there exists* $S' \supseteq S$ *such that* $(M, S') \models \mathrm{CS}^- + \mathrm{INT}$*.*

Remark

The Lemma fails, if we drop the semiregularity assumption, even assuming that I is an elementary submodel.

Lemma (Upwards Extension)

Let $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$, *where* I *is a semiregular nonstandard cut in* M. *Then there exists* $S' \supseteq S$ *such that* $(M, S') \models \mathrm{CS}^- + \mathrm{INT}$.

Remark

The Lemma fails, if we drop the semiregularity assumption, even assuming that I is an elementary submodel.

Lemma (Upwards Extension)

Let $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$, *where* I *is a semiregular nonstandard cut in* M*. Then there exists* $S' \supseteq S$ *such that* $(M, S') \models \mathrm{CS}^- + \mathrm{INT}$.

Remark

The Lemma fails, if we drop the semiregularity assumption, even assuming that I is an elementary submodel. The counterexample (joint with Roman Kossak) uses a technique calles disjunctions with stopping conditions.

Lemma (Upwards Extension)

Let $(M, S) \models \mathrm{CS}^- \upharpoonright I + \mathrm{INT}$*, where* I *is a semiregular nonstandard cut in* M*. Then there exists* $S' \supseteq S$ *such that* $(M, S') \models \mathrm{CS}^- + \mathrm{INT}$*.*

Remark

The Lemma fails, if we drop the semiregularity assumption, even assuming that I is an elementary submodel. The counterexample (joint with Roman Kossak) uses a technique calles disjunctions with stopping conditions. In the counterexample, we use a pair of models $M \preceq M'$ such that M' codes a cofinal increasing ω-sequence in M.

Uniwersytet Gdański

Now we turn to the proof of the Lemma.

Now we turn to the proof of the Lemma. We will use the recent(-ish) construction of a satisfaction class by Fedor Pakhomov.

Now we turn to the proof of the Lemma. We will use the recent(-ish) construction of a satisfaction class by Fedor Pakhomov. In the proof, we will make use of the notion of **syntactic templates**.

Now we turn to the proof of the Lemma. We will use the recent(-ish) construction of a satisfaction class by Fedor Pakhomov. In the proof, we will make use of the notion of **syntactic templates**.

Essentially, a syntactic template of a formula is its normal form:

- all terms composed of closed terms and free variables are collapsed to single distinct free variables;

Now we turn to the proof of the Lemma. We will use the recent(-ish) construction of a satisfaction class by Fedor Pakhomov. In the proof, we will make use of the notion of **syntactic templates**.
Essentially, a syntactic template of a formula is its normal form:

- all terms composed of closed terms and free variables are collapsed to single distinct free variables;
- all bound variables are distinct (although they may appear several times in the same formula; they are just quantified over exactly once);

Now we turn to the proof of the Lemma. We will use the recent(-ish) construction of a satisfaction class by Fedor Pakhomov. In the proof, we will make use of the notion of **syntactic templates**.
Essentially, a syntactic template of a formula is its normal form:

- all terms composed of closed terms and free variables are collapsed to single distinct free variables;
- all bound variables are distinct (although they may appear several times in the same formula; they are just quantified over exactly once);
- bound and free variables are chosen in some canonical way so that a formula ϕ with syntactic depth in M will have its template $\widehat{\phi} \in M$.

Fix a model $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$. We will define a mapping $f : \mathrm{Temp}(M) \to \mathrm{Temp}(I)$ preserving syntactic operations which is an identity on $\mathrm{Temp}(I)$.

Fix a model $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$. We will define a mapping $f : \mathrm{Temp}(M) \to \mathrm{Temp}(I)$ preserving syntactic operations which is an identity on $\mathrm{Temp}(I)$. Then we will set for any template ϕ:

$$S'(\phi, \alpha) :\equiv S(f(\phi), \alpha).$$

(This makes sense, since we assume that S satisfies some regularity conditions). Since f will preserve syntactic operations (in particular it will not change the variables in the outermost quantifiers), S' will satisfy compositional clauses.

Fix a model $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$. We will define a mapping $f : \mathrm{Temp}(M) \to \mathrm{Temp}(I)$ preserving syntactic operations which is an identity on $\mathrm{Temp}(I)$. Then we will set for any template ϕ:

$$S'(\phi, \alpha) :\equiv S(f(\phi), \alpha).$$

(This makes sense, since we assume that S satisfies some regularity conditions). Since f will preserve syntactic operations (in particular it will not change the variables in the outermost quantifiers), S' will satisfy compositional clauses. Since an instance of the induction axiom will be sent to an instance of the induction axiom, it will also preserve the internal induction INT.

Fix a model $(M, S) \models \mathrm{CS}^- \restriction I + \mathrm{INT}$. We will define a mapping $f : \mathrm{Temp}(M) \to \mathrm{Temp}(I)$ preserving syntactic operations which is an identity on $\mathrm{Temp}(I)$. Then we will set for any template ϕ:

$$S'(\phi, \alpha) :\equiv S(f(\phi), \alpha).$$

(This makes sense, since we assume that S satisfies some regularity conditions). Since f will preserve syntactic operations (in particular it will not change the variables in the outermost quantifiers), S' will satisfy compositional clauses. Since an instance of the induction axiom will be sent to an instance of the induction axiom, it will also preserve the internal induction INT. Finally, we extend S' to all formulae in the canonical way so that regularity conditions hold.

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M.

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M. For a template ϕ and $a \in M$, let $U(\phi, a)$ be the set of the templates of formulae ψ which occur at the syntactic tree of ϕ at the depth at most a.

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M. For a template ϕ and $a \in M$, let $U(\phi, a)$ be the set of the templates of formulae ψ which occur at the syntactic tree of ϕ at the depth at most a.
We will construct a sequence of elements of I

$$a_0 > a_1 > \dots$$

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M. For a template ϕ and $a \in M$, let $U(\phi, a)$ be the set of the templates of formulae ψ which occur at the syntactic tree of ϕ at the depth at most a.
We will construct a sequence of elements of I

$$a_0 > a_1 > \dots$$

and a sequence of functions

$$f_0, f_1, f_2, \dots$$

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M. For a template ϕ and $a \in M$, let $U(\phi, a)$ be the set of the templates of formulae ψ which occur at the syntactic tree of ϕ at the depth at most a.
We will construct a sequence of elements of I

$$a_0 > a_1 > \dots$$

and a sequence of functions

$$f_0, f_1, f_2, \dots$$

such that the following conditions are satisfied:

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M. For a template ϕ and $a \in M$, let $U(\phi, a)$ be the set of the templates of formulae ψ which occur at the syntactic tree of ϕ at the depth at most a.
We will construct a sequence of elements of I

$$a_0 > a_1 > \dots$$

and a sequence of functions

$$f_0, f_1, f_2, \dots$$

such that the following conditions are satisfied:

- $\mathrm{dom}(f_n) = U(\phi_n, a_n)$.

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M. For a template ϕ and $a \in M$, let $U(\phi, a)$ be the set of the templates of formulae ψ which occur at the syntactic tree of ϕ at the depth at most a.
We will construct a sequence of elements of I

$$a_0 > a_1 > \ldots$$

and a sequence of functions

$$f_0, f_1, f_2, \ldots$$

such that the following conditions are satisfied:

- $\mathrm{dom}(f_n) = U(\phi_n, a_n)$.
- f_n preserves the syntactic structure of the templates.

Uniwersytet Gdański

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M. For a template ϕ and $a \in M$, let $U(\phi, a)$ be the set of the templates of formulae ψ which occur at the syntactic tree of ϕ at the depth at most a.
We will construct a sequence of elements of I

$$a_0 > a_1 > \dots$$

and a sequence of functions

$$f_0, f_1, f_2, \dots$$

such that the following conditions are satisfied:

- $\mathrm{dom}(f_n) = U(\phi_n, a_n)$.
- f_n preserves the syntactic structure of the templates.
- $f_n \restriction I$ is the identity function.

Let $\phi_i, i \in \omega$ be an enumeration of the syntactic templates in M. For a template ϕ and $a \in M$, let $U(\phi, a)$ be the set of the templates of formulae ψ which occur at the syntactic tree of ϕ at the depth at most a.
We will construct a sequence of elements of I

$$a_0 > a_1 > \dots$$

and a sequence of functions

$$f_0, f_1, f_2, \dots$$

such that the following conditions are satisfied:

- $\mathrm{dom}(f_n) = U(\phi_n, a_n)$.
- f_n preserves the syntactic structure of the templates.
- $f_n \restriction I$ is the identity function.
- For an arbitrary $i \leq n$,

$$f_n \restriction U(\phi_n, a_n) \cap U(\phi_i, a_n) = f_i \restriction U(\phi_n, a_n) \cap U(\phi_i, a_n).$$

Suppose that we manage to construct such a sequence of functions.

Suppose that we manage to construct such a sequence of functions. Let

$$f(\phi_n) = f_n(\phi_n).$$

Suppose that we manage to construct such a sequence of functions. Let

$$f(\phi_n) = f_n(\phi_n).$$

We claim that f preserves the syntactic operations.

Suppose that we manage to construct such a sequence of functions. Let

$$f(\phi_n) = f_n(\phi_n).$$

We claim that f preserves the syntactic operations. Indeed, suppose that ϕ_k, ϕ_l are direct subformulae of ϕ_m.

Suppose that we manage to construct such a sequence of functions. Let

$$f(\phi_n) = f_n(\phi_n).$$

We claim that f preserves the syntactic operations. Indeed, suppose that ϕ_k, ϕ_l are direct subformulae of ϕ_m. Then the functions f_m, f_k agree on $U(\phi_m, a_n) \cap U(\phi_k, a_n)$ and the functions f_l, f_k agree on $U(\phi_m, a_n) \cap U(\phi_l, a_n)$, where $n = \max k, l, m$.

Suppose that we manage to construct such a sequence of functions. Let

$$f(\phi_n) = f_n(\phi_n).$$

We claim that f preserves the syntactic operations. Indeed, suppose that ϕ_k, ϕ_l are direct subformulae of ϕ_m. Then the functions f_m, f_k agree on $U(\phi_m, a_n) \cap U(\phi_k, a_n)$ and the functions f_l, f_k agree on $U(\phi_m, a_n) \cap U(\phi_l, a_n)$, where $n = \max k, l, m$. In particular, f agrees with f_m on these three formulae, so it preserves the syntactic structure, since f_m does.

Now, it is enough to construct the required sequence.

Now, it is enough to construct the required sequence. Suppose that we have already defined $f_0, \ldots, f_n$.

Now, it is enough to construct the required sequence. Suppose that we have already defined $f_0, \ldots, f_n$. We want to define a_{n+1} and f_{n+1}.

Now, it is enough to construct the required sequence. Suppose that we have already defined $f_0, \ldots, f_n$. We want to define a_{n+1} and f_{n+1}. Consider the relation $\trianglelefteq^*$ which is the transitive closure of the direct subformula relation $\triangleleft$ *taken within* $U(\phi_{n+1}, a_{n+1})$.

Now, it is enough to construct the required sequence. Suppose that we have already defined $f_0, \ldots, f_n$. We want to define a_{n+1} and f_{n+1}. Consider the relation $\trianglelefteq^*$ which is the transitive closure of the direct subformula relation $\triangleleft$ *taken within* $U(\phi_{n+1}, a_{n+1})$. (We have yet to define a_{n+1}.) In other words, $\phi \trianglelefteq^* \psi$ iff there exists a chain of formulae

$$\phi = \xi_0 \triangleleft \xi_1 \triangleleft \ldots \triangleleft \xi_d = \psi,$$

such that $\widehat{\xi_i} \in U(\phi_{n+1}, a_{n+1})$ for all i, where $\widehat{\xi}$ is the syntactic template of ξ.

Now, it is enough to construct the required sequence. Suppose that we have already defined $f_0, \ldots, f_n$. We want to define a_{n+1} and f_{n+1}. Consider the relation $\trianglelefteq^*$ which is the transitive closure of the direct subformula relation $\lhd$ *taken within* $U(\phi_{n+1}, a_{n+1})$. (We have yet to define a_{n+1}.) In other words, $\phi \trianglelefteq^* \psi$ iff there exists a chain of formulae

$$\phi = \xi_0 \lhd \xi_1 \lhd \ldots \lhd \xi_d = \psi,$$

such that $\widehat{\xi_i} \in U(\phi_{n+1}, a_{n+1})$ for all i, where $\widehat{\xi}$ is the syntactic template of ξ.

Since we want f_{n+1} to preserve syntactic operations, we only have to (and we are only allowed to) define it on the templates ψ which are $\trianglelefteq^*$ weakly minimal,

Uniwersytet Gdański

Now, it is enough to construct the required sequence. Suppose that we have already defined $f_0, \ldots, f_n$. We want to define a_{n+1} and f_{n+1}. Consider the relation $\trianglelefteq^*$ which is the transitive closure of the direct subformula relation $\lhd$ *taken within* $U(\phi_{n+1}, a_{n+1})$. (We have yet to define a_{n+1}.) In other words, $\phi \trianglelefteq^* \psi$ iff there exists a chain of formulae

$$\phi = \xi_0 \lhd \xi_1 \lhd \ldots \lhd \xi_d = \psi,$$

such that $\widehat{\xi_i} \in U(\phi_{n+1}, a_{n+1})$ for all i, where $\widehat{\xi}$ is the syntactic template of ξ.

Since we want f_{n+1} to preserve syntactic operations, we only have to (and we are only allowed to) define it on the templates ψ which are $\trianglelefteq^*$ weakly minimal, where a template is **weakly minimal** if at least one of its direct subformulae does not have a template in $U(\phi_{n+1}, a_{n+1})$.

Fix a formula ϕ_{n+1}.

Fix a formula ϕ_{n+1}. Consider the templates of subformulae appearing at most at the syntactic depth a_n.

Fix a formula ϕ_{n+1}. Consider the templates of subformulae appearing at most at the syntactic depth a_n. We can enumerate these formulae creating a function g from some $a \in I$ to M.

Fix a formula ϕ_{n+1}. Consider the templates of subformulae appearing at most at the syntactic depth a_n. We can enumerate these formulae creating a function g from some $a \in I$ to M. By semiregularity, the formulae whose templates are in I have syntactic depth bounded by some $b_{n+1} \in I$.

Fix a formula ϕ_{n+1}. Consider the templates of subformulae appearing at most at the syntactic depth a_n. We can enumerate these formulae creating a function g from some $a \in I$ to M. By semiregularity, the formulae whose templates are in I have syntactic depth bounded by some $b_{n+1} \in I$.
Set a_{n+1} nonstandard such that

$$a_{n+1} 2^{a_{n+1}} \leq \frac{a_n}{2}.$$

For the $\trianglelefteq^*$-weakly minimal templates ψ in $U(\phi_{n+1}, a_{n+1})$, we define $f_{n+1}(\psi)$ as follows:

- If $\psi \in \mathrm{dom}(f_k)$ for some $k \leq n$, we set $f_{n+1}(\psi) = f_k(\psi)$, where k is the greatest such index. (We take $f_{n+1}(\zeta \odot \eta) = f_k(\zeta) \odot f_{n+1}(\eta)$).

Fix a formula ϕ_{n+1}. Consider the templates of subformulae appearing at most at the syntactic depth a_n. We can enumerate these formulae creating a function g from some $a \in I$ to M. By semiregularity, the formulae whose templates are in I have syntactic depth bounded by some $b_{n+1} \in I$.
Set a_{n+1} nonstandard such that

$$a_{n+1}2^{a_{n+1}} \leq \frac{a_n}{2}.$$

For the $\trianglelefteq^*$-weakly minimal templates ψ in $U(\phi_{n+1}, a_{n+1})$, we define $f_{n+1}(\psi)$ as follows:

- If $\psi \in \mathrm{dom}(f_k)$ for some $k \leq n$, we set $f_{n+1}(\psi) = f_k(\psi)$, where k is the greatest such index. (We take $f_{n+1}(\zeta \odot \eta) = f_k(\zeta) \odot f_{n+1}(\eta)$).
- Otherwise, we set $f_{n+1}(\psi)$ to be template of the unique formula obtained by substituting $0 = 0$ for any subformula of ϕ at the syntactic depth b_{n+1} (We take $f_{n+1}(\zeta \odot \eta) =$ truncation$(\zeta) \odot f_{n+1}(\eta)$).

Let us check that f_{n+1} satisfies the required conditions.

Let us check that f_{n+1} satisfies the required conditions. It is clear by construction that f_{n+1} preserves the syntactic structure.

Let us check that f_{n+1} satisfies the required conditions. It is clear by construction that f_{n+1} preserves the syntactic structure.

Claim I $f_{n+1}(\psi) = \psi$ for templates $\psi \in I \cap \mathrm{dom}(f_{n+1})$.

Let us check that f_{n+1} satisfies the required conditions. It is clear by construction that f_{n+1} preserves the syntactic structure.
Claim I $f_{n+1}(\psi) = \psi$ for templates $\psi \in I \cap \mathrm{dom}(f_{n+1})$.
Indeed, if ψ is minimal in $U(\phi_{n+1}, a_{n+1})$, we defined $f(\psi)$ to be the truncation of ψ to the depth b_{n+1}.

Let us check that f_{n+1} satisfies the required conditions. It is clear by construction that f_{n+1} preserves the syntactic structure.

Claim I $f_{n+1}(\psi) = \psi$ for templates $\psi \in I \cap \mathrm{dom}(f_{n+1})$.

Indeed, if ψ is minimal in $U(\phi_{n+1}, a_{n+1})$, we defined $f(\psi)$ to be the truncation of ψ to the depth b_{n+1}. By definition of b_{n+1} such a truncation is the identity on the templates from $I \cap \mathrm{dom}(f_{n+1})$. Applying the compositional clauses, then preserves this property. (Similarly for the weakly minimal formulae).

Claim II

$f_{n+1} \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1}) = f_k \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.

Claim II

$f_{n+1} \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1}) = f_k \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.
Take any $\psi \in U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.

Claim II

$f_{n+1} \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1}) = f_k \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.
Take any $\psi \in U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$. By construction, $f_{n+1}(\psi)$ is uniquely determined by the values of f_{n+1} at $\trianglelefteq^*$-smaller templates in $U(\phi_{n+1}, a_{n+1})$.

Uniwersytet Gdański

Claim II

$f_{n+1} \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1}) = f_k \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.
Take any $\psi \in U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$. By construction, $f_{n+1}(\psi)$ is uniquely determined by the values of f_{n+1} at $\trianglelefteq^*$-smaller templates in $U(\phi_{n+1}, a_{n+1})$. Notice that there are at most $a_{n+1}2^{a_{n+1}}$ such templates.

Claim II

$f_{n+1} \upharpoonright U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1}) = f_k \upharpoonright U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.
Take any $\psi \in U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$. By construction, $f_{n+1}(\psi)$ is uniquely determined by the values of f_{n+1} at $\trianglelefteq^*$-smaller templates in $U(\phi_{n+1}, a_{n+1})$. Notice that there are at most $a_{n+1}2^{a_{n+1}}$ such templates. In particular, if $\psi \in U(\phi_k, a_{n+1})$, then actually all the $\trianglelefteq^*$-smaller templates are in $U(\phi_k, a_n)$.

Claim II

$f_{n+1} \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1}) = f_k \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.
Take any $\psi \in U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$. By construction, $f_{n+1}(\psi)$ is uniquely determined by the values of f_{n+1} at $\trianglelefteq^*$-smaller templates in $U(\phi_{n+1}, a_{n+1})$. Notice that there are at most $a_{n+1}2^{a_{n+1}}$ such templates. In particular, if $\psi \in U(\phi_k, a_{n+1})$, then actually all the $\trianglelefteq^*$-smaller templates are in $U(\phi_k, a_n)$.
By the induction hypothesis, for all $m \geq k$, if $\eta \in U(\phi_m, a_n) \cap U(\phi_k, a_n)$, then $f_m(\eta) = f_k(\eta)$.

Claim II

$f_{n+1} \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1}) = f_k \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.
Take any $\psi \in U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$. By construction, $f_{n+1}(\psi)$ is uniquely determined by the values of f_{n+1} at $\trianglelefteq^*$-smaller templates in $U(\phi_{n+1}, a_{n+1})$. Notice that there are at most $a_{n+1}2^{a_{n+1}}$ such templates. In particular, if $\psi \in U(\phi_k, a_{n+1})$, then actually all the $\trianglelefteq^*$-smaller templates are in $U(\phi_k, a_n)$.
By the induction hypothesis, for all $m \geq k$, if $\eta \in U(\phi_m, a_n) \cap U(\phi_k, a_n)$, then $f_m(\eta) = f_k(\eta)$. In particular for all minimal and (using induction internally in the model) weakly minimal templates η $\trianglelefteq^*$-below ψ, $f_k(\eta) = f_{n+1}(\eta)$, guaranteeing that $f_k(\psi) = f_{n+1}(\psi)$.

Claim II

$f_{n+1} \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1}) = f_k \restriction U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$.
Take any $\psi \in U(\phi_{n+1}, a_{n+1}) \cap U(\phi_k, a_{n+1})$. By construction, $f_{n+1}(\psi)$ is uniquely determined by the values of f_{n+1} at $\trianglelefteq^*$-smaller templates in $U(\phi_{n+1}, a_{n+1})$. Notice that there are at most $a_{n+1}2^{a_{n+1}}$ such templates. In particular, if $\psi \in U(\phi_k, a_{n+1})$, then actually all the $\trianglelefteq^*$-smaller templates are in $U(\phi_k, a_n)$.
By the induction hypothesis, for all $m \geq k$, if $\eta \in U(\phi_m, a_n) \cap U(\phi_k, a_n)$, then $f_m(\eta) = f_k(\eta)$. In particular for all minimal and (using induction internally in the model) weakly minimal templates η $\trianglelefteq^*$-below ψ, $f_k(\eta) = f_{n+1}(\eta)$, guaranteeing that $f_k(\psi) = f_{n+1}(\psi)$. This concludes the proof. □

Uniwersytet Gdański

It seems that the argument can be modified to prove the end-extension result for models of internal collection, but this has not yet been worked out.

It seems that the argument can be modified to prove the end-extension result for models of internal collection, but this has not yet been worked out.

- You could either try to get the analogous model-theoretic result for models of collection, since we are not using the full power of either conservativity or semiregularity.

It seems that the argument can be modified to prove the end-extension result for models of internal collection, but this has not yet been worked out.

- You could either try to get the analogous model-theoretic result for models of collection, since we are not using the full power of either conservativity or semiregularity.
- Or possibly, work only with the models in which the value of a truth predicate only depends on what happens at some uniformly fixed syntactic depth a. (Models arising from Pakhomov's construction of a satisfaction class have this property).

Thank you for your attention!