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Existential Definability with Addition and k-Regular Predicates

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42ème Journées sur les Arithmétiques Faibles

- For $k \geq 2$ consider FA $\mathcal{A}$ over Σ_k^n for $\Sigma_k = \{0, 1, ..., k-1\}$.
- The language $L(\mathcal{A})$ and the set $[\![L(\mathcal{A})]\!]_k \subseteq \mathbb{N}^n$.
- $R \subseteq \mathbb{N}^n$ is called k-FA-recognizable if there exists Σ_k^n-FA $\mathcal{A}$ such that $R = [\![L(\mathcal{A})]\!]_k$.

Theorem. Büchi [1960], Bruyère [1985], Villemaire [1992]: $R \subseteq \mathbb{N}^n$ is k-FA-recognizable if and only if it is $\exists\forall\exists$-definable in the structure $\langle \mathbb{N}; 0, 1, +, V_k, \leq \rangle$, where $V_k(x, y)$ iff y is the largest power of k that divides x.

Theorem. Haase and Różycki [2021]: $R \subseteq \mathbb{N}^n$ is k-FA-recognizable if and only if it is $\exists\forall$-definable in $\langle \mathbb{N}; 0, 1, +, V_k, \leq \rangle$, but $\exists$-formulas are less expressive.

In particular, $[\![\{10, 01\}^*]\!]_2$ is not $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, V_2, \leq \rangle$.

- Whether there is a "natural" structure where every k-FA-recognizable relation is ∃-definable, and vice versa.
- How can we describe the expressive power of the existential k-Büchi arithmetic?
- Whether there is an algorithm to decide ∃-definability in $\langle \mathbb{N}; 0, 1, +, V_k, \leq \rangle$ of a given k-FA-recognizable set.

  (Formally stated by Haase and Różycki for the sets $S \subseteq \mathbb{N}$).

- Denote by $\mathcal{F}$ the set of all finite subsets of $\mathbb{N}$.
- 2-FA-recognizability of $R \subseteq \mathcal{F}^n$ is defined similarly.

Theorem. Büchi [1960], Elgot [1961], Trakhtenbrot [1962]: $R \subseteq \mathcal{F}^n$ is 2-FA-recognizable iff it is WMSO-definable in the structure $\langle \mathbb{N}; S \rangle$. Therefore, WMSOTh$\langle \mathbb{N}; S \rangle$ is decidable.

Theorem. Klaedtke and Rueß [2003]: The existential WMSO-theory of the structure $\langle \mathbb{N}; S, EqCard \rangle$ is decidable, whereas WMSOTh$\langle \mathbb{N}; S, EqCard \rangle$ is undecidable.

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- For $m > 0$ and a finite set $D \subseteq \mathbb{N}^m$, a Σ-Parikh automaton is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a $(\Sigma \times D)$-FA and $\varphi(x_1, ..., x_m)$ is an (existential) formula of Presburger arithmetic.
- Σ-PFA $\mathcal{A}_\varphi$ accepts $w \in \Sigma^*$ iff $(q_0, w, 0, ..., 0) \to \cdots \to (q_f, \epsilon, y_1, ..., y_m)$, where q_f is a final state of $\mathcal{A}$ and $\varphi(y_1, ..., y_m)$ is true.
- $R \subseteq \mathcal{F}^n$ is 2-PFA-recognizable iff it is existentially WMSO-definable in the structure $\langle \mathbb{N}; S, EqCard \rangle$.
- Decidability of the Emptiness problem and undecidability of the Universality problem for Parikh automata.

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- FO-version? $EqNonZeroBits(x, y)$ is true iff x and y have the same number of non-zero bits.
- Question of Bès [2013]: *it would be interesting to study the expressive power of fragments of FO arithmetic which include predicates like* $EqNonZeroBits$.

Is there a "natural" structure where every k-PFA-recognizable relation is $\exists$FO-definable, and vice versa?

- In these logical characterizations the universal quantifier is bounded.
- Davis, Putnam, and Robinson [1963]: Every relation $R \subseteq \mathbb{N}^n$ is r.e. if and only if it is $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, \cdot, exp, \leq \rangle$. Proof uses multiplication, factorials, binomial coefficients etc.
- Matiyasevich's proof of DPR-theorem [1976]: Purely existential arithmetization of Turing machines. The structure $\langle \mathbb{N}; 0, 1, +, \&, \frown, \leq \rangle$, for the bitwise minimum operation & and concatenation $\frown$, where $t = x \frown y \rightleftharpoons t = x + 2^{l(x)} y$.
- Every relation $R \subseteq \mathbb{N}^n$ is definable in $\langle \mathbb{N}; 0, 1, +, V_k, \leq \rangle$ iff it is definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, \leq \rangle$.

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- Every relation $R \subseteq \mathbb{N}^n$ is definable in $\langle \mathbb{N}; 0, 1, +, V_k, \leq \rangle$ iff it is definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, \leq \rangle$.

$$y = \Theta_{k,a}(x) \Leftrightarrow \exists x_1 ... \exists x_{k-1} \Big(\bigwedge_{1 \leq i < j \leq k-1} x_i \&_k x_j = 0 \ \wedge (x_1 + ... + x_{k-1}) \preccurlyeq_k \mathbf{1}_k(x) \wedge$$

$$x_1 + 2x_2 + ... + (k-1)x_{k-1} = x \wedge y = x_a \Big).$$

$y = \Theta_{k,0}(t, x)$. **Example:** $\Theta_{3,0}(100000, 1020) = 110101$

Existential characterization of k-FA-recognizable languages

Theorem 1

For an integer $k \geq 2$ every relation is k-FA-recognizable if and only if it is $\exists$-definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, \leq \rangle$.

- k-FA $\mathcal{A} = (Q, q_0, F, \delta)$.
- Variables $\overline{q} = q_0, \ldots, q_s$ for every $q_i \in Q$.
- For a state $p \in Q$, denote by $\nu(p)$ its number from $[0..s]$.

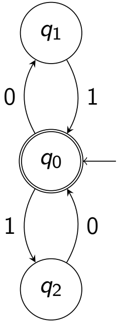
$$K_k(t, \overline{q}) \rightleftharpoons \bigwedge_{0 \leq i < j \leq s} q_i \&_k q_j = 0 \wedge q_0 + \ldots + q_s = \mathbf{1}_k(t) \wedge 1 \preccurlyeq_k q_0 \wedge \bigvee_{p \in F} t \preccurlyeq_k q_{\nu(p)}.$$

- For every $(p, \overline{a}) \in Q \times \Sigma_k^n$

$$\Delta_{(p,\overline{a})}(t, \overline{q}, \overline{x}) \rightleftharpoons \left(q_{\nu(p)} \&_k \underset{i \in [1..n]}{\&_k} \Theta_{k,a_i}\left(\frac{t}{k}, x_i\right)\right) \preccurlyeq_k \left(\Big|_{k}\,{}_{\widetilde{p} \in \delta(p,\overline{a})} \frac{q_{\nu(\widetilde{p})}}{k} \right).$$

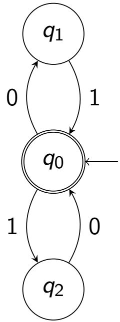
$$R_{L(\mathcal{A})}(\overline{x}) \Leftrightarrow \exists t \exists \overline{q} \Big(P_k(t) \wedge \bigwedge_{i \in [1..n]} x_i < t \wedge K_k(t, \overline{q}) \wedge \bigwedge_{(p,\overline{a}) \in Q \times \Sigma_k^n} \Delta_{(p,\overline{a})}(t, \overline{q}, \overline{x}) \Big).$$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$



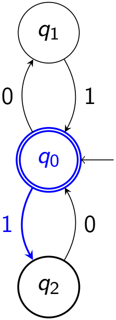
Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	1	1	0	0	1	0	1	x
···	0										t
···	0										q_0
···	0										q_1
···	0										q_2



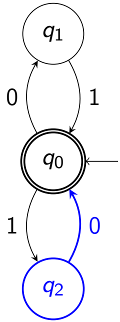
Example: ∃-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	1	1	0	0	1	0	**1**	x
···	0										t
···	0									**1**	q_0
···	0										q_1
···	0								**1**		q_2



Example: ∃-formula for the set $[\![\{10, 01\}^*]\!]_2$

⋯	0	0	0	1	1	0	0	1	**0**	1	x
⋯	0										t
⋯	0							**1**		1	q_0
⋯	0										q_1
⋯	0								**1**		q_2



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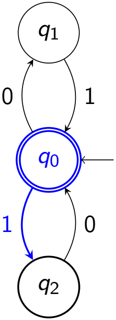
$\cdots$	0	0	0	1	1	0	0	**1**	0	1	x
$\cdots$	0										t
$\cdots$	0							**1**		1	q_0
$\cdots$	0										q_1
$\cdots$	0						**1**		1		q_2



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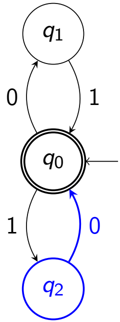
···	0	0	0	1	1	0	**0**	1	0	1	x
···	0										t
···	0					**1**		1		1	q_0
···	0										q_1
···	0						**1**		1		q_2



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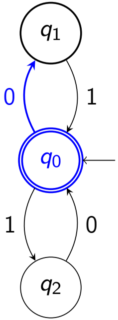
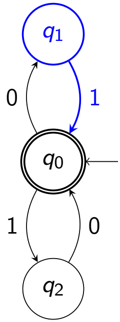
···	0	0	0	1	1	**0**	0	1	0	1	x
···	0										t
···	0					**1**		1		1	q_0
···	0				**1**						q_1
···	0						1		1		q_2



Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	1	**1**	0	0	1	0	1	x
···	0										t
···	0			**1**		1		1		1	q_0
···	0				**1**						q_1
···	0						1		1		q_2



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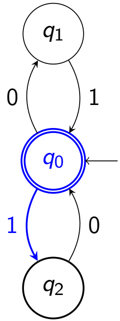
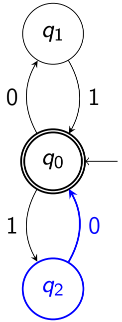
$\cdots$	0	0	0	**1**	1	0	0	1	0	1	x
$\cdots$	0										t
$\cdots$	0			**1**		1		1		1	q_0
$\cdots$	0				1						q_1
$\cdots$	0		**1**				1		1		q_2



Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	**0**	1	1	0	0	1	0	1	x
···	0	1									t
···	0	**1**		1		1		1		1	q_0
···	0				1						q_1
···	0		**1**				1		1		q_2



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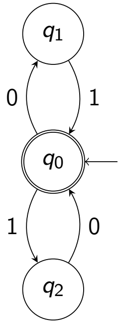
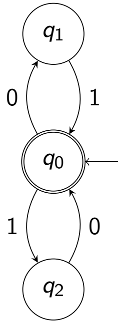
···	0	0	0	1	1	0	0	1	0	1	x
···	0	1									t
···	0	1		1		1		1		1	q_0
···	0				1						q_1
···	0		1				1		1		q_2



$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	1	1	0	0	1	0	1	x
···	0	1									t
···	0	1		1		1		1		1	q_0
···	0				1						q_1
···	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$
$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$

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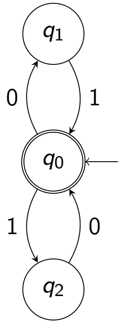
⋯	0	0	0	1	1	0	0	1	0	1	x
⋯	0	1									t
⋯	0	1		1		1		1		1	q_0
⋯	0				1						q_1
⋯	0		1				1		1		q_2



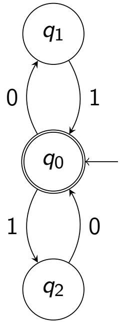
$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$

$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$

$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge$$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	**1**	1	0	0	**1**	0	**1**	x
···	0	1									t
···	0	1		**1**		1		**1**		**1**	q_0
···	0				1						q_1
···	0			**1**				**1**		**1**	$\frac{q_2}{2}$



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \land x < t \land q_0 + q_1 + q_2 = 2t - 1 \land$$

$$q_0 \& q_1 = 0 \land q_0 \& q_2 = 0 \land q_1 \& q_2 = 0 \land$$

$$q_0 \& 1 = 1 \land q_0 \& t = t \land q_0 \& x \preccurlyeq \frac{q_2}{2} \;\land$$

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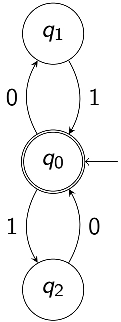
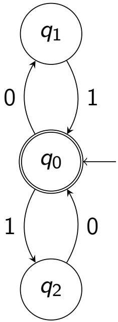
⋯	0	0	0	1	1	0	0	1	0	1	x
⋯	0	1									t
⋯	0	1		1		1		1		1	q_0
⋯	0				1						q_1
⋯	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$
$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$
$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge q_0 \& x \preccurlyeq \frac{q_2}{2} \wedge$$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

$\cdots$	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**0**	$\sim(\frac{t}{2}, x)$
$\cdots$	0	1									t
$\cdots$	0	1		1		1		1		1	q_0
$\cdots$	0					1					$\frac{q_1}{2}$
$\cdots$	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \land x < t \land q_0 + q_1 + q_2 = 2t - 1 \land$$
$$q_0 \& q_1 = 0 \land q_0 \& q_2 = 0 \land q_1 \& q_2 = 0 \land$$
$$q_0 \& 1 = 1 \land q_0 \& t = t \land q_0 \& x \preccurlyeq \frac{q_2}{2} \land q_0 \& \sim(\frac{t}{2}, x) \preccurlyeq \frac{q_1}{2} \land$$

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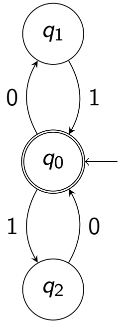
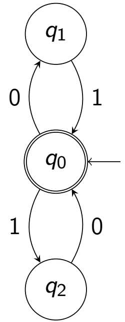
$\cdots$	0	0	0	1	1	0	0	1	0	1	x
$\cdots$	0	1									t
$\cdots$	0	1		1		1		1		1	q_0
$\cdots$	0				1						q_1
$\cdots$	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$
$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$
$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge q_0 \& x \preccurlyeq \frac{q_2}{2} \wedge q_0 \& \sim (\frac{t}{2}, x) \preccurlyeq \frac{q_1}{2} \wedge$$
$$q_1 \& x \preccurlyeq \frac{q_0}{2} \wedge q_1 \& \sim (\frac{t}{2}, x) \preccurlyeq 0 \wedge q_2 \& x \preccurlyeq 0 \wedge$$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

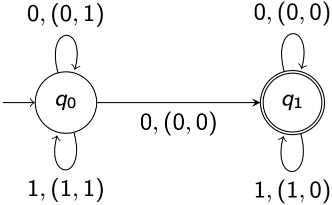
···	0	0	0	1	1	0	0	1	0	1	x
···	0	1									t
···	0	1		1		1		1		1	q_0
···	0				1						q_1
···	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$
$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$
$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge q_0 \& x \preccurlyeq \frac{q_2}{2} \wedge q_0 \& \sim(\frac{t}{2}, x) \preccurlyeq \frac{q_1}{2} \wedge$$
$$q_1 \& x \preccurlyeq \frac{q_0}{2} \wedge q_1 \& \sim(\frac{t}{2}, x) \preccurlyeq 0 \wedge q_2 \& x \preccurlyeq 0 \wedge q_2 \& \sim(\frac{t}{2}, x) \preccurlyeq \frac{q_0}{2} \Big).$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

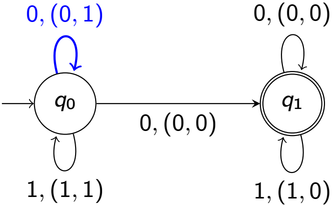
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$$
\begin{aligned}
x &= 10011011100 \\
y_1 &= 0 \\
y_2 &= 0
\end{aligned}
$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

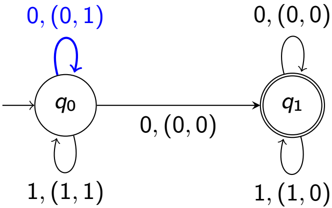
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 0$
$y_2 = 1$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

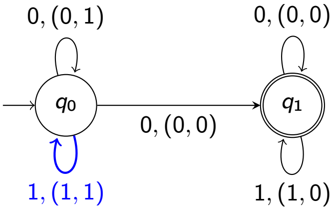
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 0$
$y_2 = 2$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

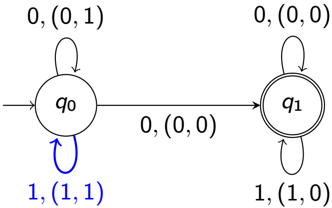
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 1$
$y_2 = 3$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

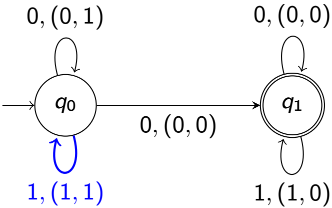
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 2$
$y_2 = 4$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

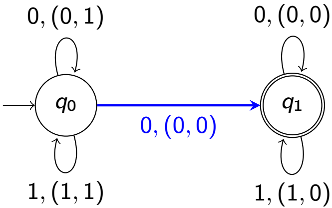
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$$x = 10011011100$$
$$y_1 = 3$$
$$y_2 = 5$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

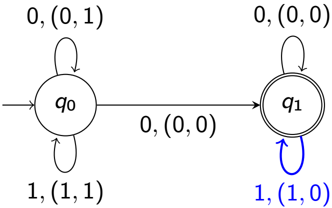
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 3$
$y_2 = 6$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

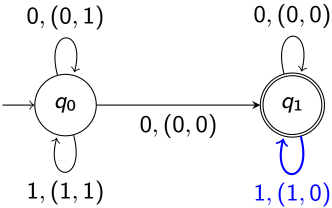
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$$x = 10011011100$$
$$y_1 = 4$$
$$y_2 = 6$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 5$
$y_2 = 6$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

$\{0,1\}$-PFA with
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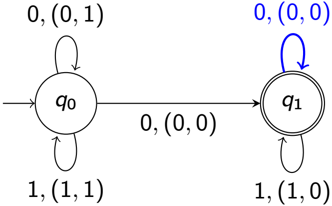

$$x = 10011011100$$
$$y_1 = 5$$
$$y_2 = 6$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

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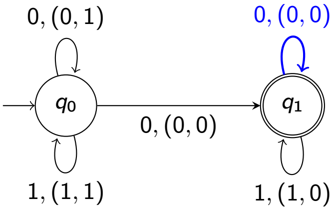

$x = 10011011100$
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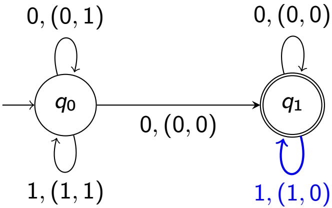
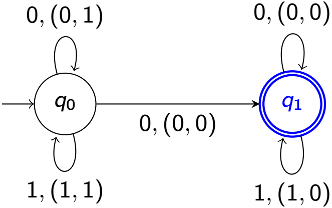


$x = 10011011100$
$y_1 = 6$
$y_2 = 6$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

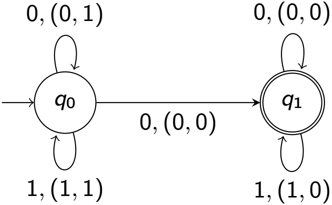
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = \mathbf{10011011100}$
$y_1 = 6$
$y_2 = 6$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



- Parikh map $\Phi_k : \mathbb{N} \to \mathbb{N}^k$ such that $\Phi_k(x) = (\#_{k,0}(x), \ldots, \#_{k,k-1}(x))$, where $\#_{k,i}$ counts the number of occurrences of i in k-ary expansion of x.
- $R(x_1, \ldots, x_n)$ is $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, \leq \rangle$, and $\bar{a} \in \{0, \ldots, k-1\}^n$. Then $R(\#_{k,a_1}(x_1), \ldots, \#_{k,a_n}(x_n))$ is $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, \&_k, EqNZB_k, \leq \rangle$.

$$\#_{k,a}(x) + \#_{k,b}(y) = \#_{k,c}(z) \Leftrightarrow \exists x' \exists y' (EqNZB_k(x' + y', \Theta_{k,c}(z)) \wedge x' \&_k y' = 0 \wedge EqNZB_k(x', \Theta_{k,a}(x)) \wedge EqNZB_k(y', \Theta_{k,b}(y))).$$

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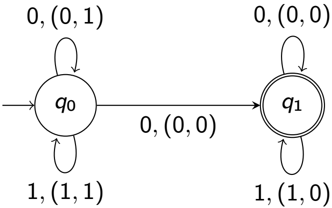
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



Theorem 2

For every integer $k \geq 2$ a relation $R \subseteq \mathbb{N}^n$ is k-PFA-recognizable if and only if it is ∃-definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, EqNZB_k, \leq \rangle$.

Corollary 1. The ∃-theory of $\langle \mathbb{N}; 0, 1, +, \&_k, EqNZB_k, \leq \rangle$ is decidable and the ∀∃-theory of this structure is undecidable. [Klaedtke and Rueß, 2003]

Corollary 2. The problem of deciding whether a set existentially definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, EqNZB_k, \leq \rangle$ is ∃-definable in $\langle \mathbb{N}; 0, 1, +, \&_k, \leq \rangle$ is undecidable. [Cadilhac, Finkel and McKenzie, 2011]

- Two-way multi-counter machine over Σ_k^n (k-MCM): (m, Q, q_0, F, δ).
- Transition function $\delta : Q \times (\Sigma \cup \{\vdash, \dashv\}) \times \{0,1\}^m \to 2^{Q \times \{-1,0,1\}^{m+1}}$.
- The same arguments as in the cases of k-FA and k-PFA for existential characterization of r.e. sets. Introduce concatenation $t = x \frown_k y \rightleftharpoons t = x + k^{l_k(x)} y$ and use **byte**wise multiplication instead of **bit**wise to encode δ.
- Predicate $\Delta_k(u, t, x)$, which is true when u is a power of k greater than k^2, x has the same u-byte-length as t and has the form

Multi-counter machines, concatenation, and DPR-theorem

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$$x = \underline{0000001}\ 0000010\ 0000100\ 0000010\ 0000100\ 0000010\ \underline{0000001}$$

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$$x = \underline{0000001}\ 0000010\ 0000100\ 0000010\ 0000100\ 0000010\ \underline{0000001}$$

$$x = 0000001 \ldots 0..0001000..0\ 0..abcdefg..0 \ldots \quad \ldots \quad 0000001$$

$$\tfrac{(kx)}{u} = 0000000 \ldots \quad \ldots \quad 0..0010000..0$$

$$\tfrac{(x)}{u} = 0000000 \ldots \quad \ldots \quad 0..0001000..0$$

$$\tfrac{(x)}{ku} = 0000000 \ldots \quad \ldots \quad 0..0000100..0$$

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$$x = \underline{0000001}\ 0000010\ 0000100\ 0000010\ 0000100\ 0000010\ \underline{0000001}$$

$x =$ 0000001 ...0..0001000..0 0..*abcdefg*..0 0000001

$\binom{kx}{u} =$ 0000000 ... ... 0..0010000..0 0..0*cde*000..0 ...

$\binom{x}{u} =$ 0000000 ... ... 0..0001000..0 0..00**cde**00..0 ...

$\binom{x}{ku} =$ 0000000 ... ... 0..0000100..0 0..000*cde*0..0 ...

Theorem 3

For every integer $k \geq 2$ a relation is k-MCM-recognizable if and only if it is $\exists$-definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, \frown_k, \leq \rangle$. Therefore, every relation $R \subseteq \mathbb{N}^n$ is r.e. iff it is $\exists$-definable in this structure.

Corollary 1 (DPR-theorem). Every relation $R \subseteq \mathbb{N}^n$ is r.e. if and only if it is $\exists$-definable in the structure $\langle \mathbb{N}; 0, 1, +, \cdot, exp, \leq \rangle$.

- Fix $k = 2$, then $z = x \&_2 y \Leftrightarrow z \preccurlyeq y \wedge y \preccurlyeq x + y - z$
- $x \preccurlyeq y \Leftrightarrow \binom{y}{x} \equiv 1 (\text{mod } 2)$
- $x \preccurlyeq y \Leftrightarrow s_2(y) = s_2(x) + s_2(y - x) \Leftrightarrow EqNZB(y, x \frown (y - x))$

Corollary 2. Every relation $R \subseteq \mathbb{N}^n$ is r.e. if and only if it is $\exists$-definable in the structure $\langle \mathbb{N}; 0, 1, +, EqNZB, \frown, \leq \rangle$.

Open Problem. We know that for multiplicatively independent k and l multiplication is definable in $\langle \mathbb{N}; 0, 1, +, V_k, V_l, \leq \rangle$ (Villemaire [1992]). Whether multiplication is **existentially** definable in $\langle \mathbb{N}; 0, 1, +, \&_k, \&_l, \leq \rangle$?

How can we describe the sets $S \subseteq \mathbb{N}$ that are $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, V_k, \leq \rangle$?

How can we describe the sets $S \subseteq \mathbb{N}$ that are $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, V_k, \leq \rangle$? Quantifier elimination (partial)

Theorem. Semënov [1980]: A set $R \subseteq \mathbb{N}$ is $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, P_k, \leq \rangle$ if and only if R can be represented as a finite union of expressions of the form $u_1 v_1^* u_2 \ldots u_n v_n^* \Sigma_{l,m,c}$, where $u_1, v_1, u_2, \ldots, u_n, v_n \in \Sigma_k^*$, and $l, m, c \in \mathbb{N}$.

- $\Sigma_{l,m,c}$ – all k-ary representations of $x \in \mathbb{N}$ s.t. $m \mid x - c$ and bit-length of x is divisible by l. $\Sigma_{2,2,1} = \{\epsilon, 11, 1001, 1011, 1101, 1111, \ldots\}$.
- Two sorts of variables: ordinary $x_1, x_2, \ldots$ and special $k^{\alpha_1}, k^{\alpha_2}, \ldots$, which are $\exists$-quantified.

Presburger arithmetic with P_k and k-Semënov arithmetic

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- Two sorts of variables: ordinary $x_1, x_2, \dots$ and special $k^{\alpha_1}, k^{\alpha_2}, \dots$, which are $\exists$-quantified.
- The structure $\langle \mathbb{N}; 0, 1, +, k^x, \leq \rangle$ is more expressive. Its FO-theory is k-Semënov arithmetic, which is decidable (Semënov [1983]), and $\exists$-theory is in **NexpTime** (Benedikt, Chistikov, Mansutti [2023]).
- $\exists\forall$-theory of $\langle \mathbb{N}; 0, 1, +, V_k, k^x, \leq \rangle$ (k-Büchi-Semënov arithmetic) is undecidable.
- $\exists$-theory of $\langle \mathbb{N}; 0, 1, +, \&_k, k^x, \leq \rangle$ is decidable in **ExpSpace** (Draghichi, Haase, Manea [2023]). (automata-theoretic techniques)

Quantifier elimination for the existential k-Büchi arithmetic?

- Two sorts of variables: ordinary $x_1, x_2, \dots$ and special $k^{\alpha_1}, k^{\alpha_2}, \dots$, which are $\exists$-quantified.
- Function $Rem_k(x, k^\alpha)$ for the remainder of x modulo k^α instead of $V_k(x, k^\alpha)$; $Rem_2(110101, 1000) = 101$.
- Specify an order $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n$.
- $x \to x_n k^{\alpha_n} + x_{n-1} k^{\alpha_{n-1}} + \dots + x_1 k^{\alpha_1} + x_0$ and for $i \in [1..n]$ $Rem_k(x, k^{\alpha_i}) \to x_{i-1} k^{\alpha_{i-1}} + \dots + x_1 k^{\alpha_1} + x_0$.
- k-ary expansions: x is split into
$$\boldsymbol{x_n} \cdot 0^{d_{n-1}} \boldsymbol{x_{n-1}} \cdot \dots \cdot 0^{d_1} \boldsymbol{x}_1 \cdot 0^{d_0} \boldsymbol{x}_0.$$
- Eliminate $\widetilde{x}_i \rightleftharpoons x_i k^{\alpha_i}$ for $i = n, \dots, 1, 0$ in the same way as in Presburger arithmetic; $k^{\alpha_i} \mid \widetilde{x}_i$ are handled by using Rem_k.

Theorem 4

A k-regular set $R \subseteq \mathbb{N}$ is $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, V_k, \leq \rangle$ if and only if there are positive integers l and m such that R has a $\{\cdot, \cup\}$-representation over $\mathscr{C}_{l,m}$.

- $\mathscr{C}_{l,m}$ comprises $\{w\}$ and w^* for every $w \in \{0, \ldots, k-1\}^*$ of length at most l, and $\Sigma_{l',m',c'}$ for $l' \leq l$, $m' \leq m$, $c' \in [0..m'-1]$.
- Proof idea: (1) eliminate all ordinary variables;
  (2) k-split x into $\boldsymbol{x_n} \cdot 0^{d_{n-1}} \boldsymbol{x_{n-1}} \cdot \ldots \cdot 0^{d_1} \boldsymbol{x_1} \cdot 0^{d_0} \boldsymbol{x_0}$;
  (3) each fragment $\boldsymbol{x_i}$ can be described using a $\exists$-formula of Presburger arithmetic with P_k;
  (4) apply Semënov's theorem to describe $\boldsymbol{x_0}, \boldsymbol{x_1}, \ldots, \boldsymbol{x_t}$.

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- Apply $\{\cdot, \cup\}$-representation theorem by Hashiguchi [1983] **?**

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- Apply $\{\cdot, \cup\}$-representation theorem by Hashiguchi [1983] **?**
- Combination with QE for k-Semënov arithmetic:

Theorem 5

The $\exists$-theory of $\langle \mathbb{N}; 0, 1, +, V_k, k^x, \leq \rangle$ is decidable in **NExpTime***.*

- Whether $\exists\mathrm{Def}\langle\mathbb{N};0,1,+,V_k,\leq\rangle\cap\exists\mathrm{Def}\langle\mathbb{N};0,1,+,k^x,\leq\rangle$ coincides with $\exists\mathrm{Def}\langle\mathbb{N};0,1,+,P_k,\leq\rangle$?
- For a pair of classes $\mathscr{R}$ and $\mathscr{S}$ of k-FA-recognizable predicates the property of $\exists$-definability in $\langle\mathbb{N};0,1,+,\mathscr{R},\leq\rangle$ is decidable in the sets $\exists$-definable in $\langle\mathbb{N};0,1,+,\mathscr{S},\leq\rangle$?
- Improve the **ExpSpace** upper bound for $\exists\mathrm{Th}\langle\mathbb{N};0,1,+,\&_k,k^x,\leq\rangle$.
- Decidable $\exists\mathrm{Th}\langle\mathbb{N};0,1,+,\&_k,k^x,EqNZB_k,\leq\rangle$?

- Whether $\exists\mathrm{Def}\langle\mathbb{N};0,1,+,V_k,\leq\rangle\cap\exists\mathrm{Def}\langle\mathbb{N};0,1,+,k^x,\leq\rangle$ coincides with $\exists\mathrm{Def}\langle\mathbb{N};0,1,+,P_k,\leq\rangle$?
- For a pair of classes $\mathscr{R}$ and $\mathscr{S}$ of k-FA-recognizable predicates the property of $\exists$-definability in $\langle\mathbb{N};0,1,+,\mathscr{R},\leq\rangle$ is decidable in the sets $\exists$-definable in $\langle\mathbb{N};0,1,+,\mathscr{S},\leq\rangle$?
- Improve the **ExpSpace** upper bound for $\exists\mathrm{Th}\langle\mathbb{N};0,1,+,\&_k,k^x,\leq\rangle$.
- Decidable $\exists\mathrm{Th}\langle\mathbb{N};0,1,+,\&_k,k^x,EqNZB_k,\leq\rangle$?

Thanks for your attention !