Metamathematics of the Global Reflection Principle.

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Introduction

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$$\forall\phi(\mathrm{Pr}_B(\phi) \to T(\phi)) \qquad \text{(GR(B))}$$

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Our basic B will be the elementary arithmetic EA ($\mathrm{I}\Delta_0$ + "exp is total".) We assume that all B's extends EA and are formulated in the language $\mathcal{L} := \{\leq, +, \times, 0, 1\}$.

Familiarize yourself with $\mathrm{CT}^-[\mathrm{EA}]$

Theorem (Enayat-Visser, Leigh)

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Theorem (Kotlarski-Krajewski-Lachlan)

$\mathrm{CT}^-[\mathrm{EA}] + \mathrm{INT} \vdash \mathrm{PA}$ *and* $\mathrm{CT}^-[\mathrm{EA}] + \mathrm{INT}$ *is conservative over* PA.

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Theorem (Enayat-Ł.-Wcisło)

There exists a PTIME function f such that if p is a proof of an arithmetical sentence ϕ in $\mathrm{CT}^-[\mathrm{PA}]$, then $f(p)$ is a proof of ϕ in PA.

Disjunctions that are too long for CT^-.

For a natural number n and a sentence ϕ let

$$\bigvee_{i\leq n} \phi := (\dots(\phi \vee \phi) \vee \phi) \vee \dots \vee \phi).$$

Theorem (Kotlarski-Krajewski-Lachlan)

If $\mathcal{M} \models \mathrm{PA}$ is countable and recursively saturated and $a \in M$ is nonstandard, then there is $T \subseteq M$ such that

$$(\mathcal{M}, T) \models \mathrm{CT}^-[\mathrm{PA}] + T\left(\bigvee_{i\leq a} 0 = 1\right).$$

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The above theory is called CT_0.

Theorem (Kotlarski-Smoryński)

The arithmetical consequences of CT_0 *coincide with* $\mathrm{RFN}^{<\omega}(\mathrm{PA})$.

Main course: disjunctive correctness

For a sequence of sentences $\phi_0, \ldots, \phi_n$, $\bigvee_{i \leq n} \phi_i$ denotes the disjunction

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Theorem (Cieśliński-Wcisło-Ł.)

$\mathrm{CT}^-[\mathrm{EA}] + \text{DC-out}$ *coincides with* CT_0.

Main course: Σ_1-reflection over UTB^-

UTB^- denote the following collection of $\mathcal{L}_T$ sentences (extending PA)

$$\forall x \big(T(\ulcorner \phi(\dot{x}) \urcorner) \equiv \phi(x) \big).$$

Theorem (Ł.)

$\Sigma_1^{\mathcal{L}_T}$-RFN$(\mathrm{UTB}^-) + \mathrm{CT}^-$ *coincides with* CT_0.

The main result

Easy implications

The following are very easy:

- $\mathsf{CT}^- + \Sigma_1^{\mathcal{L}_T}\text{-RFN}(\mathsf{UTB}^-) \vdash \forall \phi(\mathrm{Pr}_{\mathsf{PA}}(\phi) \to T(\phi))$

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- $\mathrm{CT}_0 \vdash$ DC-out.
  Assume $\forall i < |\bar{\psi}|\, T(\neg\psi_i)$ and use bounded induction for the formula $\phi(x) := T\left(\neg \bigvee_{i<x} \neg\psi_i\right)$.

The main result

DC-out implies CT_0

The core argument: DC-out $\Rightarrow$ Sind

Let Sind be the following statement

$$(T(s_0) \wedge \forall j < |s| - 1(T(s_j) \rightarrow T(s_{j+1})) \rightarrow \forall j < |s| T(s_j)).$$

Lemma (Cieśliński)

Over $\mathrm{CT}^-[\mathrm{EA}]$ DC-out *implies* Sind*.*

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So assume $T(\neg\phi_i)$. Then $T(\bigvee_{j \leq i} \neg\psi_j)$. By DC-out this contradicts the Claim.

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Fix a sequence $\phi_0, \ldots, \phi_a$ and assume that we have $T(\phi_j)$. Define a sequence s by putting

$$|s| := a - j - 1$$
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An application of Sind yields the thesis.

$(\mathrm{DC} + \mathrm{Sind}) \Rightarrow \mathrm{CT}_0.$

Lemma

$\mathrm{CT}^-[\mathrm{EA}] + \mathrm{DC} + \mathrm{Sind}$ *implies* CT_0.

Working in $\mathrm{CT}^-[\mathrm{EA}] + \mathrm{DC} + \mathrm{Sind}$ we show that T is coded, i.e. for every a there is a c such that

$$\forall x < a\big(T(x) \equiv x \in c\big).$$

Fix a and consider the following sequence:

$$\psi_i = \exists c \bigwedge_{\phi < i} (\phi \equiv \phi \in c).$$

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Fix a and consider the following sequence:

$$\psi_i = \exists c \bigwedge_{\phi < i} (\phi \equiv \phi \in c).$$

We have (using DC) $T(\psi_0)$ and $T(\psi_i) \to T(\psi_{i+1})$.

$(\mathsf{DC} + \mathsf{Sind}) \Rightarrow \mathsf{CT}_0$.

Lemma

$\mathsf{CT}^-[\mathsf{EA}] + \mathsf{DC} + \mathsf{Sind}$ *implies* CT_0.

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We have (using DC) $T(\psi_0)$ and $T(\psi_i) \to T(\psi_{i+1})$. So $T(\psi_a)$ and DC yields the thesis.

The main result

CT_0 implies $\Sigma_1^{\mathcal{L}_T}$-$\mathsf{RFN}(\mathsf{UTB}^-)$.

Key lemmata

$\mathrm{Pr}^{T}_{\mathrm{Th}}(x)$ is the canonical $\mathcal{L}_T$ formula expressing "There is a proof of x from the axioms of Th and the true sentences."

Lemma (Δ_0-reflection$^+$)

For every $\phi(x) \in \Delta_0^{\mathcal{L}_T}$,

$$\mathrm{CT}_0 \vdash \forall x \left[\mathrm{Pr}^{T}_{\mathrm{UTB}}(\phi(\dot{x})) \to \phi(x)\right].$$

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Lemma (Bounding lemma)

For every $\phi(x) \in \Delta_0^{\mathcal{L}_T}$,

$$\mathrm{CT}_0 \vdash \mathrm{Pr}^{T}_{\mathrm{UTB}^-}(\exists v \phi(v)) \rightarrow \exists y \mathrm{Pr}^{T}_{\mathrm{UTB}^-}(\exists v < \underline{y} \phi(v)).$$

The most useful property of CT_0

Suppose $(\mathcal{M}, T) \models \mathsf{CT}^-[\mathsf{EA}]$ and $d \in M$.

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Obviously if $(\mathcal{M}, T) \models \mathrm{CT}^-$, then $(\mathcal{M}, T_d) \models \mathrm{CT}^-(d)$.

Theorem (Essentially Wcisło)

Suppose $(\mathcal{M}, T) \models \mathrm{CT}_0$. *Then for every* d, $(\mathcal{M}, T_d) \models \mathrm{Ind}(\mathcal{L}_T)$.

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Fix $\Delta_0^{\mathcal{L}_T}$ formula $\phi(x)$.

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Fix $\Delta_0^{\mathcal{L}_T}$ formula $\phi(x)$. Fix $(\mathcal{M}, T) \models \mathrm{CT}_0$, $a \in M$ and assume $\mathrm{Pr}^T_{\mathrm{UTB}}(\phi(\dot{x}))$. Since $\phi(x)$ is bounded there exists $b \in M$ such that

$$(\mathcal{M}, T) \models \phi(a) \iff \exists(\mathcal{N}, T') \supseteq_e (\mathcal{M}, T\restriction_b) \ \ (\mathcal{N}, T') \models \phi(a).$$

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Fix $\Delta_0^{\mathcal{L}_T}$ formula $\phi(x)$. Fix $(\mathcal{M}, T) \models \mathrm{CT}_0$, $a \in M$ and assume $\mathrm{Pr}^T_{\mathrm{UTB}}(\phi(\dot{x}))$. Since $\phi(x)$ is bounded there exists $b \in M$ such that

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Bounding

Fix $\phi(x)$ and $(\mathcal{M}, T) \models \mathrm{CT}_0$.

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$$\mathrm{Th} := F \cup \{\forall v < \underline{a}\neg\phi(v) \ \mid \ a \in M\}.$$

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1. $(\mathcal{M}, T'\restriction_M) \models \forall y \neg \phi(y)$.
2. $(\mathcal{M}, T'\restriction_M) \models \mathrm{Ind}(\mathcal{L}_T)$.

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Let $(\mathcal{N}, T') \models \mathrm{Th}$ be a model of Th (which exists by ACT). Consider $(\mathcal{M}, T'\restriction_M)$. Then

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So by induction, $(\mathcal{M}, T' \restriction_M) \models \mathrm{Con}_{F \cup \forall y \neg \phi(y)}$.

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So by induction, $(\mathcal{M}, T'\restriction_M) \models \mathrm{Con}_{F\cup\forall y\neg\phi(y)}$. Since F was arbitrary, we conclude that $\neg\mathrm{Pr}^{T}_{\mathrm{UTB}^-}(\exists y\phi(y))$.

Some bibliography

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Thank you for your attention.