$(\mathcal{L}, n)$-Models: Producing Weaknesses of Arithmetic

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Introduction

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The purpose of this talk is to introduce the concept $(\mathcal{L}, n)$-*models* which provide a flexible framework for proving independence from PA of true “mathematical” Π^0_1 statements. In particular, statements not concerning fast growing functions. This idea was originally due to Shelah ([1]) and, independently in a slightly different form, Kripke (unpublished). We mostly follow Shelah, though with some modifications and strengthenings of his original set up.

Partial Structures

Let's fix a finite first order language $\mathcal{L}$. A *partial* $\mathcal{L}$ *structure* is a set with interpretations for constants, relations and function symbols from $\mathcal{L}$ defined on it in the usual way except that functions can be partial.

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Let $\mathcal{L}$ be the language of PA, potentially enriched with some extra relation symbols. Let $n \in \omega$ (possibly non standard) the structure $\mathcal{M}_n$ is the structure whose universe is $n = \{0, 1, ..., n-1\}$ with $+$, $\times$ etc defined in the natural way.

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For instance, $\mathcal{M}_6 \models 1 + 1 = 2$ but $\mathcal{M}_6$ cannot say anything about the term $5 + 3$. We treat such terms as being syntactically incorrect.

$(\mathcal{L}, n)$-Models

Given two partial $\mathcal{L}$ structures $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \subseteq \mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$ in the normal sense and for each function symbol f in $\mathcal{L}$ or arity k (say), $f^{\mathcal{B}} \upharpoonright [\mathcal{A}]^k$ is total. In other words, $\mathcal{B}$ closes functions under $\mathcal{A}$.

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Example (Key Example Continued)

Let $\mathcal{L}$ be as before and let $n > m^2$. Then $\mathcal{M}_m \subseteq \mathcal{M}_n$ since for all $k, l < m$, $kl < n$.

$(\mathcal{L}, n)$-Models

Fix a natural number n. The following definition is the main character of the talk.

Definition ($(\mathcal{L}, n)$-Model)

An $(\mathcal{L}, n)$-*model* $\vec{\mathcal{A}} = \langle \mathcal{A}_0, ..., \mathcal{A}_{n-1} \rangle$ is a sequence of partial $\mathcal{L}$-structures of length n so that for all $i < n - 1$ $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$.

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Example (Key Example Continued Again)

Let $\vec{m} = m_0 < m_1 < ... < m_{n-1}$ be a sequence of natural numbers so that for all $i < n - 1$, $m_i^2 < m_{i+1}$. The associated $(\mathcal{L}, n)$-model is $\vec{M}_{\vec{m}} = \langle \mathcal{M}_{m_0}, ..., \mathcal{M}_{m_{n-1}} \rangle$. We call such a model *square increasing*.

Fulfillment, Part I

Definition (Fulfillment)

Let $\varphi(\vec{x})$ be an $\mathcal{L}$ formula, $\mathcal{A}$ an $(\mathcal{L}, n)$-model and $\vec{a}$ a tuple of elements of the same arity as $\vec{x}$, all belonging to some $\mathcal{A}_i$ for $i + dp(\varphi) < n - 1$ with i least with every term $t(\vec{x})$ appearing in φ is so that $t(\vec{a})$ is defined in $\mathcal{A}_{i+1}$. We define recursively $\mathcal{A} \models^* \varphi(\vec{a})$ (read as $\mathcal{A}$ *fulfills* $\varphi(\vec{a})$).

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1. If φ is atomic, then $\mathcal{A} \models^* \varphi(\vec{a})$ if and only if $\mathcal{A}_{n-1} \models \varphi(\vec{a})$.
2. If $\varphi := \psi_1 \wedge \psi_2$, then $\mathcal{A} \models^* \varphi(\vec{a})$ if and only if $\mathcal{A} \models^* \psi_1(\vec{a})$ and $\mathcal{A} \models^* \psi_2(\vec{a})$ and the same for $\vee$ and $\neg$.
3. If $\varphi := \exists y \psi(y, \vec{x})$, then $\mathcal{A} \models^* \varphi(\vec{a})$ if and only if there is a $b \in \mathcal{A}_{i+1}$ and $\mathcal{A} \models^* \psi(b, \vec{a})$
4. If $\varphi := \forall y \psi(y, \vec{x})$, then $\mathcal{A} \models^* \varphi(\vec{a})$ if and only if for all $j \in [0, n - dp(\psi)]$, and all $b \in \mathcal{A}_j$ we have that $\mathcal{A} \models^* \psi(b, \vec{a})$.

Fulfillment, Part II

Roughly speaking, we think of $\vec{\mathcal{A}}$ as some approximation to a bigger (non partial) structure we are trying to build. If $\vec{A} \models^* \varphi(\bar{a})$ represents our "best guess" at the n^{th} stage of the construction of what will be true in the limit.

Fulfillment, Part II

Roughly speaking, we think of $\vec{\mathcal{A}}$ as some approximation to a bigger (non partial) structure we are trying to build. If $\vec{A} \models^* \varphi(\bar{a})$ represents our "best guess" at the n^{th} stage of the construction of what will be true in the limit.

Example (Key Example Continued Once More)

Let $\vec{\mathcal{M}}_{\vec{m}}$ be a square increasing model. Then $\vec{\mathcal{M}}_{\vec{m}}$ fulfills PA minus induction, plus "$<$ is a linear order with no greatest element".

Fulfillment, Part III

Even though $\models^*$ doesn't seem exactly like a satisfaction relation, it comes with a completeness theorem.

Theorem (The Completeness Theorem for $(\mathcal{L}, n)$-models)

(PA) For any $\mathcal{L}$-sentence φ, $\vdash \varphi$ if and only if for all sufficiently large n and all $(\mathcal{L}, n)$-models $\vec{A}$, $\vec{A} \models^* \varphi$.

Proving the completeness theorem primarily boils down to induction on formulae, working through the definitions. However, there is one catch. Since PA can only quantify over finite objects, we need to find a way to "finitize" the theory of $(\mathcal{L}, n)$-models. This justifies the following lemma which turns out to be the key ingredient for both the completeness theorem and, later we'll see, eliminating fast growth of functions in independence.

Fulfillment, Part IV

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Lemma (The Finite Model Lemma)

There is a primitive recursive function $f(i, k, \mathcal{L})$ so that for any $\mathcal{L}$-sentence φ and any $(\mathcal{L}, n)$-model $\vec{\mathcal{A}}$ there is an $(\mathcal{L}, n)$-model $\vec{\mathcal{B}}$ so that any subformula ψ of φ $\mathcal{B} \models^* \psi$ if and only if $\vec{\mathcal{A}} \models^* \psi$ and $\mathcal{B}_0$ has cardinality $|\mathcal{L}| = f(0, |\varphi|, \mathcal{L})$ and $\mathcal{B}_{i+1}$ has cardinality at most $f(i+1, |\varphi|, \mathcal{L})$ and $\mathcal{B}_i \subseteq \mathcal{A}_i$ (as a subset, not a substructure) for each $i < n$. Moreover, given φ, $\mathcal{L}$ and $\vec{\mathcal{A}}$, the procedure for producing $\mathcal{B}$ is computable.

Fulfillment, Part V

In the contrapositive the completeness theorem states that a sentence φ is consistent if and only if for all sufficiently large n there is an $(\mathcal{L}, n)$-model which fulfills it. Consequently we obtain in PA a new characterization of $\mathrm{con}(\mathrm{PA})$ which is the main tool in applications of PA.

Corollary

(PA) For any model $\mathcal{M} \models \mathrm{PA}$, $\mathcal{M} \models$ *"*PA *is consistent" if and only if* $\mathcal{M} \models$ *"For all finite* $\Phi \subseteq \mathrm{PA}$, *and all sufficiently large n there is a (finite)* $(\mathcal{L}, n)$*-model which fulfills* Φ*".*

An Application: Bounded Colorings

Using these tools we can now present our example of a true but unprovable Π^0_1 sentence. The idea is to use a Paris-Harrington type statement about colorings on $(\mathcal{L}, n)$-models and use the finite model lemma to bound the existential quantifier.

An Application: Bounded Colorings

Definition (Bounded Colorings)

Let r be a natural number. A *bounded coloring in r colors* is a function F with domain a set of $(\mathcal{L}, n)$-models (n fixed) and range $r = \{0, ..., r-1\}$ so that

1. F is invariant under isomorphism: if $\vec{\mathcal{A}}_0 \cong \vec{\mathcal{A}}_1$ then $F(\vec{\mathcal{A}}_0) = F(\vec{\mathcal{A}}_1)$ and
2. F is *weakly hereditary*: given any linearly-ordered collection of partial $\mathcal{L}$ structures of length $k > n$ $\vec{\mathcal{A}} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq ... \subseteq \mathcal{A}_{k-1}$ so that F is defined on all n-length subtuples of $\vec{\mathcal{A}}$, there is a sentence φ of syntactic length at most n so that $\vec{\mathcal{A}} \models^* \varphi$ and if $\vec{\mathcal{B}}$ is the procedure from the finite model lemma for $\mathcal{A}$ and φ then F applied to any subtuple of the $\mathcal{A}$'s of length n is already determined by F applied to the corresponding $\mathcal{B}$'s.

We say that a bounded coloring is *on* some finite number N if the domain of the coloring is the set of codes of $(\mathcal{L}, n)$-models of size less than N for some fixed coding.

An Application: Bounded Colorings

The *Bounded Coloring Principle* (BCP) states that for all n, r, $\mathcal{L}$ and k if F is a bounded coloring on $N = kf(k, n, \mathcal{L}) + 1$ then there is a $\subseteq$-linearly ordered set of partial $\mathcal{L}$-structures of size k $H = \{\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq ... \subseteq \mathcal{A}_{k-1}\}$ so that $|\mathcal{A}_0| \leq k$ and F is *homogeneous* on H: for any n-tuple of elements from H, $\vec{\mathcal{A}}$, $F(\vec{\mathcal{A}})$ is defined and F assigns all such tuples the same color. Note that this is Π^0_1.

Requiring $\mathcal{A}_0$ to have cardinality less than or equal to k plays the exact same role as requiring homogeneous sets to be relatively large plays in the original Paris-Harrington statement.

An Application: Bounded Colorings

Theorem

(S.) The bounded coloring principle is true in $\mathbb{N}$ *but over* PA *it implies* $\mathrm{con}(\mathrm{PA})$*. In particular, it is independent of* PA*.*

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The proof of this theorem boils down to three lemmas. The first states that BCP is equivalent over PA to its Π^0_2 version, where the primitive recursive bound is removed. The second states that this statement is true in $\mathbb{N}$ and the third states that using BCP suffices to find, for any sufficiently large $n < \omega$ $(\mathcal{L}, n)$-models of any finite fragment of PA desired. By the corollary to the completeness theorem, this implies $\mathrm{con}(\mathrm{PA})$.

An Application: Bounded Colorings

Lemma

(PA) BCP *is equivalent to the seemingly weaker statement* BCP$'$*: for all n, r, $\mathcal{L}$ and k there is an N so that if F is a bounded coloring on N then there is a $\subseteq$-linearly ordered set of partial $\mathcal{L}$-structures of size k on which F is homogeneous and the least such model has cardinality at most k.*

Lemma

$\mathbb{N} \models$ BCP$'$ *and hence* BCP *is true.*

Lemma

(PA) Assume BCP*, let $\varphi(\bar{x})$ be a formula in the language $\mathcal{L}$ of* PA*. Then for any sufficiently large n there is a square-increasing $(\mathcal{L}, n)$ model $\vec{\mathcal{M}}_{\vec{m}} \models^* LNP(\varphi)$, where $LNP(\varphi)$ is the least number principle for φ.*

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Proof.

Fix n, r, $\mathcal{L}$ and k suppose we have a bounded coloring F on some N which witnesses BCP$'$. Let $H = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq ... \mathcal{A}_{k-1}$ be homogeneous. This is a linearly ordered set of models so by the definition of bounded coloring we can apply the finite model lemma to find a collection of "small" models $H' = \mathcal{A}_0 \subseteq ... \subseteq \mathcal{B}_{k-1}$ which are also homogenous for the coloring. Pushing forward isomorphically these structures onto an initial segment of the natural numbers and using isomorphism invariance then allows one to find such a homogenous set of codes all less than $kf(k, n, \mathcal{L}) + 1$.

□

Thank You!

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