Applications of Model Theory to Families of Integer Sequences

Arseniy (Senia) Sheydvasser

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Acknowledgements:

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- Special thanks to the organizers of SUMRY 2017, to Stefan Steinerberger for introducing me to the problem, and to Nathan Fox and Kevin O'Bryant for valuable insight and examples.

General Setting:

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Question

What if instead we have an algorithm $\mathcal{A}$ so that it can accept as inputs non-standard integers n and k; what information does this give us about the family S_n?

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def A(n,k):
    i = 0

    while(i < n):
        if i == k:
            print True

    print False
```

Bad

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def A(n,k):
    print (k >= 0) and (k < n)
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- Thus, we shall insist that the algorithm halt in finite time.

First Example:

Definition (Hofstader, "Gödel, Escher, Bach")

The *Hofstader Q-sequence* is defined by
$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$ and initial conditions
$Q(1) = 1$ and $Q(2) = 1$.

- The first few terms are
  $1, 1, 2, 3, 3, 4, 5, 5, 6, 6, 6, 8, 8, 8, 10, 9, 10, 11, 11 \ldots$
- Open question whether this sequence is infinite or not.

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- Open question whether this sequence is infinite or not.

Definition (Fox 2018)

Define the sequence Q_r by the recurrence relation $Q_r(n) = Q_r(n - Q_r(n-1)) + Q_r(n - Q_r(n-2))$ and initial conditions $Q_r(1) = 1, Q_r(2) = 2, \dots Q_r(r) = r$.

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- Recall that $Q_N(n) = Q_N(n - Q_N(n-1)) + Q_N(n - Q_N(n-2))$.
- We can keep computing in this way until we hit the $(N+29)$-nd term.

$$\begin{aligned} Q_N = & 1, 2, 3, \ldots N-1, N, 3, N+1, N+2, 5, N+3, 6, 7, N+4, \\ & N+6, 10, 8, N+6, N+10, 12, N+7, 14, N+12, 11, \\ & N+11, N+15, 16, 13, 17, 15, N+14, 20, 20, 2N+8. \end{aligned}$$

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$$\begin{aligned} Q_N(N+29) &= Q_N(N+29-Q_N(N+28)) + Q_N(N+29-Q_N(N+27)) \\ &= Q_N(N+29-2N-8) + Q_N(N+29-20) \\ &= Q_N(21-N) + Q_N(N+9) \text{ ☹} \end{aligned}$$

Non-standard Algorithm:

- We can now write down what $\mathcal{A}(N, k)$ does if N is non-standard:
  1. If $k < 1$, return F.
  2. If $k \leq N$, return T.
  3. Otherwise, compute the 28 terms on the previous slide.
  4. If k is one of the terms, return T. Otherwise, return F.

Logical Consequences:

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- What we have therefore proved over the hyper-naturals is that for all N sufficiently large, the sequence Q_N has $N+28$ terms.
- This can be phrased in a first-order way, and so we conclude that for all *naturals* N sufficiently large, the sequence Q_N has $N+28$ terms!
- The bad news is that this isn't exciting: there is a completely elementary proof of an even stronger result in *A New Approach to the Hofstadter Q-Recurrence*, Fox 2018.

Second Example:

Definition

A *Sidon set* is a set $S \subset \mathbb{N}$ such that $\forall w, x, y, z \in S$, $w + x = y + z$ if and only if $\{w, x\} = \{y, z\}$.

An (A, B)-*form Sidon set* is a set $S \subset \mathbb{N}$ such $\forall w, x, y, z \in S$, $Aw + Bx = Ay + Bz$ if and only if $\{w, x\} = \{y, z\}$.

The *greedy* (A, B)-*form Sidon sequence* $S_{A,B}$ is the sequence starting with 0, such that each subsequent term is the next smallest term such that the sequence is an (A, B)-form Sidon set.

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$$S_{1,1} = 0, 1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97 \ldots$$
$$S_{1,2} = 0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69 \ldots$$
$$S_{1,3} = 0, 1, 2, 9, 10, 11, 18, 19, 20, 81, 82, 83 \ldots$$

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- Because the extension to hyper-naturals preserves first-order statements, each term t in $S_{1,N}$ is the smallest such that for all $w, x, y, z \in S_{1,N} \cap [1, t]$, $w + xN = y + zN$ if and only if $\{w, x\} = \{y, z\}$.

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- Thus, at each step, we need to check if $t = x + (y - z)N$ or $t = x + (y - z)/N$ for $x, y, z \in S_{1,N} \cap [1, t - 1]$. This can be done recursively.

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S_{1,N} = [1, 2N^2 + N] = & 0, 1, 2, \ldots N - 1, \\
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Non-standard Algorithm:

- In this recursive fashion, we can prove that

$$x \in S_{1,N} \Leftrightarrow \exists T \in {}^*\mathbb{N} \text{ s.t. } x = \sum_{l=0}^{T} a_l N^{2l},\ 0 \leq a_l < N.$$

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- Unfortunately, it is a theorem in the folklore (due to Kevin O'Bryant) that this is true for all $N \geq 2$, and this is again proved by elementary means.

Third Example:

Definition

An *Ulam sequence* is an increasing sequence $U(a, b)$ of integers defined by

- $u_0 = a$, $u_1 = b$, and
- u_k (for $k > 1$) is the smallest integer that can be written as the sum of two distinct smaller terms u_m, u_n in exactly one way.

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Examples:

- $U(1, 2) : 1, 2, 3, 4, 6, 8, 11, 13, 16, 18 \ldots$
- $U(1, 3) : 1, 3, 4, 5, 6, 8, 10, 12, 17, 21 \ldots$
- $U(2, 3) : 2, 3, 5, 7, 8, 9, 13, 14, 18, 19 \ldots$

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- Introduced in 1964 by Ulam, who wanted to understand their growth properties.
- Despite their apparent simplicity, almost nothing is known about Ulam sequences.

Rigidity of the $U(1, n)$ Sequences:

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$U(1,5)$:	1,	5,	6,	7,	8,	9,	10,	12,	20,	22,	23,	24 ...
$U(1,6)$:	1,	6,	7,	8,	9,	10,	11,	12,	14,	24,	26,	27 ...

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U(1,n): & \boxed{1}, & \boxed{n,\dots 2n}, & \boxed{2n+2}, & \boxed{4n}, & \boxed{4n+2,\dots 5n-1}, & \boxed{5n+1}\dots
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The Rigidity Conjecture:

Conjecture

There exists a positive integer N and integer coefficients m_i, p_i, k_i, r_i such that for all $n \geq N$,

$$U(1, n) = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i].$$

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- This is very well supported numerically (more on that later).
- Note that the coefficients don't depend on n, and can be calculated using any two consecutive Ulam sequences.
- Effectively, the conjecture says that once you have seen two (sufficiently large) Ulam sequences $U(1, n)$, you have seen them all.

Next Best Result:

Theorem (Weak Rigidity Theorem)

There exist integer coefficients m_i, p_i, k_i, r_i *such that for every* $C > 0$, *there exists a positive integer* N *such that for all* $n \geq N$,

$$U(1, n) \cap [1, Cn] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, Cn].$$

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- We shall prove this by making use of the machinery we have developed.

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- Fix a standard natural $C > 0$.
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- To make this formal, argue by induction on C and i.
- We thus construct m_i, p_i, k_i, r_i such that

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- In fact, we produce an algorithm capable of constructing these coefficients up to C!

Consequences:

- We have therefore proved over the hyper-naturals that for all sufficiently large N,

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- It follows that the same is true over the naturals, proving the theorem.

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- This is the first example of an algorithm where we needed to restrict the domain.
- Also the first example where the theorem is not known independently.
- The proof is vaguely non-constructive, but we can make the result completely constructive.

Growth Rate of Coefficients:

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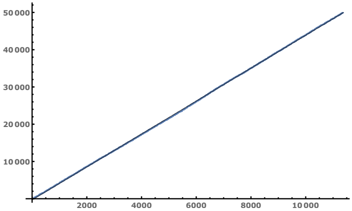
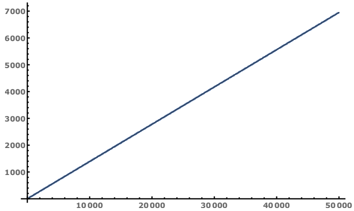
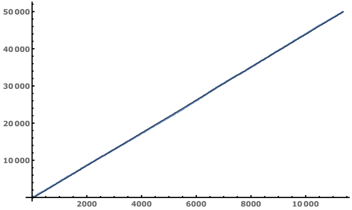


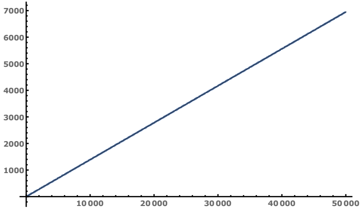
i vs. m_i



m_i vs. p_i



i vs. k_i



k_i vs. r_i

Growth Rate of Coefficients:

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- In particular, for all i such that $k_i \leq 50000$, we find that

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- This is useful, because we can use this statement about the growth rate to make the weak rigidity theorem effective.

Effective Estimates:

Theorem

Suppose that for some positive integer M,

$$U(1,N_0)\cap[1,k_M N_0+r_M+1]=\bigsqcup_{i=1}^{M}[m_i N_0+p_i,k_i N_0+r_i]$$

where for some $B,\epsilon>0$, $|p_i-m_iB|,|r_i-k_iB|<\epsilon$, and $N_0>4(1+\epsilon)-B$. Then for all $N>N_0$,

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- The proof proceeds by induction over M and N.

Effective Estimates:

- This yields an algorithm for computing a bound N_0 that works in the statement of the weak rigidity theorem.

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  1. Calculate coefficients m_i, p_i, k_i, r_i.
  2. Do a linear regression to fit the best value of B to the computed coefficients. Calculate the corresponding maximum error ϵ.
  3. Compute $N_0' = \lceil 4(1+\epsilon) - B \rceil$.
  4. Use the coefficients m_i, p_i, k_i, r_i to predict the first CN terms of $U(1, N_0'), U(1, N_0' \pm 1) \ldots$.
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  5. Halt once you find the smallest N_0 such that $U(1, n)$ matches the prediction for all $N_0 \leq n \leq \max\{N_0, N_0'\}$.
- Using this, we prove that for all $n \geq 4$,

$$U(1, n) \cap [1, 50000n] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, 50000n].$$

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- Are there any general theorems that we can prove about integer sequences coming from an algorithm extendable to non-standard inputs?
- If we can prove some restrictions on the growth rate of the sequences, does this tells us something, like it does for the Ulam sequence?
- Does there exist any $\epsilon > 0$ such that there are integer coefficients m_i, p_i, k_i, r_i so that for any $C > 0$, there is an $N > 0$ such that for all $n \geq N$,

$$U(1, n) \cap [1, Cn^{1+\epsilon}] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, Cn^{1+\epsilon}]?$$