A Model–theoretic Framework for Conservation Results in Arithmetic

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Introduction

- **General Question**: To obtain 'nice' characterizations of the class $Th_\Gamma(T)$ of the Γ–consequences of an arithmetical theory.

  T = Σ_n–induction $\mathbf{I}\Sigma_n$, Σ_n–collection $\mathbf{B}\Sigma_n$

  Γ = class of formulas in the arithmetic hierarchy

$$\Delta_0 \subset \begin{matrix}\Sigma_1 \\ \\ \Pi_1\end{matrix} \subset \begin{matrix}\Sigma_2 \\ \\ \Pi_2\end{matrix} \subset \begin{matrix}\Sigma_3 \\ \\ \Pi_3\end{matrix} \subset \dots$$

- Equivalently, to find a 'nice' theory $T' \subset T$ satisfying
  - T' is Γ–axiomatizable, and
  - T is Γ–conservative over T'.

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But, what does it mean to be a 'nice characterization'?

- **R. Kaye**, *Using Herbrand–type Theorems to Separate Strong Fragments*, in Oxford Logic Guides 23 (1993):

  "... the most interesting fragments of arithmetics are the *natural* fragments (...) which are typically interesting because of their elegant axiomatizations and because of their combinatorial and number–theorteic consequences."

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  "Theorems 2.1 and 2.2 give natural axiomatizations of the Σ_{n+2} and Σ_{n+1} consequences of $\mathbf{I}\Sigma_n$. These axiomatizations are **especially nice** in that they themselves have the form of induction axioms."

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Introduction

$\mathbf{I}\Sigma_n$ and $\mathbf{B}\Sigma_n$ are given by axiom schemes:

- $\mathbf{I}\Sigma_n$ is P^- together with

  $$\forall \bar{v}\,(\varphi(0,\bar{v}) \land \forall x\,(\varphi(x,\bar{v}) \rightarrow \varphi(x+1,\bar{v})) \rightarrow \forall x\,\varphi(x,\bar{v}))$$

  where φ runs over Σ_n

- $\mathbf{B}\Sigma_n$ is $\mathbf{I}\Delta_0$ together with

  $$\forall \bar{v}\,(\forall x\,\exists y\,\varphi(x,y,\bar{v}) \rightarrow \forall z\,\exists u\,\forall x \leq z\,\exists y \leq u\,\varphi(x,y,\bar{v}))$$

  where φ runs over Σ_n

- Parameters $\bar{v}$ are allowed to occur in φ

Introduction

General Question: To find natural restrictions on an axiom scheme to obtain axiomatizations of its Σ_k/Π_k–consequences.

Axiom Scheme	Γ	Restriction
$\mathbf{I}\Sigma_n, \mathbf{B}\Sigma_n$	Π_{n+1}	Inference rule version
$\mathbf{I}\Sigma_n, \mathbf{B}\Sigma_n$	Σ_{n+2}	Parameter free version
$\mathbf{I}\Sigma_n, \mathbf{B}\Sigma_n$	Σ_{n+1}	??*

(*): Kaye–Paris–Dimitracopoulos [JSL'88] and Beklemishev–Visser [APAL'05] obtained axiomatizations of the Σ_{n+1}–consequences of $\mathbf{I}\Sigma_n$

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Outline

Part I: We develop a model–theoretic framework for obtaining conservation results, based on an arithmetic version of the notion of an existentially closed model.

This method allows for characterizing the Π_{n+1} and Σ_{n+2}–consequences of each axiom scheme enjoying certain *logical/syntactical* properties.

Part II: We introduce axiom schemes restricted "up to" definable elements and show that this restriction captures the Σ_{n+1}–consequences of $\mathbf{I}\Sigma_n$ and $\mathbf{B}\Sigma_n$.

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Part I: Two inspiring works

Beklemishev's work ([APAL'97],[AML'98],[JSL'03])

- To reduce induction/collection *schemes* to a version of induction/collection *rule*, typically by *cut–elimination*.
- To derive conservation results for parameter free schemes.

Avigad's work ([APAL'02])

- To use the so–called *Herbrand saturated* models as a uniform method for proving conservation results.
- Such models do exist for *universal* theories
- "*Skolemization*"

A general definition

Fix a first–order language L and a new k–ary predicate symbol P.

- A *k-scheme* **E** is a sentence of $L \cup \{P\}$ of the form $A \rightarrow B$.
- $\mathbf{E}_\varphi$ denotes the L–formula obtained by substituting $\varphi(t_1, \ldots, t_k, v)$ for each atomic subformula of **E** of the form $P(t_1, \ldots, t_k)$, where t_i are L–terms.

Examples

- Induction is a 1–scheme for:
  - $A \equiv P(0) \wedge \forall x\,(P(x) \rightarrow P(x+1))$
  - $B \equiv \forall x\, P(x)$
- Collection is a 2–scheme for:
  - $A \equiv \forall x\, \exists y\, P(x, y)$
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Theories associated to a k–scheme

- Parametric version:

$$\mathbf{E}\Gamma \equiv \{\forall v\,(\mathbf{A}_\varphi(v) \rightarrow \mathbf{B}_\varphi(v))\ :\ \varphi(\bar{x}, v) \in \Gamma\}$$

- Uniform or separated parameter version:

$$\mathbf{UE}\Gamma \equiv \{\forall v\,\mathbf{A}_\varphi(v) \rightarrow \forall v\,\mathbf{B}_\varphi(v)\ :\ \varphi(\bar{x}, v) \in \Gamma\}$$

- Parameter free version:

$$\mathbf{E}\Gamma^- \equiv \{\mathbf{A}_\varphi \rightarrow \mathbf{B}_\varphi\ :\ \varphi(\bar{x}) \in \Gamma^-\}$$

- Inference rule version:
  $T + \Gamma$–$\mathbf{E}$R is the closure of T under nested applications of the $\mathbf{E}$–rule restricted to Γ–formulas:

$$\mathbf{E}\text{–R:}\qquad \frac{\forall v\,(\mathbf{A}_\varphi(v))}{\forall v\,(\mathbf{B}_\varphi(v))}$$

$\exists\Pi$–closed models

Fix $\Pi \subseteq Form(L)$ containing all atomic formulas, closed under $\wedge$, $\vee$, term substitution and subformulas, and satisfying $\neg\Pi \subseteq \exists\Pi$.

Definition

Let $\mathfrak{A}$ be an L–structure. We say that $\mathfrak{A}$ is $\exists\Pi$–closed for T if,

1. $\mathfrak{A} \models T$, and
2. for each $\mathfrak{B} \models T$, $\quad \mathfrak{A} \prec_{\Pi} \mathfrak{B} \Longrightarrow \mathfrak{A} \prec_{\exists\Pi} \mathfrak{B}$.

- Existentially closed models of arithmetic theories were studied in the early 70's: Hirschfeld–Wheeler('75).
- Applications: Dimitracopoulos('89), Adamowicz–Bigorajska('01), Beckmann('04), Adamowicz–Kołodziejczyk('07)
- Used to prove conservativity: Zambella-Visser('96), Avigad('02).

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$\exists\Pi$–closed models. Properties

Lemma (Existence)

Suppose $T \subseteq \forall\exists\Pi$. Each $\mathfrak{A} \models T$ has a Π–elementary extension which is $\exists\Pi$–closed for T.

Lemma (Niceness)

Suppose $\mathfrak{A}$ is $\exists\Pi$–closed for T. Then

$$T + D_{\Pi}(\mathfrak{A}) \vdash D_{\forall\neg\Pi}(\mathfrak{A})$$

where $D_{\Gamma}(\mathfrak{A})$ denotes the Γ–diagram of $\mathfrak{A}$.

Monotonic schemes

$\mathbf{E} = \mathbf{A} \rightarrow \mathbf{B}$ is T–monotonic over Π and Γ if, for each $\varphi(\bar{x}, v) \in \Gamma$ and $\theta(\bar{w}) \in \Pi$:

- Syntactical conditions

  (S1) $\theta(\bar{w}) \rightarrow \varphi(\bar{x}, v) \in \Gamma$

  (S2) $\mathbf{A}_\varphi \in \forall\neg\Pi$

  (S3) $T + \Gamma$–$\mathbf{E}$R is $\forall\exists\Pi$–axiomatizable

- Logical conditions

  (L1) $T \vdash (\theta \rightarrow \mathbf{A}_\varphi) \rightarrow \mathbf{A}_{\theta\rightarrow\varphi}$

  (L2) $T \vdash \mathbf{B}_{\theta\rightarrow\varphi} \rightarrow (\theta \rightarrow \mathbf{B}_\varphi)$

Remark: Induction and collection are $\mathbf{I}\Delta_0$–monotonic over Π_n,Σ_n.

Axiom scheme vs Inference rule

Lemma

Suppose $\mathbf{E}$ *is* T*–monotonic over* Π *and* Γ *and* $\mathfrak{A}$ *is* $\exists\Pi$*–closed model for* T*. Then*

$$\mathfrak{A} \models T + \Gamma\text{–}\mathbf{E}R \quad \Longrightarrow \quad \mathfrak{A} \models \mathbf{E}\Gamma$$

Proof: Assume $\mathfrak{A} \models \mathbf{A}_\varphi(a)$. By (S2) and "niceness", it follows that

$$T + D_\Pi(\mathfrak{A}) \vdash \mathbf{A}_\varphi(a)$$

There is $\theta(v, w) \in \Pi$ such that $\mathfrak{A} \models \theta(a, b)$ and

$$T \vdash \theta(v, w) \rightarrow \mathbf{A}_\varphi(v)$$

By (L1), $T \vdash \mathbf{A}_{\theta\rightarrow\varphi}$ and so, by (S1),(L2)

$$(T + \Gamma\text{–}\mathbf{E}\mathrm{R}) \vdash \theta(v, w) \rightarrow \mathbf{B}_\varphi(v)$$

Hence, $\mathfrak{A} \models \mathbf{B}_\varphi(a)$ since $\mathfrak{A} \models \theta(a, b)$. □

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A general conservation result

Theorem

Suppose $T \subseteq \forall\exists\Pi$ and $\mathbf{E}$ is a T–monotonic scheme over Π and Γ.

1. $T + \mathbf{E}\Gamma$ *is* $\forall\neg\Pi$*–conservative over* $T + \Gamma\text{–}\mathbf{E}R$.
2. $T + \mathbf{E}\Gamma$ *is* $\exists\forall\neg\Pi$*–conservative over* $T + \mathbf{UE}\Gamma$.
3. *If* $T + \mathbf{E}\Gamma^{-} \subseteq \forall\exists\Pi$ *and its extensions are closed under* $\Gamma\text{–}\mathbf{ER}$*, then* $T + \mathbf{E}\Gamma$ *is* $\exists\forall\neg\Pi$*–conservative over* $T + \mathbf{E}\Gamma^{-}$.

Proof: (1): Assume $\mathfrak{A} \models (T + \Gamma\text{–}\mathbf{E}R) + \neg\varphi$, where $\varphi \in \forall\neg\Pi$. By (S3) and "existence", there is $\mathfrak{A} \prec_{\Pi} \mathfrak{B}$ such that $\mathfrak{B}$ is $\exists\Pi$–closed for $T + \Gamma\text{–}\mathbf{E}R$. Hence, $\mathfrak{B} \models T + \mathbf{E}\Gamma + \neg\varphi$.

(2,3): Similar □

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Suppose $T \subseteq \forall\exists\Pi$ and $\mathbf{E}$ is a T–monotonic scheme over Π and Γ.

1. *$T + \mathbf{E}\Gamma$ is $\forall\neg\Pi$–conservative over $T + \Gamma\text{–}\mathbf{E}R$.*
2. *$T + \mathbf{E}\Gamma$ is $\exists\forall\neg\Pi$–conservative over $T + \mathbf{UE}\Gamma$.*
3. *If $T + \mathbf{E}\Gamma^- \subseteq \forall\exists\Pi$ and its extensions are closed under $\Gamma\text{–}\mathbf{ER}$, then $T + \mathbf{E}\Gamma$ is $\exists\forall\neg\Pi$–conservative over $T + \mathbf{E}\Gamma^-$.*

Proof: (1): Assume $\mathfrak{A} \models (T + \Gamma\text{–}\mathbf{E}\mathrm{R}) + \neg\varphi$, where $\varphi \in \forall\neg\Pi$. By (S3) and "existence", there is $\mathfrak{A} \prec_{\Pi} \mathfrak{B}$ such that $\mathfrak{B}$ is $\exists\Pi$–closed for $T + \Gamma\text{–}\mathbf{E}\mathrm{R}$. Hence, $\mathfrak{B} \models T + \mathbf{E}\Gamma + \neg\varphi$.

(2,3): Similar $\square$

Applications (I)

Language: $L = \{0, S, +, \cdot, \leq\}$ $\Pi = \Pi_n$, $\Gamma = \Sigma_n$

Schemes: Σ_n–induction, Σ_n–collection

- Induction and collection are $\mathbf{I}\Delta_0$–monotonic over Π_n and Σ_n.
- If $T \vdash \mathbf{I}\Sigma_n^-$, T is closed under Σ_n–induction rule.
- If $T \vdash \mathbf{B}\Sigma_n^-$, T is closed under Σ_n–collection rule.

Then, $\forall\neg\Pi = \Pi_{n+1}$ and $\exists\forall\neg\Pi = \Sigma_{n+2}$ and so...

Theorem ($n \geq 1$)

1. $Th_{\Pi_{n+1}}(\mathbf{I}\Sigma_n) \equiv \mathbf{I}\Delta_0 + \Sigma_n\text{–}IR$.
2. $Th_{\Pi_{n+1}}(\mathbf{B}\Sigma_n) \equiv \mathbf{I}\Delta_0 + \Sigma_n\text{–}CR \equiv \mathbf{I}\Sigma_{n-1}$.
3. $Th_{\Sigma_{n+2}}(\mathbf{I}\Sigma_n) \equiv \mathbf{I}\Sigma_n^-$.
4. $Th_{\Sigma_{n+2}}(\mathbf{B}\Sigma_n) \equiv \mathbf{B}\Sigma_n^-$.

Applications (II)

Language: $L = \{0, S, +, \cdot, \leq\}$ $\Pi = \Pi_n$, $\Gamma = \Sigma_n$

Scheme: Δ_n–induction

Theorem $(n \geq 1)$

1. $Th_{\Pi_{n+1}}(\mathbf{I}\Delta_n) \equiv \mathbf{I}\Delta_0 + \Delta_n\text{–}IR \equiv \mathbf{I}\Sigma_{n-1}$.
2. $Th_{\Sigma_{n+2}}(\mathbf{I}\Delta_n) \equiv \mathbf{UI}\Delta_n$.

- Beklemishev[JSL'03] proved it for $n = 1$ and posed as a pending question to extend it to $n > 1$.
- From Slaman's theorem it immediately follows that

$$\mathbf{UI}\Delta_n + exp \Longleftrightarrow \mathbf{B}\Sigma_n^- + exp$$

So, $\mathbf{I}\Delta_n^-$ is strictly weaker than $\mathbf{UI}\Delta_n$.

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Applications (III)

L = Buss's Bonded Arithmetic + $\{MSP, -\}$, $\Pi = \hat{\Pi}^b_i$, $\Gamma = \hat{\Sigma}^b_i$

Schemes: Σ^b_i–induction T^i_2, Σ^b_i–polyinduction S^i_2.

- Both schemes are *LIOpen*–monotonic over $\hat{\Pi}^b_i$ and $\hat{\Sigma}^b_i$.
- If $T \vdash T^{i.-}_2$ then T is closed under $\hat{\Sigma}^b_i$–induction rule.
- If $T \vdash S^{i,-}_2$ then T is closed under $\hat{\Sigma}^b_i$–polyinduction rule.

Theorem ($i \geq 1$)

1. T^i_2 *is* $\exists\forall\Sigma^b_i$*–conservative over* $T^{i,-}_2$.
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- This improves previous $\forall\Sigma^b_i$–conservativity obtained by Bloch.

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Part II: Axiom scheme "up to" definable elements

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- Collection

$$\forall \bar{v}\,(\forall x\,\exists y\,\varphi(x,y,\bar{v}) \rightarrow \forall z\,\exists u\,\forall x \leq z\,\exists y \leq u\,\varphi(x,y,\bar{v}))$$

Definition ("Up to" schemes)

1. $\mathbf{E}(\Gamma, A, B)$ denotes the $\mathbf{E}$–scheme up to elements in A restricted to Γ–formulas with parameters in B.

2. $\mathbf{E}(\Gamma^-, A)$ denotes the $\mathbf{E}$–scheme up to elements in A restricted to *parameter free* Γ–formulas.

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Definable and minimal elements

- a is Γ–definable in $\mathfrak{A}$ with parameters in $X \subseteq \mathfrak{A}$ if there are $\varphi(x, \bar{v}) \in \Gamma$ and $\bar{b} \in X$ satisfying

$$\mathfrak{A} \models \varphi(a, \bar{b}) \wedge \exists! x\, \varphi(x, \bar{b})$$

- a is Γ–minimal in $\mathfrak{A}$ with parameters in $X \subseteq \mathfrak{A}$ if there are $\varphi(x, \bar{v}) \in \Gamma$ and $\bar{b} \in X$ satisfying

$$\mathfrak{A} \models a = (\mu x)\,(\varphi(x, \bar{b}))$$

- $\mathcal{K}_n(\mathfrak{A}, X) = \{a \in \mathfrak{A} : a \text{ is } \Sigma_n\text{–definable with parameters in } X\}$
- $\mathcal{I}_n(\mathfrak{A}, X) = \{b \in \mathfrak{A} : \exists a \in \mathcal{K}_n(\mathfrak{A}, X) \text{ such that } b \leq a\}$

Expressing "$\forall x \in \mathcal{K}_n$" in the language of Aritmetic

- Suppose $\mathfrak{A} \models \mathbf{I}\Sigma_n^-$. For each $a \in \mathcal{K}_n(\mathfrak{A})$ there is b Π_{n-1}–minimal such that $a = (b)_0$.

$$\text{“}\forall x \in \mathcal{K}_n \, \Phi(x, \bar{v})\text{”}$$

$$\Updownarrow$$

$$\{\forall z, x \left(\left\{ \begin{array}{ll} z = (\mu t)\,(\delta(t)) \\ \wedge \quad x = (z)_0 \end{array} \right\} \rightarrow \Phi(x, \bar{v}) \right) \,:\, \delta(t) \in \Pi_{n-1}\}$$

$$\text{“}\forall x \in \mathcal{I}_n \, \Phi(x, \bar{v})\text{”}$$

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$$\{\forall z, x \left(\left\{ \begin{array}{rr} z = (\mu t)\,(\delta(t)) \\ \wedge \quad\quad x = (z)_0 \end{array} \right\} \rightarrow \Phi(x, \bar{v})\right) \,:\, \delta(t) \in \Pi_{n-1}\}$$

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$$\{\forall z, x \left(\left\{ \begin{array}{rr} z = (\mu t)\,(\delta(t)) \\ \wedge \quad\quad x \leq (z)_0 \end{array} \right\} \rightarrow \Phi(x, \bar{v})\right) \,:\, \delta(t) \in \Pi_{n-1}\}$$

A "nice" axiomatization of $Th_{\Sigma_{n+1}}(B\Sigma_n)$

Theorem ($n \geq 1$)

Over $\mathbf{I}\Sigma_{n-1}^{-}$ *the following theories are equivalent:*

1. $Th_{\Sigma_{n+1}}(\mathbf{B}\Sigma_n)$
2. $B(\Sigma_n^{-}, \mathcal{K}_n)$

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Theorem ($n \geq 1$)

Over $\mathbf{I}\Sigma_{n-1}^-$ *the following theories are equivalent:*

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2. $B(\Sigma_n^-, \mathcal{K}_n)$

Proof: ($1 \Longrightarrow 2$):

$$\text{“}\forall x \in \mathcal{K}_n\, \Phi(x)\text{”}$$

$$\Updownarrow$$

$$\{\exists z, x\, (\forall t\, \neg\delta(t) \vee (\left\{ \begin{array}{cc} z = (\mu t)\,(\delta(t)) \\ \wedge \quad x = (z)_0 \end{array} \right\} \wedge \Phi(x)))\ :\ \delta(t) \in \Pi_{n-1}\}$$

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Proof: (2⟹1): Assume $\mathfrak{A} \models B(\Sigma_n, \mathcal{K}_n, \mathcal{K}_n)$.

Case 1: $\mathcal{I}_n(\mathfrak{A}) = \mathfrak{A}$. Then, $\mathfrak{A} \models \mathbf{B}\Sigma_n^- \vdash Th_{\Sigma_{n+1}}(\mathbf{B}\Sigma_n)$.

Case 2: $\mathcal{I}_n(\mathfrak{A}) \neq \mathfrak{A}$. Then

- $\mathcal{I}_n(\mathfrak{A}) \models \mathbf{B}\Sigma_n^-$ (end–extension properties)
- $\mathcal{I}_n(\mathfrak{A}) \models Th_{\Pi_{n+1}}(\mathfrak{A})$, by $B(\Sigma_n^-, \mathcal{K}_n)$.

So, $\mathfrak{A} \models Th_{\Sigma_{n+1}}(\mathbf{B}\Sigma_n)$ □

A “nice” axiomatization of $Th_{\Sigma_{n+1}}(B\Sigma_n)$

Corollary ($n \geq 1$)

Let $\mathfrak{A}$ be a model of $\mathbf{I}\Sigma_{n-1}$. The following are equivalent:

1. $\mathfrak{A} \models Th_{\Sigma_{n+1}}(\mathbf{B}\Sigma_n)$
2. $\mathfrak{A} \models B(\Sigma_n^-, \mathcal{K}_n)$
3. $\mathfrak{A} \models \mathbf{L}\Delta_n^-$.
4. (+exp) *Every “locally increasing” Σ_n–definable function in $\mathfrak{A}$ is “globally increasing”.*

$$\begin{array}{lcl} \textit{Locally increasing} & \rightsquigarrow & \forall x\,(f(x) \leq f(x+1)) \\ \textit{Globally increasing} & \rightsquigarrow & \forall x, y\,(x \leq y \rightarrow f(x) \leq f(y)) \end{array}$$

What about $Th_{\Sigma_{n+1}}(\mathbf{I}\Sigma_n)$?

- **Goal**: To obtain an "up to" restriction on the Σ_n–induction scheme that captures its Σ_{n+1}–consequences.
- Does $I(\Sigma_n^-, \mathcal{K}_n)$ axiomatize the $Th_{\Sigma_{n+1}}(\mathbf{I}\Sigma_n)$? NO Because...
  - Over $\mathbf{I}\Sigma_{n-1}^-$ it holds that $I(\Sigma_n^-, \mathcal{K}_n) \equiv \mathbf{I}\Pi_n^-$.
  - $\mathbf{I}\Pi_n^-$ is strictly weaker than $Th_{\Sigma_{n+1}}(\mathbf{I}\Sigma_n)$.
- **Question**: How can we extend $I(\Sigma_n^-, \mathcal{K}_n)$ to capture the Σ_{n+1}–consequences of $\mathbf{I}\Sigma_n$?

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Iterating Σ_n–definability: $\mathcal{H}_n^\infty$

Definition

- $\mathcal{H}_n^0(\mathfrak{A}) = \mathcal{I}_n(\mathfrak{A})$
- For each k, $\mathcal{H}_n^{k+1}(\mathfrak{A}) = \mathcal{I}_n(\mathfrak{A}, \mathcal{H}_n^k(\mathfrak{A}))$
- $\mathcal{H}_n^\infty(\mathfrak{A}) = \bigcup_{k\geq 0} \mathcal{H}_n^k(\mathfrak{A})$

Lemma

1. *If* $\mathfrak{A} \models \mathbf{I}\Sigma_{n-1}$ *then* $\mathcal{H}_n^\infty(\mathfrak{A}) \prec_n^e \mathfrak{A}$.
2. $\mathcal{H}_n^\infty(\mathfrak{A})$ *is the least initial segment of* $\mathfrak{A}$ *containing all the* Σ_n*–definable elements and closed under* Σ_n*–definability.*

Expressing "$\forall x \in \mathcal{H}_n^{\infty}$" in the language of Aritmetic

- Suppose $\mathfrak{A} \models \mathbf{I}\Sigma_{n-1}$. For each $a \in \mathcal{K}_n(\mathfrak{A}, X)$ there is b Π_{n-1}–minimal (with parameters in X) such that $a = (b)_0$.

$$\text{“}\forall x \in \mathcal{H}_n^k \, \Phi(x, \bar{v})\text{”}$$

$$\Updownarrow$$

$$\forall \bar{a}, \bar{b} \left(\left\{ \begin{array}{lcl} a_0 = (\mu x)\,(\delta_0(x)) & \wedge & b_0 \leq a_0 \\ a_1 = (\mu x)\,(\delta_1(x, b_0)) & \wedge & b_1 \leq a_1 \\ & \vdots & \\ a_k = (\mu x)\,(\delta_k(x, b_{k-1})) & \wedge & b_k \leq a_k \end{array} \right\} \rightarrow \Phi(b_k, \bar{v}) \right)$$

where $\delta_0, \ldots, \delta_k$ run over Π_{n-1}.

A "nice" axiomatization of $Th_{\Sigma_{n+1}}(\mathbf{I}\Sigma_n)$

Theorem ($n \geq 1$)

Over $\mathbf{I}\Sigma_{n-1}$ the following theories are equivalent:

1. $Th_{\Sigma_{n+1}}(\mathbf{I}\Sigma_n)$
2. $I(\Sigma_n^-, \mathcal{H}_n^\infty)$
3. $I(\Sigma_n, \mathcal{H}_n^\infty, \mathcal{H}_n^\infty)$

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Proof: (1$\Longrightarrow$2): For each k, $\mathcal{H}_n^k(\mathfrak{A})$ is not cofinal in $\mathfrak{A}$.

(2$\Longrightarrow$3): It follows from a general property:

$$\left.\begin{array}{c} \mathfrak{A} \models I(\Sigma_n, \{a\}, \{b\}) \\ 2^{\langle b,b \rangle} \leq a \end{array}\right\} \Longrightarrow \mathfrak{A} \models I(\Sigma_n, (\leq a), (\leq b))$$

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Proof: (3$\Longrightarrow$1): Assume $\mathfrak{A} \models I(\Sigma_n, \mathcal{H}_n^\infty, \mathcal{H}_n^\infty)$.

Case 1: $\mathcal{H}_n^\infty(\mathfrak{A}) = \mathfrak{A}$. Then, $\mathfrak{A} \models \mathbf{I}\Sigma_n$.
Case 2: $\mathcal{H}_n^\infty(\mathfrak{A}) \neq \mathfrak{A}$. Then

- $\mathcal{H}_n^\infty(\mathfrak{A}) \prec_n^e \mathfrak{A}$ proper.
- $\mathcal{H}_n^\infty(\mathfrak{A}) \models \mathbf{B}\Sigma_{n+1} \vdash \mathbf{I}\Sigma_n$ (end–extension properties)

So, $\mathfrak{A} \models Th_{\Sigma_{n+1}}(\mathbf{I}\Sigma_n)$. □

Kaye–Paris–Dimitracopoulos' theories [JSL'88]

For each $k \geq 1$, $\mathbf{L}\Sigma_n^{(k),-}$ denotes

$$\exists x_1, \dots, x_k \, \varphi(x_1, \dots, x_k)$$
$$\Downarrow$$
$$\exists x_1, \dots, x_k \left\{ \begin{array}{c} x_1 = (\mu t)\,(\exists x_2, \dots, x_k \, \varphi(t, x_2, \dots, x_k)) \quad \wedge \\ x_2 = (\mu t)\,(\exists x_3, \dots, x_k \, \varphi(x_1, t, \dots, x_k)) \quad \wedge \\ \vdots \\ x_k = (\mu t)\,(\varphi(x_1, x_2, \dots, t)) \end{array} \right\}$$

where $\varphi(x_1, \dots, x_k)$ runs over Σ_n.

- **Theorem:** $Th_{\Sigma_{n+1}}(\mathbf{I}\Sigma_n) \equiv \bigcup_{k \geq 1} \mathbf{L}\Sigma_n^{(k),-}$

Beklemishev–Visser's theories [APAL'05]

- The Σ_n^-–LIMR is given by:

$$\frac{\exists u\, \forall x > u\, (f(x+1) \leq f(x))}{\exists u\, \forall x > u\, (f(x) = f(u))},$$

  where f runs over the Σ_n^-–functions provably total in $\mathbf{I}\Sigma_{n-1}$.

- $[\mathbf{I}\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_0 \equiv \mathbf{I}\Sigma_{n-1}$
  $[\mathbf{I}\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_{k+1} = [[\mathbf{I}\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_k, \Sigma_n^-\text{–LIMR}]$

- **Theorem:** $Th_{\Sigma_{n+1}}(\mathbf{I}\Sigma_n) \equiv \bigcup_{k \geq 1} [\mathbf{I}\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_k$

The equivalence theorem

Theorem $(k \geq 0)$

Let $\mathfrak{A} \models \mathbf{I}\Sigma_{n-1}$. The following are equivalent:

1. $\mathfrak{A} \models I(\Sigma_n^-, \mathcal{H}_n^k)$
2. $\mathfrak{A} \models [\mathbf{I}\Sigma_{n-1}, \Sigma_n^- \text{–} LIMR]_{k+1}$
3. $\mathfrak{A} \models \mathbf{L}\Sigma_n^{(k+1),-}$

- We prove a hierarchy theorem for these families:

  $$\mathcal{K}_n(\mathfrak{A}, \mathcal{H}_n^k(\mathfrak{A})) \models I(\Sigma_n^-, \mathcal{H}_n^k) + \neg I(\Sigma_n^-, \mathcal{H}_n^{k+1})$$

- In [APAL'05] Beklemishev–Visser posed the question of characterizing the theories $[\mathbf{I}\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_k$ for $k > 1$.

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Conclusions

- $\exists\Pi$–closed models provide a simple method to obtain conservation results between theories described by *axiom schemes* and their *inference rule* or *parameter free* versions.
  - It leans upon the syntactical structure of the axiom scheme and no *ad hoc* model–theoretic construction is involved.
  - The notion of a *monotonic* scheme isolates general *syntactical conditions* sufficient for the method to become applicable.
  - Most of the "classic" conservation results between fragments of Arithmetic can be derived in this framework.

Conclusions

- $\exists\Pi$–closed models provide a simple method to obtain conservation results between theories described by *axiom schemes* and their *inference rule* or *parameter free* versions.

- Axiom schemes "up to" definable elements are interesting and useful fragments of Arithmetic.
  - They capture the Σ_{n+1}–consequences of Σ_n–induction and Σ_n–collection schemes.
  - They provide nice reformulation of "classic" fragments and suggest new techniques for studying them: to consider *inference rule "up to" definable elements*.

A promising application: Π_n^-–induction scheme

- $\mathbf{I}\Pi_n^- \equiv I(\Sigma_n^-, \mathcal{K}_n)$
- By our general conservation result,

  $$\mathbf{I}\Pi_n^- \text{ is } \Pi_{n+1}\text{–conservative over } \mathbf{I}\Sigma_{n-1}^- + (\Sigma_n, \mathcal{K}_n)\text{–IR}.$$

- "Σ_n–IR *up to definable elements*" suggests a new point of view to study $Th_{\Pi_{n+1}}(\mathbf{I}\Pi_n^-)$.
- This approach seems to provide new, uniform proofs of...
  - Theorem(KPD): $\mathbf{I}\Pi_1^-$ is Π_2–conservative over $\mathbf{I}\Delta_0 + exp$.
  - Theorem(Bek): $\mathbf{I}\Pi_2^-$ is $Bool(\Sigma_2)$–conservative over $\mathbf{I}\Sigma_1^-$.

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- Roughly speaking...

  "closed models + definability = cut–elimination + reflection"

  How can we make explicit this apparent relation?