Node-to-Node Disjoint Paths in k-ary n-cube with Faulty Edges

Yonghong Xiang, Iain Stewart
School of Engineering and Computing Sciences
Durham University
Durham, United Kingdom, DH1 3LE
Email: {yonghong.xiang, i.a.stewart}@dur.ac.uk

Florent Madelaine
Département Informatique
Institut Universitaire de Technologie
63172 AUBIERE Cédex, FRANCE
florent.madelaine@u-clermont1.fr

***Abstract*—In a k-ary n-cube Q_n^k with at most $2n-2$ faulty edges, let u and v be any two given nodes. Suppose the number of healthy links of u is no more than that of v, and denote by m. In this paper, we find m disjoint paths between u and v.**

***Keywords*-interconnection networks; disjoint paths; k-ary n-cube; fault tolerance;**

I. Introduction

k-ary n-cubes have been used as interconnection networks for distributed-memory parallel computers [1] and are popular choices for networks-on-chips [2]. k-ary n-cube has the following basic properties: it is vertex- and edge-symmetric [3]; it is Hamiltonian [4]; it has diameter $n\lfloor\frac{k}{2}\rfloor$; it has a recursive decomposition. Moreover, it has admirable properties in relation to routing, broadcasting and communication in general (see, for example, [3], [5]).

Of particular relevance to us is the one-to-one node-disjoint paths problem, which enable parallel communication between source and destination nodes, provide alternative routing paths when faults occurs or for the purpose of avoiding traffic jam. Whilst Menger's Theorem [6] implies that there exist $2n$ node-disjoint paths between two given nodes in a graph of node-connectivity $2n$, it is by no means easy to identify and actually construct the paths. Moreover, it will be harder to find as many as possible disjoint paths when there exist faults in the graph.

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As more and more processors are incorporated into parallel machines, faults become more common, be it faults in the processors or on the connections between processors. Given the significant cost of parallel machines, we would prefer to be able to tolerate (small numbers of) faults and still be able to use our parallel machine.

Bose et al [5] find $2n$ node-disjoint paths between any two given nodes in a healthy k-ary n-cube. In this paper, we are going to find node-disjoint paths between any two given nodes in a faulty k-ary n-cube. In particular, in a k-ary n-cube Q_n^k with at most $2n-2$ faulty edges, let $\mathbf{u}$ and $\mathbf{v}$ be any two given nodes; denote the number of healthy links of a node $\mathbf{u}$ as $deg_H(\mathbf{u})$; we find $\min\{deg_H(\mathbf{u}), deg_H(\mathbf{v})\}$ disjoint paths between $\mathbf{u}$ and $\mathbf{v}$.

The paper is organized as follows: the next section is the basic definition; we describe how to find disjoint paths in faulty k-ary 2-cube in Section III; by induction, we find disjoint paths for k-ary n-cube for $n \geq 3, k \geq 4$, and this is described in Section IV; Section V is the conclusion of the paper.

II. Preliminaries

The *k-ary n-cube, denoted by* Q_n^k, for $k \geq 3$ and $n \geq 2$, has k^n nodes indexed by $\{0, 1, \ldots, k-1\}^n$, and there is a link $((u_n, u_{n-1}, \ldots, u_1), (v_n, v_{n-1}, \ldots, v_1))$ if, and only if, there exists $d \in \{1, 2, \ldots, n\}$ such that $min\{|u_d - v_d|, k - |u_d - v_d|\} = 1$, and $u_i = v_i$, for every $i \in \{1, 2, \ldots, n\} \setminus \{d\}$. Throughout, we assume that addition on tuple elements is modulo k.

An index $d \in \{1, 2, \ldots, n\}$ is often referred to as a *dimension*. We can *partition Q_n^k over dimension d* by fixing the dth element of any node tuple at some value v, for every $v \in \{0, 1, \ldots, k-1\}$. This results in k copies $Q_0, Q_1, \ldots, Q_{k-1}$ of Q_{n-1}^k (with Q_v obtained by fixing the dth element at v), with corresponding nodes in $Q_0, Q_1, \ldots, Q_{k-1}$ joined in a cycle of length k (in dimension d) (we suppress d, n and k in the notation as they are always understood).

Denote F the set of faulty edges in Q_n^k, F_i the set of faulty edges in subgraph $Q_i, i \in \{0, 1, \ldots, k-1\}$. $F^{\mathbf{v}}$ (resp. $H^{\mathbf{v}}$) denotes the set of faulty (resp. healthy) edges that incident with vertex $\mathbf{v}$. $F_i^{\mathbf{v}}$ (resp. $H_i^{\mathbf{v}}$) denotes the set of faulty (resp. healthy) edges that lie in Q_i and incident with $\mathbf{v}$. The set of edges incident with node $\mathbf{v}$ is denoted by $N^{\mathbf{v}}$. Denote the set of faulty edges that are not incident with $\mathbf{u}$ and $\mathbf{v}$ as F^F. Given a node $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_2, v_1)$, which lies in Q_{v_1}. Denote the node $(v_n, v_{n-1}, \ldots, v_2, v_1 + i)$ as $\mathbf{v}^i$, $-(k-1) \leq i \leq k-1$. In addition, $\mathbf{v}^1$ (resp. $\mathbf{v}^{-1}$) is also denoted as $\mathbf{v}^+$ (resp. $\mathbf{v}^-$). The distance between two node $\mathbf{u}$ and $\mathbf{v}$ is denoted by $dist(\mathbf{u}, \mathbf{v})$. A path can be written as a list of nodes or as a list of nodes and edges mixed or sub-paths with nodes and edges mixed. We put them in a pair of squared brackets to denote that is a path.

III. Node-to-Node Disjoint Paths in Faulty k-ary 2-cube, $k \geq 4$

We first find disjoint paths in Q_2^4 with at most two faulty nodes. Based on this, we find disjoint paths for $Q_2^k, k \geq 5$ with at most two faulty nodes by mapping the graph Q_2^k to Q_2^4.

As the procedure of finding disjoint paths for Q_2^4 with two faulty edges is long and tedious, we put it in the appendix A.

In Q_2^k, one node associates with one row and one column, and one edge associates with one row and two columns, or one column and two rows. Denote the two faulty edges as f_1 and f_2. If there are at most four rows and at most four columns associate with $\mathbf{u}, \mathbf{v}, f_1$, and f_2, it can be reduced to finding disjoint paths in Q_2^4 by removing some of the non-associated rows and columns, and then extend the disjoint paths by inserting back the rows and columns. So, we will only consider the cases that there are more than four rows or columns associated. We will consider the following cases:

(1) six columns are associated;
(2) five columns are associated.

The cases with more than four rows associated can be mapped to the above two cases by the symmetric property of Q_n^k.

If there are 6 columns associated, then there are at most 4 rows associated, and the configuration can be described as follows: $\mathbf{u} = (0,0), \mathbf{v} = (i_1, j_1), f_1 = ((i_2, j_2), (i_2, j_3)), f_2 = ((i_3, j_4), (i_3, j_5))$, where $1 \leq i_1, i_2, i_3, j_1, j_2, j_3, j_4, j_5 \leq k, j_3 = j_2 + 1, j_5 = j_4 + 1$ and all the $j_t, 1 \leq t \leq 5$ are different from each other. To reduce the case to Q_2^4, we keep all of the associated rows and some other rows if necessary to have 4 rows; to have 4 columns, we set $j_1 = j_2 = 1, j_4 = 2$, if $j_1 < j_2 < j_4$, or we set $j_2 = 0, j_1 = j_4 = 2$, if $j_2 < j_1 < j_4$. The other cases can be easily mapped to these two cases by the symmetric property of Q_2^k.

If there are 5 columns associated, the configuration can be described as: $\mathbf{u} = (0,0), \mathbf{v} = (i_1, j_1), f_1 = ((i_2, j_2), (i_2, j_3)), f_2 = ((i_3, j_4), (i_4, j_4))$, where $1 \leq i_t, j_t \leq k, i_s \neq i_t, j_s \neq j_t$ for all $1 \leq s, t \leq 4$. To reduce the case to Q_2^4 (to have 4 columns), we set $j_1 = 1, j_2 = 2, j_4 = 3$, if $j_1 < j_2 < j_4$, and set $j_2 = 0, j_1 = 2, j_4 = 3$, if $j_2 < j_1 < j_4$; by the symmetric property of Q_2^k, the four rows can be kept in the same way as the columns if there are five rows associated, or we just keep all of associated rows and some other rows if necessary to have four rows. The other cases can be easily mapped to these two cases.

Thus, we have the following lemma.

Lemma 1: In a k-ary 2-cube Q_2^k, when there are at most two faulty edges, for any two given nodes $\mathbf{u}$ and $\mathbf{v}$, there exist $\min\{deg_{\mathbf{u}}, deg_{\mathbf{v}}\}$ disjoint paths between. ■

IV. NODE-TO-NODE DISJOINT PATHS IN FAULTY k-ARY n-CUBE, $n \geq 4, k \geq 4$

There are at most $2n-2$ faulty edges in Q_n^k, and there are n dimensions. Hence, we can partition Q_n^k on a dimension, such that there is at most one faulty edge lies on that dimension.

Given an edge $e = (\mathbf{s}, \mathbf{t})$ on some path, define *U-jump(e)* as replacing the edge e by a sub-path of length 3: $[\mathbf{s}, \mathbf{s}^-, \mathbf{t}^-, \mathbf{t}]$; similarly, *D-jump($e$)* is defined as replacing e by another sub-path of length 3: $[\mathbf{s}, \mathbf{s}^+, \mathbf{t}^+, \mathbf{t}]$.

W.l.o.g., suppose we partition Q_n^k through dimension one. So that there is either one faulty edge or none of the faulty edges lie in this dimension.

The proof is technical and complicated. There are a number of cases to be considered. For example, w.l.o.g., suppose $\mathbf{u} = (0, 0, \ldots, 0)$, we need to consider several different cases depends on where the node $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$ lies and whether there is a faulty edge lies on dimension 1 or not. Suppose there is one faulty edge lies on the dimension 1, then we have to distinguish the value of v_1:

(1) $v_1 = 0$, which means $\mathbf{u}$ and $\mathbf{v}$ lie in the same sub-graph Q_0;
(2) $v_1 \neq 0$, which means $\mathbf{u}$ and $\mathbf{v}$ lie in different sub-graphs;

The technique in building disjoint paths for these two cases are very different. For example, for case (1), we can use induction on Q_0 by assuming some of the faulty edges are healthy if necessary so as to have at most $2n-4$ faulty edges (if a path includes an edge, say e, that is originally fault, we need to do adjustment to avoid the faulty edge e, for example, do a *U-jump(e)* or *D-jump(e)*), and then find some more paths to link the dimension 0 edges between them; while for case (2), as $\mathbf{u}$ and $\mathbf{v}$ lie in different sub-graphs, we can apply induction on each sub-graph by making some assumptions; however, we need to link these disjoint paths in different sub-graphs together, which makes the procedure complicated.

Furthermore, if we go into details for case (2), we have the following sub-cases to consider:

(2.1) $dist(\mathbf{u}, \mathbf{v}^{-v_1}) = 0$, which means the node $\mathbf{u}$ and the node $\mathbf{v}$ lie on the same dimension 1 circle. Node $\mathbf{u}$ or its neighbor can reach node $\mathbf{v}$ or its corresponding neighbor along the dimension 1 circle.
(2.2) $dist(\mathbf{u}, \mathbf{v}^{-v_1}) = 1$, which means node $\mathbf{u}$ and node $\mathbf{v}$ lie on different dimension 1 circles, but they have one overlap incident edge $(\mathbf{u}, \mathbf{v}^{-v_1})$.
(2.3) $dist(\mathbf{u}, \mathbf{v}^{-v_1}) \geq 2$, which means node $\mathbf{u}$ and node $\mathbf{v}$ lie on different dimension 1 circles, and they have no overlap incident edges.

These cases are similar to each other. However, the complexity of each case varies much. Due to the space limited, we will not include all of the proof in the paper, but the representative ones and the remainder will be left in the appendix.

We claim that case (2.2) is representative as most of the techniques used in the other cases will be used here. For example, use induction on Q_0 to find disjoint paths is similar to case (1); to extend these disjoint paths to Q_{v_1} is similar to case (2.1) when $\exists e_1 \in F^{\mathbf{u}}, e_1^{v_1} \notin F^{\mathbf{v}}$ and $\exists e_2 \notin F^{\mathbf{u}}, e_2^{v_1} \in F^{\mathbf{v}}$, and case (2.3). Also, similar discussion on the case that the dimension one faulty edge does not block any path can be applied to the case when there are no dimension one faulty edges.

The condition for case (2.2) is:

(1) There is one dimension one faulty edge $f = (\mathbf{r}^h, \mathbf{r}^{h+1})$;
(2) $dist(\mathbf{u}, \mathbf{v}^{-v_1}) = 1$

In what follows, we will discuss case (2.2) in details.

Case 2.2 $dist(\mathbf{u}, \mathbf{v}^{-v_1}) = 1$, i.e., There exists some $j \in \{2, 3, \ldots, n\}, v_j = 1$, and for all $i \in \{2, 3, \ldots, n\} \backslash \{j\}, v_i = 0$. W.l.o.g., suppose $j = 2$.

Let $e \in N^{\mathbf{t}}$. For a given $i \in \{1, 2, \ldots, k-1\}$, we define $map(\mathbf{t}, i)$ as temporarily mark e^i as faulty if e^i is faulty, or healthy otherwise.

Suppose $\mathbf{r}, \mathbf{s}$ lie in a same sub-graph $Q_i, i \in \{0, 1, \ldots, k-1\}$, and $|F_i| \leq 2n-4$. Denote $DP(\mathbf{r}, \mathbf{s})$ as a procedure of finding disjoint paths between $\mathbf{r}$ and $\mathbf{s}$ in the sub-graph Q_i by induction.

Please note, if the following two paths exist, and are not blocked, we will always keep them in the set of disjoint paths, and thus we may not mention them in the following discussion unless necessary:

$\sigma_1 : [\mathbf{u}, \mathbf{v}^-, \mathbf{v}]$; and

$\sigma_2 : [\mathbf{u}, \mathbf{u}^+, \mathbf{v}]$.

(A22-1) Assume f is healthy.

As there are now at most $2n-3$ faulty edges in Q_n^k, we have the following cases to consider. Recall that we suppose $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}|$.

Suppose $v_1 = 1$.

(A22-1.1) $|F^{\mathbf{u}}| = 2n-3$.

We only need to find 3 disjoint paths. Let $(\mathbf{u}, \mathbf{s})$ be the only healthy edge in Q_0 that incident with $\mathbf{u}$. The two more paths are:

(a) $[\mathbf{u}, \mathbf{u}^-, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$; and

(b) $[\mathbf{u}, \mathbf{s}, \mathbf{s}^-, \ldots, \mathbf{s}^2, \mathbf{u}^2, \mathbf{v}^2, \mathbf{v}^+, \mathbf{v}]$;

(A22-1.2) After doing $map(\mathbf{v}, 0)$, we have $|F_0| = 2n-3$.

If $|F^{\mathbf{u}} \cup F^{\mathbf{v}^-}| = 2n-3$, we must have $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}^-}| + 1$. Assume a faulty edge $(\mathbf{u}, \mathbf{s})$ is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Remove the node $\mathbf{v}^-$ from each disjoint path, and extend to $\mathbf{v}$. For simplicity, this procedure after doing $DP(\mathbf{u}, \mathbf{v}^-)$ will be called "extend" the paths to $\mathbf{v}$ in throughout the remainder part of this section. Two more paths can be:

(a) $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \mathbf{v}^+, \mathbf{v}]$; and

(b) $[\mathbf{u}, \mathbf{u}^-, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$.

If there is a faulty edge e that is neither incident with $\mathbf{u}$ nor with $\mathbf{v}^{-v_1}$, assume e is healthy and do $DP(\mathbf{u}, \mathbf{v}^-)$; extend the paths to $\mathbf{v}$. Do *U-jump(e)* if e lies on some of the disjoint paths. Two more paths are:

(a) $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \mathbf{v}^+, \mathbf{v}]$; and

(b) $[\mathbf{u}, \mathbf{u}^-, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$;

(A22-1.3) $|F_1| \leq 2n-4$, and after doing $map(\mathbf{v}, 0)$, we have $|F_0| \leq 2n-4$.

Do $map(\mathbf{v}, 0)$ and $DP(\mathbf{u}, \mathbf{v}^-)$. Extend the paths to $\mathbf{v}$. Two more paths are:

(a) $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \mathbf{v}^+, \mathbf{v}]$; and

(b) $[\mathbf{u}, \mathbf{u}^-, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$;

From our above construction method, in a similar way to the case $v_1 = 1$ but extending the dimension 1 edges, we will be able to find disjoint paths for the case $v_1 \geq 2$.

If the dimension 1 edge f lies on some of the above path, we will rebuild disjoint paths as follows.

(A22-2) f is blocking some path.

Suppose $v_1 = 1$.

(A22-2.1) $f = (\mathbf{u}, \mathbf{u}^+)$.

(A22-2.1.1) $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$

We first do $map(\mathbf{v}, 0)$. But $(\mathbf{u}, \mathbf{v}^-)$ is marked as faulty if either $(\mathbf{u}, \mathbf{v}^-)$ is originally faulty or $(\mathbf{u}^+, \mathbf{v})$ is faulty.

If $(\mathbf{u}, \mathbf{v}^-) \in F$, and $(\mathbf{u}^+, \mathbf{v}) \notin F$, assume two of $\mathbf{v}^-$'s faulty links e_1, e_2 are healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Let edge $(\mathbf{u}, \mathbf{w})$ lies on the path that e_1 lies on, and $(\mathbf{u}, \mathbf{z})$ lies on the path that e_2 lies on. Remove the paths that contains e_1 or e_2. For all of the other paths, extend them to $\mathbf{v}$. Build the following three paths:

(a) $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$;

(b) $[\mathbf{u}, \mathbf{z}, \mathbf{z}^+, \mathbf{z}^2, \rho_2, \mathbf{v}^+, \mathbf{v}]$; and

(c) $[\mathbf{u}, \mathbf{w}, \mathbf{w}^+, \rho_3, \mathbf{u}^+, \mathbf{v}]$,

where $\rho_1 \in Q_{k-1}, \rho_2 \in Q_2$ and $\rho_3 \in Q_1$, and they can be found by induction if not straightforward. Specifically, ρ_3 can be build by finding disjoint paths between $\mathbf{w}^+$ and $\mathbf{v}$ (note that there are at most $2n-4$ faulty edges lie in Q_1).

If $(\mathbf{u}, \mathbf{v}^-) \notin F$, and $(\mathbf{u}^+, \mathbf{v}) \in F$, do $map(\mathbf{v}, 0)$. If there are more than $2n-4$ faulty edges in Q_0, we assume $(\mathbf{u}, \mathbf{v}^-)$ is healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$; remove path $[\mathbf{u}, \mathbf{v}^-]$extend the paths to $\mathbf{v}$. Build one more path:

$$[\mathbf{u}, \mathbf{u}^-, \ldots, \mathbf{u}^{v_1+1}, \rho, \mathbf{v}^+, \mathbf{v}],$$

where $\rho \in Q_{v_1+1}$.

If either $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$, or $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$, do $map(\mathbf{v}, 0)$; assume $e \in F^{\mathbf{v}^-}$ is healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Let $(\mathbf{u}, \mathbf{w})$ is on the path containing e. Remove this path, and extend all other paths to $\mathbf{v}$. Build the following two paths:

(a) $[\mathbf{u}, \mathbf{w}, \mathbf{w}^+, \rho_1, \mathbf{u}^+, \mathbf{v}]$, where ρ can be chosen from the disjoint paths between $\mathbf{w}^+$ and $\mathbf{v}$ in Q_1; and

(b) $[\mathbf{u}, \mathbf{u}^{k-1}, \ldots, \mathbf{u}^{v_1+1}, \rho_2, \mathbf{v}^+, \mathbf{v}]$, where $\rho_2 \in Q_{v_1+1}$ and can be found by induction.

If $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$, path (a) is not needed.

(A22-2.1.2) $|F^{\mathbf{u}}| > |F^{\mathbf{v}}|$

(A22-2.1.2.1) $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$.

If $|F^{\mathbf{u}}| = |F^{\mathbf{v}}| + 1$, then after $map(\mathbf{v}, 0)$, we have $|F_0^{\mathbf{u}}| = |F_0^{\mathbf{v}^-}| + 1$. Assume faulty edge e that incident with $\mathbf{v}^-$ is healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Let $(\mathbf{u}, \mathbf{w})$ lie on the path that passes the edge e. Remove this path and extend all other paths to $\mathbf{v}$. Build the following two paths:

(a) $[\mathbf{u}, \mathbf{w}, \mathbf{w}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$, where $\rho_1 \in Q_{k-1}$ can be found by finding disjoint paths between $\mathbf{u}^-$ and $\mathbf{v}^{-2}$ by induction, and

(b) $[\mathbf{u}, \mathbf{u}^-, \ldots, \mathbf{u}^{v_1+1}, \rho_2, \mathbf{v}^+, \mathbf{v}]$, where $\rho_2 \in Q_{v_1+1}$

If $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}|+2$, then after $map(\mathbf{v}, 0)$, assume one of the faulty edges e that incidents with $\mathbf{u}$ is healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Remove the path that passes the edge e and extend all other paths to $\mathbf{v}$. Build the following path:

$$[\mathbf{u}, \mathbf{u}^-, \ldots, \mathbf{u}^{v_1+1}, \rho_2, \mathbf{v}^+, \mathbf{v}],$$

where $\rho_2 \in Q_{v_1+1}$.

(A22-2.1.2.2) $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$; or $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$; or $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$.

Do $map(\mathbf{v}, 0)$. If there are at most $2n-4$ faulty edges in Q_0, do $DP(\mathbf{u}, \mathbf{v}^-)$, and extend the paths to $\mathbf{v}$. Build another path:

$$[\mathbf{u}, \mathbf{u}^-, \ldots, \mathbf{u}^{v_1+1}, \rho_2, \mathbf{v}^+, \mathbf{v}],$$

where $\rho_2 \in Q_{v_1+1}$.

If there are $2n-3$ faulty edges in Q_0 and $(\mathbf{u}, \mathbf{v}^-) \in F$ after $map(\mathbf{v}, 0)$, we assume $(\mathbf{u}, \mathbf{v}^-)$ is healthy and do $DP(\mathbf{u}, \mathbf{v}^-)$. Remove the path $[\mathbf{u}, \mathbf{v}^-]$. Extend all other paths to $\mathbf{v}$, and build one more path as above.

If there are $2n-3$ faulty edges in Q_0 and $(\mathbf{u}, \mathbf{v}^-) \notin F$ after $map(\mathbf{v}, 0)$, we have $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$. Hence, we assume one of $\mathbf{v}^-$'s faulty link e is healthy and do $DP(\mathbf{u}, \mathbf{v}^-)$. Remove the path that passes edge e and extend all other path to $\mathbf{v}$. Build one more path as above.

(A22-2.2) $f = (\mathbf{u}, \mathbf{u}^-)$.

(A22-2.2.1) $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$

(A22-2.2.1.1) $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$.

Do $map(\mathbf{v}, 0)$, and assume faulty edges e_1, e_2 incident with $\mathbf{v}^-$ are healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Let $(\mathbf{u}, \mathbf{w})$ lies on the path that passes e_1, and $(\mathbf{u}, \mathbf{z})$ lies on the path that passes e_2. Remove these two paths and extend all other paths to $\mathbf{v}$. Build the following two paths:

(a) $[\mathbf{u}, \mathbf{w}, \mathbf{w}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$, where $\rho_2 \in Q_{k-1}$, and

(b) $[\mathbf{u}, \mathbf{z}, \mathbf{z}^+, \mathbf{z}^2, \rho_2, \mathbf{v}^+, \mathbf{v}]$, where $\rho_2 \in Q_2$.

ρ_1 and ρ_2 can be built by induction in their corresponding sub-graph.

(A22-2.2.1.2) $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$

Do $map(\mathbf{v}, 0)$ but mark $(\mathbf{u}, \mathbf{v}^-)$ as healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Extend the paths to $\mathbf{v}$. Build one more path:

$$[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{v}^+, \mathbf{v}],$$

where $\rho_2 \in Q_2$.

(A22-2.2.1.3) $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$.

Do $map(\mathbf{v}, 0)$, and $DP(\mathbf{u}, \mathbf{v}^-)$. Extend the disjoint paths to $\mathbf{v}$. Build the same one more path as in case (A22-2.2.1.2) .

(A22-2.2.1.4) $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$.

Do $map(\mathbf{v}, 0)$. There exists one faulty edge e incident with $\mathbf{v}^-$. We assume e is healthy, and find do $DP(\mathbf{u}, \mathbf{v}^-)$. Remove the path that passes the edge e (if such path exists), and extend all other paths to $\mathbf{v}$.

(A22-2.2.2) $|F^{\mathbf{u}}| > |F^{\mathbf{v}}|$

(A22-2.2.2.1) $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$.

Do $map(\mathbf{v}, 0)$. If $|F^{\mathbf{u}}| = |F^{\mathbf{v}}| + 1$, assume edge e, a faulty edge incident with $\mathbf{v}^-$, is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Let $(\mathbf{u}, \mathbf{w})$ lies on the path passes e. Remove this path and expand all other paths to reach $\mathbf{v}$. Build one more path:

$$[\mathbf{u}, \mathbf{w}, \mathbf{w}^-, \rho, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}].$$

If $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 2$, assume edge $(\mathbf{u}, \mathbf{v}^-)$ is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Remove the path $[\mathbf{u}, \mathbf{v}^-]$, and extend all other paths to $\mathbf{v}$.

(A22-2.2.2.2) $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$.

Do $map(\mathbf{v}, 0)$. If there are no more than $2n-4$ faulty edges in Q_0, do $DP(\mathbf{u}, \mathbf{v}^-)$, and extend the disjoint paths to $\mathbf{v}$.

If there are $2n-3$ faulty edges in Q_0 and $|F_0^{\mathbf{u}}| = |F_0^{\mathbf{v}^-}|$, then there must exist e that is neither incident with $\mathbf{u}$ nor incident with $\mathbf{v}^-$. Assume e is healthy and do $DP(\mathbf{u}, \mathbf{v}^-)$. If some path passes edge e, do *U-jump(e)*. Extend these paths to $\mathbf{v}$.

(A22-2.2.2.3) $(\mathbf{u}^+, \mathbf{v}) \in F$.

No matter whether $(\mathbf{u}, \mathbf{v}^-)$ is faulty or not, we do $map(\mathbf{v}, 0)$, and assume $(\mathbf{u}, \mathbf{v}^-)$ is healthy and do $DP(\mathbf{u}, \mathbf{v}^-)$. Extend the disjoint paths to $\mathbf{v}$. Build one more path:

$$[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho, \mathbf{v}^+, \mathbf{v}].$$

(A22-2.3) $f = (\mathbf{v}, \mathbf{v}^-)$.

If $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$, we assume two of $\mathbf{u}$'s incident faulty edges e_1, e_2 (lie in Q_0) are healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Suppose $\mathbf{w}_1$ (resp. $\mathbf{w}_2$) lies on the path that edge e_1 (resp. e_2) lies on, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are neighbors of $\mathbf{v}^-$. Remove these two paths, extend all other paths to $\mathbf{v}$, and build the following three paths:

(a) $[\mathbf{u}, \mathbf{v}^-, \mathbf{v}^{-2}, \ldots, \mathbf{v}^+, \mathbf{v}]$;
(b) $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{w}_1^-, \mathbf{w}_1, \mathbf{w}_1^+, \mathbf{v}]$, $\rho_1 \in Q_{k-1}$; and
(c) $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{w}_2^2, \mathbf{w}_2^+, \mathbf{v}]$, $\rho_2 \in Q_2$.

Based on the above paths, if $(\mathbf{u}^+, \mathbf{v}) \notin F$, no need to build the above path (c) as $\mathbf{u}, \mathbf{u}^+, \mathbf{v}$ is already one of the disjoint paths; if $(\mathbf{u}, \mathbf{v}^-) \in F$, no need to build the above path (a).

(A22-2.4) $f = (\mathbf{v}, \mathbf{v}^+)$.

(A22-2.4.1) $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$.

There exist faulty edges e_1, e_2 incident with node $\mathbf{u}$. Assume they are healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Let $\mathbf{w}_i$ lies on the path that passes edge e_i, for $i = 1, 2$. Remove path that passes e_1 and e_2. Extend all of the other paths to $\mathbf{v}$. Build the following two more paths:

(a) $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{w}_1^-, \mathbf{w}_1, \mathbf{w}_1^+, \mathbf{v}]$, $\rho_1 \in Q_{k-1}$;
(b) $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{w}_2^2, \mathbf{w}_2^+, \mathbf{v}]$, $\rho_2 \in Q_2$.

(A22-2.4.2) $(\mathbf{u}, \mathbf{v}^-) \notin F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$.

Similar to case (A22-2.4.1), but only assume one faulty edge e_1 is healthy, and only need to find one more path (a) as in case (A22-2.4.1).

(A22-2.4.3) $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \in F$.

There exists $e_1 \in F^{\mathbf{u}}$. Assume e_1 is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Let $\mathbf{w}_1$ lies on the path that passes the edge e_1. Remove this path and extend all other paths to $\mathbf{v}$. Build the following two paths:

(a) $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}_-, \mathbf{v}]$;
(b) $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{w}_1^2, \mathbf{w}_1^+, \mathbf{v}]$.

(A22-2.4.4) $(\mathbf{u}, \mathbf{v}^-) \in F$ and $(\mathbf{u}^+, \mathbf{v}) \notin F$.

Similar to case (A22-2.4.3), but no need to make any assumption, except when $|F_0| = 2n - 3$ after $map(\mathbf{v}, 0)$. If the exceptional case happens, we assume $(\mathbf{u}, \mathbf{v}^-)$ is healthy and then do $DP(\mathbf{u}, \mathbf{v}^-)$. Extend the paths to $\mathbf{v}$.

(A22-2.5) $f = (\mathbf{u}^h, \mathbf{u}^{h+1})$, or $f = (\mathbf{v}^h, \mathbf{v}^{h+1}), h \neq 0, -1$.

In this case, the two paths below are included in the disjoint paths set:

(a) $[\mathbf{u}, \mathbf{v}^-, \mathbf{v}]$; and
(b) $[\mathbf{u}, \mathbf{u}^+, \mathbf{v}]$.

We need to find a path linking edge $(\mathbf{u}, \mathbf{u}^-)$ and edge $(\mathbf{v}, \mathbf{v}^+)$ but avoid the faulty edge f. It can be either

(a) $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}_-, \mathbf{v}]$, $\rho_1 \in Q_{k-1}$,

if $f = (\mathbf{u}^h, \mathbf{u}^{h+1})$ for some $h \neq 0, -1$, or

(b) $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{w}_1^2, \mathbf{w}_1^+, \mathbf{v}]$, $\rho_2 \in Q_2$,

if $f = (\mathbf{v}^h, \mathbf{v}^{h+1})$ for some $h \neq 0, -1$.

The other paths can be build similar to case (A22-1).

As we assume $v_1 = 1$ at the beginning of case (A22-2), we now consider $v_1 > 1$.

Compare to the case (A22-1), where we assume f is healthy and when $v_1 > 1$, here we only need to care about the case when $f = (\mathbf{r}^h, \mathbf{r}^{h+1})$ with $0 \leq h \leq v_1$ but f is neither incident with $\mathbf{u}$ nor incident with $\mathbf{v}$, and blocks some of the paths as constructed in case (A22-1) (when $v_1 > 1$). We consider the following cases.

- If $\mathbf{r} = \mathbf{u}$, there exists $\mathbf{s}$, a neighbor of $\mathbf{u}$, such that $(\mathbf{u}^j, \mathbf{s}^j)$ is healthy for every $j = 0, 1, \ldots, k-1$. This is true, as there are at most $2n-3$ faulty edges in all $Q_i, i = 0, 1, \ldots, k-1$, and each node in Q_i has $2n-2$ neighbors inside Q_i. Thus, we adjust the blocked path by replacing the faulty edge with a sub-path: $[\mathbf{r}^h, \mathbf{s}^h, \mathbf{s}^{h+1}, \mathbf{r}^{h+1}]$. Note that if $h = v_1 - 1$ or $h = 0$, some slight adjustment is necessary, which will be simple and is not stated here.
  However, it might be the case that the only healthy edge in Q_h for node $\mathbf{r}^h$ is $\mathbf{v}^{h-v_1}$, and $\mathbf{v}^{h-v_1}$ is already on some path. In this case, we change to partition the graph Q_n^k through dimension 2. This will result in one dimension two faulty edge. For convenience,

we also call a sub-graph Q_i if the dimension two number is i; we also denote the a node $(v_n, v_{n-1}, \ldots, v_2 + i, v_1)$ as $\mathbf{v}^i$ if $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_2, v_1)$, for $-k+1 \leq i \leq k-1$. Now we have $dist(\mathbf{u}, \mathbf{v}^-) > 1$, and the only dimension 2 faulty edge is: $((0, 0, \ldots, 1, i), (0, 0, \ldots, 2, i))$. As there are no faulty edges in the sub-graph Q_0, we find disjoint paths by induction between $\mathbf{u}$ and $(0, 0, \ldots, 0, v_1)$, and extend these paths to $\mathbf{v}$. Thus we have found $2n-4$ disjoint paths. Two more paths can be:

(a) $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$, $\rho_1 \in Q_{k-1}$ can be found by induction;

(b) $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{v}^+, \mathbf{v}]$, $\rho_2 \in Q_2$ can be found by induction.

This is essentially a small case of **Case 2.3**: $dist(\mathbf{u}, \mathbf{v}^{-v_1}) > 1$ in appendix B. For the case that node $\mathbf{r}$ is a neighbor of $\mathbf{v}^-$, it can be done similar to the case $\mathbf{r} = \mathbf{u}$.

- If $\mathbf{r} = \mathbf{v}^{-v_1}$, and the only healthy edge incident with $\mathbf{r}^h$ is $\mathbf{u}^h$, we then repartition the graph similar to the case above. Otherwise, there exists $(\mathbf{r}^h, \mathbf{s}^h) \notin F$ and $(\mathbf{r}^{h+1}, \mathbf{s}^{h+1}) \notin F$, replace the faulty edge f by a sub-path of length three: $[\mathbf{r}^h, \mathbf{s}^h, \mathbf{s}^{h+1}, \mathbf{r}^{h+1}]$. If $(\mathbf{s}^h, \mathbf{s}^{h+1})$ is already on some path and whose dimension one part is $\mathbf{s}^x, \mathbf{s}^{x+1}, \ldots, \mathbf{s}^y$, we replace these edges by $\mathbf{s}^x, \mathbf{s}^{x-1}, \ldots, \mathbf{s}^{y+1}, \mathbf{s}^y$.

Based on Lemma 1 and the above discussion include the sections in the appendix, we obtain the following result.

Theorem 1: In a k-ary n-cube with at most $2n-2$ faulty edges, there exist $\min\{deg_H(\mathbf{u}), deg_H(\mathbf{v})\}$ disjoint paths between any two given nodes $\mathbf{u}$ and $\mathbf{v}$. ■

V. CONCLUSION

We have proved that there exist $\min\{deg_H(\mathbf{u}), deg_H(\mathbf{v})\}$ node-disjoint paths in k-ary n-cube when there exist at most $2n-2$ faulty edges. This will make the k-ary n-cube interconnection network robust in communicating and routing even though there have up to $2n-2$ faulty edges. Our future research will focus on finding shortest node-disjoint paths for k-ary n-cubes with faulty edges and/or faulty nodes.

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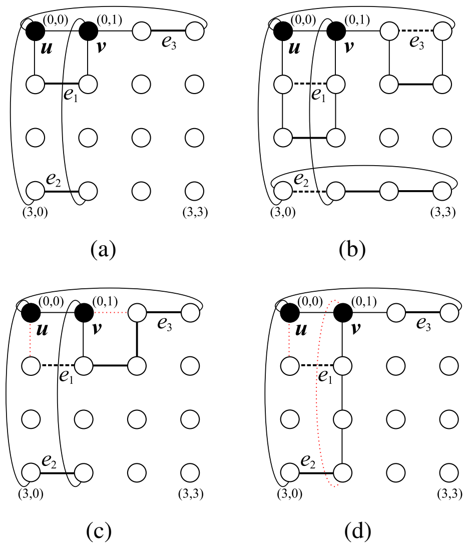

Figure 1. Disjoint paths between $\mathbf{u}$ and $\mathbf{v}$ in Q_2^4 for case $\mathbf{v} = (0,1)$

To the referee: the appendix part will be removed if the paper being accepted by the conference, but we will put the related information on our website: http://www.dur.ac.uk/yonghong.xiang/ for public access, and we will direct the reader to our website in the final version of the paper as well.

APPENDIX A. DISJOINT PATHS IN Q_2^4

Without lose of generality (W.l.o.g.), suppose $\mathbf{u} = (0,0)$. By the vertex symmetric property of Q_n^k, for node $\mathbf{v}$, only the following five cases need to be considered: $(0,1), (0,2), (1,1), (2,1)$, and $(2,2)$.

When there are no faulty edges in Q_2^4, if a path passes node $(0,1)$ (resp. $(1,0), (3,0)$, and $(0,3)$), then we name the path ρ_{10} (resp. ρ_{20}, ρ_{30}, and ρ_{40}). We call these four of $\mathbf{u}$'s neighbors the *residual nodes*. Denote ρ_{ij} as the jth alternative path to path ρ_{i0} if it passes the same residual vertex as ρ_{i0}.

1) $\mathbf{v} = (0,1)$*:* Suppose there are no faulty edges in Q_2^4, then we can build four disjoint paths between $\mathbf{u}$ and $\mathbf{v}$ as follows (see Fig.1(a)).

$\rho_{10} : [\mathbf{u}, \mathbf{v}]$; $\rho_{20} : [\mathbf{u}, (1,0), (1,1), \mathbf{v}]$
$\rho_{30} : [\mathbf{u}, (3,0), (3,1), \mathbf{v}]$; $\rho_{40} : [\mathbf{u}, (0,3), (0,2), \mathbf{v}]$.

Let $e_1 = ((1,0),(1,1))$, $e_2 = ((3,0),(3,1))$, $e_3 = ((0,2),(0,3))$. From Fig.1(b), we can see that if any two of e_1, e_2 and e_3 are faulty, we have the following two sets of alternative paths ρ_{21}, ρ_{31} and ρ_{41} (see Fig. 1(b)) and ρ_{22}, ρ_{32} and ρ_{42}.

$\rho_{21} : [\mathbf{u}, (1,0), (2,0), (2,1), (1,1), \mathbf{v}]$;
$\rho_{31} : [\mathbf{u}, (3,0), (3,3), (3,2), (3,1), \mathbf{v}]$;
$\rho_{41} : [\mathbf{u}, (0,3), (1,3), (1,2), (0,2), \mathbf{v}]$;

and

$\rho_{22} : [\mathbf{u}, (1,0), (1,3), (1,2), (1,1), \mathbf{v}]$;
$\rho_{32} : [\mathbf{u}, (3,0), (2,0), (2,1), (3,1), \mathbf{v}]$;
$\rho_{42} : [\mathbf{u}, (0,3), (3,3), (3,2), (0,2), \mathbf{v}]$.

ρ_{ij} is disjoint with $\rho_{it}, t \neq j$. So, if e_i is faulty for some $i \in \{1,2,3\}$ and the other faulty edge lies on one of its alternative path, then we can use the other alternative path.

Suppose $\{f_1, f_2\} \cap (F^{\mathbf{u}} \cup F^{\mathbf{v}}) \neq \emptyset$. (a) If $(\mathbf{u}, \mathbf{v}) \in F$, then, the problem reduces to the case discussed above (just remove the path $[\mathbf{u}, \mathbf{v}]$). If only one of the faulty edges linked with $\mathbf{u}$ or $\mathbf{v}$, then we only need to build 3 disjoint paths and this again reduces to the case discussed above. (b) If the both faulty edges are linked with $\mathbf{u}$ or both linked with $\mathbf{v}$, or one links with $\mathbf{u}$ and one links with $\mathbf{v}$ but they both lie on ρ_{i0} for some $i \in \{1,2,3,4\}$. All we need to do is to remove the path(s) that contain faulty edges. (c) If the both faulty edges are linked with different nodes and are not lying on the same path, then we need to adjust some of the ρ_{i0} to obtain 3 disjoint paths. When $(\mathbf{u}, (1,0))$ and $(\mathbf{v}, (0,2))$ are faulty, we keep path ρ_{10} and ρ_{30}, and build the following path (see Fig.1(c)):

$\rho_{43} : [\mathbf{u}, (0,3), (0,2), (1,2), (1,1), \mathbf{v}]$.

When $(\mathbf{u}, (1,0))$ and $(\mathbf{v}, (3,1))$ are faulty, keep path ρ_{10} and ρ_{40}, and build a new path (see Fig.1(d)):

$\rho_{33} : [\mathbf{u}, (3,0), (3,1), (2,1), (1,1), \mathbf{v}]$.

2) $\mathbf{v} \in \{(0,2), (1,1), (1,2), (2,2)\}$*:* If there are no faulty edges in Q_2^4, we can easily build four disjoint paths $\rho_{10}, \rho_{20}, \rho_{30}, \rho_{40}$ between $\mathbf{u}$ and $\mathbf{v}$.

For $\mathbf{v} = (0,2)$, we have: (See Fig.2(a))

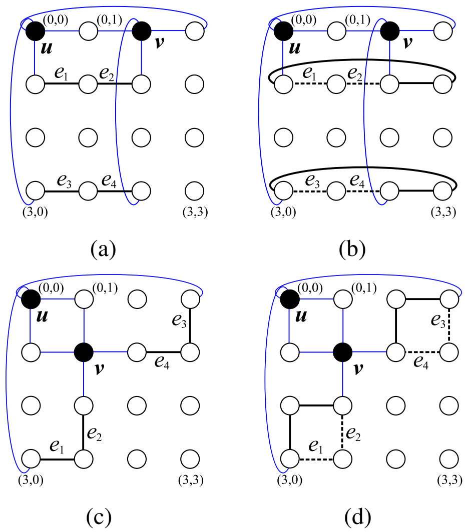

Figure 2. Four disjoint paths between $\mathbf{u}$ and $\mathbf{v}$ in Q_2^4, (a), (b) for case $\mathbf{v} = (0, 2)$, and (c), (d) for case $\mathbf{v} = (1, 1)$

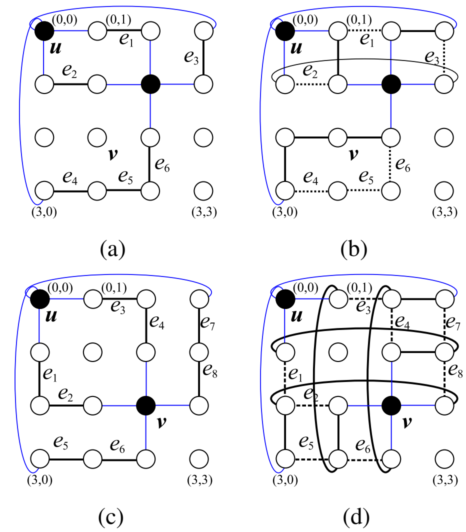

Figure 3. Four disjoint paths between $\mathbf{u}$ and $\mathbf{v}$ in Q_2^4, (a), (b) for case $\mathbf{v} = (1, 2)$, and (c), (d) for case $\mathbf{v} = (2, 2)$

$\rho_{10} : [\mathbf{u}, (0, 1), \mathbf{v}]$;
$\rho_{20} : [\mathbf{u}, (1, 0), (1, 1), (1, 2), \mathbf{v}]$;
$\rho_{30} : [\mathbf{u}, (3, 0), (3, 1), (3, 2), \mathbf{v}]$;
$\rho_{40} : [\mathbf{u}, (0, 3), \mathbf{v}]$;
$\rho_{21} : [\mathbf{u}, (1, 0), (1, 1), (1, 2), \mathbf{v}]$;
$\rho_{31} : [\mathbf{u}, (3, 0), (3, 3), (3, 2), \mathbf{v}]$.

For $\mathbf{v} = (1, 1)$, we have: (See Fig.2(c))

$\rho_{10} : [\mathbf{u}, (0, 1), \mathbf{v}]$;
$\rho_{20} : [\mathbf{u}, (1, 0), \mathbf{v}]$;
$\rho_{30} : [\mathbf{u}, (3, 0), (3, 1), (2, 1), \mathbf{v}]$;
$\rho_{40} : [\mathbf{u}, (0, 3), (1, 3), (1, 2), \mathbf{v}]$;
$\rho_{31} : [\mathbf{u}, (3, 0), (2, 0), (2, 1), \mathbf{v}]$;
$\rho_{41} : [\mathbf{u}, (0, 3), (0, 2), (1, 2), \mathbf{v}]$.

For $\mathbf{v} = (1, 2)$, we have: (See Fig.3(a))

$\rho_{10} : [\mathbf{u}, (0, 1), (0, 2), \mathbf{v}]$;
$\rho_{20} : [\mathbf{u}, (1, 0), (1, 1), \mathbf{v}]$;
$\rho_{30} : [\mathbf{u}, (3, 0), (3, 1), (3, 2), (2, 2), \mathbf{v}]$;
$\rho_{40} : [\mathbf{u}, (0, 3), (1, 3), \mathbf{v}]$;
$\rho_{11} : [\mathbf{u}, (0, 1), (1, 1), \mathbf{v}]$;
$\rho_{21} : [\mathbf{u}, (1, 0), (1, 3), \mathbf{v}]$;
$\rho_{31} : [\mathbf{u}, (3, 0), (2, 0), (2, 1), (2, 2), \mathbf{v}]$;
$\rho_{41} : [\mathbf{u}, (0, 3), (0, 2), \mathbf{v}]$.

For $\mathbf{v} = (2, 2)$, we have: (See Fig.3(c))

$\rho_{10} : [\mathbf{u}, (0, 1), (0, 2), (1, 2), \mathbf{v}]$;
$\rho_{20} : [\mathbf{u}, (1, 0), (2, 0), (2, 1), \mathbf{v}]$;
$\rho_{30} : [\mathbf{u}, (3, 0), (3, 1), (3, 2), \mathbf{v}]$;
$\rho_{40} : [\mathbf{u}, (0, 3), (1, 3), (2, 3), \mathbf{v}]$;
$\rho_{11} : [\mathbf{u}, (0, 1), (3, 1), (2, 1), \mathbf{v}]$;
$\rho_{21} : [\mathbf{u}, (1, 0), (1, 3), (1, 2), \mathbf{v}]$;
$\rho_{31} : [\mathbf{u}, (3, 0), (2, 0), (2, 3), \mathbf{v}]$;
$\rho_{41} : [\mathbf{u}, (0, 3), (0, 2), (3, 2), \mathbf{v}]$.

If none of the $F^{\mathbf{u}} \cup F^{\mathbf{v}} = \emptyset$, and one faulty edge is on some path, one not, then we can find an alternative path for the blocked path as shown above. Hence, we only need to consider when one faulty edge lies on some path, say $f_1 \in \rho_{i0}, i \in \{1, 2, 3, 4\}$, and the other one lies on its alternative path $f_2 \in \rho_{i1}$.

1) $\mathbf{v} = (0, 2)$ (please refer to Fig.2(a)): if ρ_{20} and ρ_{21} are both blocked, i.e., one of $e_1 = ((1, 0), (1, 1))$ and $e_2 = ((1, 1), (1, 2))$, and one of $((1, 0), (1, 3))$ and $((1, 2), (1, 3))$ are faulty, then define
$\rho_{22} : [\mathbf{u}, (1, 0), (2, 0), (2, 1), (2, 2), (1, 2), \mathbf{v}]$;
if ρ_{30} and ρ_{31} are both blocked, then define
$\rho_{32} : [\mathbf{u}, (3, 0), (2, 0), (2, 1), (2, 2), (3, 2), \mathbf{v}]$.
2) $\mathbf{v} = (1, 1)$ (please refer to Fig.2(c)): if ρ_{30} and ρ_{31} are both blocked, then define
$\rho_{32} : [\mathbf{u}, (3, 0), (3, 3), (2, 3), (2, 2), (2, 1), \mathbf{v}]$;
if ρ_{40} and ρ_{41} are both blocked, then define
$\rho_{42} : [\mathbf{u}, (0, 3), (3, 3), (3, 2), (2, 2), (1, 2), \mathbf{v}]$;

3) $\mathbf{v} = (2,1)$ (please refer to Fig.3(a)): if ρ_{10} and ρ_{11} are both blocked, then define $\rho_{12} : [\mathbf{u}, (0,1), (0,2), (1,2), (1,1), \mathbf{v}]$; if ρ_{20} and ρ_{21} are both blocked, then define $\rho_{22} : [\mathbf{u}, (1,0), (1,3), (2,3), (2,0), \mathbf{v}]$ and ρ_{42} be $\mathbf{u}, (0,3), (0,2), (1,2), (2,2), \mathbf{v}]$; if ρ_{30} and ρ_{31} are both blocked, then define $\rho_{32} : [\mathbf{u}, (3,0), (3,3), (3,2), (3,1), \mathbf{v}]$; if ρ_{40} and ρ_{41} are both blocked, then define $\rho_{42} : [\mathbf{u}, (0,3), (3,3), (3,2), (2,2), \mathbf{v}]$.
4) $\mathbf{v} = (2,2)$ (please refer to Fig.3(c)): if ρ_{10} and ρ_{11} are both blocked, then define $\rho_{12} : [\mathbf{u}, (0,1), (1,1), \mathbf{v}]$; if ρ_{20} and ρ_{21} are both blocked, then define $\rho_{22} : [\mathbf{u}, (1,0), (1,1), \mathbf{v}]$; if ρ_{30} and ρ_{31} are both blocked, then define $\rho_{32} : [\mathbf{u}, (0,3), (3,3), (2,3), \mathbf{v}]$; if ρ_{40} and ρ_{41} are both blocked, then define $\rho_{42} : [\mathbf{u}, (3,0), (3,3), (3,2), \mathbf{v}]$.

If only one faulty edge is linked with $\mathbf{u}$ or $\mathbf{v}$, then we only need to build 3 disjoint paths between $\mathbf{u}$ and $\mathbf{v}$, and this reduces to what we have discussed above.

If both faulty edges are of $\mathbf{u}$ and/or $\mathbf{v}$'s neighbors, then we have the following several cases.

1) both belong to either $\mathbf{u}$ or $\mathbf{v}$'s neighbors, then we only need to find two disjoint paths and it has been done;
2) one is $\mathbf{u}$'s neighbor, and the other is $\mathbf{v}$'s neighbor, and they are on the same path as shown in Fig.1~Fig.3, then we have done as well;
3) one is $\mathbf{u}$'s neighbor, and the other is $\mathbf{v}$'s neighbor, and they are not on the same path as shown in the above Figures, we need to rebuild some paths.
   - $\mathbf{v} = (0,1)$ (refer to Fig.1(a)): if ρ_{20} and ρ_{30} are blocked, then define $\rho_{32} : [\mathbf{u}, (3,0), (3,1), (2,1), (1,1), \mathbf{v}]$. if ρ_{20} and ρ_{40} are blocked, then define $\rho_{42} : [\mathbf{u}, (0,3), (1,3), (1,2), (1,1), \mathbf{v}]$. if ρ_{30} and ρ_{40} are blocked, then define $\rho_{42} : [\mathbf{u}, (0,3), (3,3), (3,2), (3,1), (0,1), \mathbf{v}]$
   - $\mathbf{v} = (0,2)$ (refer to Fig.2(a)): if ρ_{10} and ρ_{20} are blocked, then define $\rho_{22} : [\mathbf{u}, (1,0), (1,1), (0,1), \mathbf{v}]$; if ρ_{10} and ρ_{30} are blocked, then define $\rho_{32} : [\mathbf{u}, (3,0), (3,1), (0,1), \mathbf{v}]$; if ρ_{10} and ρ_{40} are blocked, then define $\rho_{22} : [\mathbf{u}, (1,0), (1,1), (0,1), \mathbf{v}]$, and $\rho_{42} : [\mathbf{u}, (0,3), (1,3), (1,2), \mathbf{v}]$; if ρ_{20} and ρ_{30} are blocked, then define $\rho_{32} : [\mathbf{u}, (3,0), (3,1), (3,2), (2,2), (2,1), \mathbf{v}]$; if ρ_{20} and ρ_{40} are blocked, then define $\rho_{42} : [\mathbf{u}, (0,3), (1,3), (1,2), \mathbf{v}]$; if ρ_{30} and ρ_{40} are blocked, then define $\rho_{42} : [\mathbf{u}, (0,3), (3,3), (3,2), \mathbf{v}]$.
   - $\mathbf{v} = (1,1)$ (refer to Fig.2(c)): if ρ_{10} and ρ_{20} are blocked, then define $\rho_{32} = [\mathbf{u}, (3,0), (3,1), (0,1), \mathbf{v}]$, and $\rho_{22} = [\mathbf{u}, (1,0), (2,0), (2,1), \mathbf{v}]$; if ρ_{10} and ρ_{30} are blocked, then define $\rho_{32} = [\mathbf{u}, (3,0), (3,1), (0,1), \mathbf{v}]$; if ρ_{10} and ρ_{40} are blocked, then define $\rho_{42} = [\mathbf{u}, (0,3), (0,2), (0,1), \mathbf{v}]$; if ρ_{20} and ρ_{30} are blocked, then define $\rho_{32} = [\mathbf{u}, (3,0), (2,0), (1,0), \mathbf{v}]$; if ρ_{20} and ρ_{40} are blocked, then define $\rho_{42} = [\mathbf{u}, (0,3), (1,3), (1,0), \mathbf{v}]$; if ρ_{30} and ρ_{40} are blocked, then define $\rho_{42} = [\mathbf{u}, (0,3), (1,3), (2,3), (2,2), (2,1), \mathbf{v}]$.
   - $\mathbf{v} = (2,1)$ (refer to Fig.3(a)): if ρ_{10} and ρ_{20} are blocked, define $\rho_{22} = [\mathbf{u}, (1,0), (1,1), \mathbf{v}]$; if ρ_{10} and ρ_{30} are blocked, define $\rho_{32} = [\mathbf{u}, (3,0), (3,1), (0,1), (1,1), \mathbf{v}]$; if ρ_{10} and ρ_{40} are blocked, define $\rho_{42} = [\mathbf{u}, (0,3), (0,2), (0,1), (1,1), \mathbf{v}]$; if ρ_{20} and ρ_{30} are blocked, define $\rho_{32} = [\mathbf{u}, (3,0), (2,0), \mathbf{v}]$; if ρ_{20} and ρ_{40} are blocked, define $\rho_{42} = [\mathbf{u}, (0,3), (1,3), (2,3), (2,0), \mathbf{v}]$; if ρ_{30} and ρ_{40} are blocked, define $\rho_{42} = [\mathbf{u}, (0,3), (3,3), (3,2), (3,1), \mathbf{v}]$.
   - $\mathbf{v} = (2,2)$ (refer to Fig.3(c)), we have the following cases to consider: if ρ_{10} and ρ_{20} are blocked, define $\rho_{22} = [\mathbf{u}, (1,0), (1,1), (1,2), \mathbf{v}]$; if ρ_{10} and ρ_{30} are blocked, define $\rho_{22} = [\mathbf{u}, (1,0), (1,1), (1,2), \mathbf{v}]$, and

$\rho_{32} = [\mathbf{u}, (3,0), (2,0), (2,1), \mathbf{v}]$;
if ρ_{10} and ρ_{40} are blocked, define
$\rho_{42} = [\mathbf{u}, (0,3), (1,3), (1,2), \mathbf{v}]$;
if ρ_{20} and ρ_{30} are blocked, define
$\rho_{32} = [\mathbf{u}, (3,0), (2,0), (2,1), \mathbf{v}]$;
if ρ_{20} and ρ_{40} are blocked, define
$\rho_{42} = [\mathbf{u}, (0,3), (1,3), (2,3), (2,0), (2,1), \mathbf{v}]$;
if ρ_{30} and ρ_{40} are blocked, define
$\rho_{42} = [\mathbf{u}, (0,3), (3,3), (3,2), \mathbf{v}]$.

APPENDIX B. THERE IS ONE DIMENSION ONE FAULTY EDGE

Let the dimension one faulty edge $f = (\mathbf{r}, \mathbf{r}^+)$. We have three cases to consider: $v_1 = 0$; and $v_1 \neq 0$.

Case 1 $v_1 = 0$. This means that $\mathbf{u}$ and $\mathbf{v}$ lie in Q_0.

Case 1.1 $|F_0| = 2n - 3$.

(A11-1) $f \in F^{\mathbf{u}} \cup F^{\mathbf{v}}$. W.l.o.g., let $f = (\mathbf{u}, \mathbf{u}^-)$.

(A11-1.1) $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 1$.

If there exists $e \in F^F$, we assume e is healthy, and find disjoint paths in Q_0 by induction. If e lies on some path, then do a *U-jump*. Build one more path as follows: $[\mathbf{u}, \mathbf{u}^+, \rho, \mathbf{v}^+, \mathbf{v}]$, where the sub-path ρ is a path in Q_1 that links with $\mathbf{u}^+$ and $\mathbf{v}^+$.

Please note that ρ can be easily obtained by induction. For example, to find ρ in Q_1, even if $|F_1| > 2n - 4$, we can still find such a path by assume some of the faulty edges are healthy until $|F_1| \leq 2n-4$ is satisfied and then use induction to find disjoint paths. Let ρ be one path that doesn't include originally fault edges. This is true, as there are at most $2n-3$ faulty edges (f lies outside any Q_i, for $i = 0, 1, \ldots, k-1$) in Q_1, and each node has $2n - 2$ neighbors.

If $F^F = \emptyset$, and $(\mathbf{u}, \mathbf{v}) \in F$, assume $(\mathbf{u}, \mathbf{v})$ is healthy. Find disjoint paths in Q_0 by induction. Remove the path $[\mathbf{u}, \mathbf{v}]$; find a path: $[\mathbf{u}, \mathbf{u}^+, \rho, \mathbf{v}^+, \mathbf{v}]$, where ρ is a path in Q_1. We have done.

If $F^F = \emptyset$, and $(\mathbf{u}, \mathbf{v}) \notin F$, we have $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 2$. Assume faulty edge $(\mathbf{u}, \mathbf{w})$ is healthy, find disjoint path by induction in Q_0, and remove the path that passes $\mathbf{w}$; find a path: $[\mathbf{u}, \mathbf{u}^+, \rho, \mathbf{v}^+, \mathbf{v}]$, where ρ is a path in Q_1. We have done.

(A11-1.2) $|F^{\mathbf{u}}| \leq |F^{\mathbf{v}}|$.

We have $|F_0^{\mathbf{u}}| \leq |F_0^{\mathbf{v}}| - 1$. Assume faulty edge $(\mathbf{w}, \mathbf{v})$ is healthy, and find disjoint paths in Q_0 by induction. There is a path that passes the node $\mathbf{w}$, do a *U-jump*. Find one more path: $[\mathbf{u}, \mathbf{u}^+, \rho, \mathbf{v}^+, \mathbf{v}]$.

(A11-2) $f \notin F^{\mathbf{u}} \cup F^{\mathbf{v}}$.

(A11-2.1) $|F^F| \geq 1$.

Let $e = (\mathbf{s}, \mathbf{t}) \in F^F$, such that $(\mathbf{s}, \mathbf{s}^+) \notin F$ and $(\mathbf{t}, \mathbf{t}^+) \notin F$, find disjoint paths in Q_0 by induction. If e lies on some path, do a $D-jump$. Find a path in Q_1 that links with $\mathbf{u}^+$ and $\mathbf{v}^+$ and avoid e^+. Find a path in Q_{k-1} that links with $\mathbf{u}^-$ and $\mathbf{v}^-$.

(A11-2.2) $|F^F| = 0$.

If $|F^{\mathbf{u}}| > |F^{\mathbf{v}}|$, assume faulty edge $(\mathbf{u}, \mathbf{w})$ is healthy, and find disjoint paths by induction in Q_0. Remove the path that passes the node $\mathbf{w}$. Find two paths: $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^-, \mathbf{v}]$, and $[\mathbf{u}, \mathbf{u}^+, \rho_2, \mathbf{v}^+, \mathbf{v}]$, $\rho_1 \in Q_{k-1}$ and $\rho_2 \in Q_1$.

If $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$, as $|F^F| = 0$, we must have $(\mathbf{u}, \mathbf{v}) \in F$; assume it is healthy, and find disjoint paths in Q_0 by induction. Remove the path $[\mathbf{u}, \mathbf{v}]$. Find two paths: $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^-, \mathbf{v}]$, and $[\mathbf{u}, \mathbf{u}^+, \rho_2, \mathbf{v}^+, \mathbf{v}]$, $\rho_1 \in Q_{k-1}$ and $\rho_2 \in Q_1$.

Case 1.2 $|F_0| \leq 2n - 4$.

(A12-1) $f \in F^{\mathbf{u}} \cup F^{\mathbf{v}}$. W.l.o.g., let $f = (\mathbf{u}, \mathbf{u}^-)$.

(A12-1.1) $|F_{\mathbf{u}}| \geq |F_{\mathbf{v}}| + 1$.

By induction, find disjoint paths in Q_0, and find a path $[\mathbf{u}, \mathbf{u}^+, \rho, \mathbf{v}^+, \mathbf{v}]$, $\rho \in Q_1$.

(A12-1.2) $|F_{\mathbf{u}}| \leq |F_{\mathbf{v}}|$.

Assume faulty edge $(\mathbf{w}, \mathbf{v})$ is healthy, and by induction, find disjoint paths in Q_0. Suppose $(\mathbf{u}, \mathbf{s})$ lies on the path that passes node $\mathbf{w}$. Remove this path. Find two more paths: $[\mathbf{u}, \mathbf{s}, \mathbf{s}^-, \rho_1, \mathbf{v}^-, \mathbf{v}]$, $\rho_1 \in Q_{k-1}$, and $[\mathbf{u}, \mathbf{u}^+, \rho_2, \mathbf{v}^+, \mathbf{v}]$, $\rho_2 \in Q_1$.

(A12-2) $f \notin F^{\mathbf{u}} \cup F^{\mathbf{v}}$.

By induction, find disjoint paths in Q_0. Now, we need to find another two disjoint paths.

(A12-2.1) It is possible that there exist $i = 1$ or $i = k-1$, such that $|F_i| > 2n - 4$. W.l.o.g., suppose $F_{k-1} > 2n - 4$. Hence, $F_1 < 2$. So, in Q_1, by induction, we find disjoint paths between $\mathbf{u}^+$ and $\mathbf{v}^+$, and choose the shortest one as the one we need. To find a path between $\mathbf{u}^-$ and $\mathbf{v}^-$, we have the following two cases.

(A12-2.1.1) $|F_{k-1}| = 2n-2$.

A special case is that $|F_{k-1}^{\mathbf{u}^-}| = 2n-2$ or $|F_{k-1}^{\mathbf{v}^-}| = 2n-2$. It can be solved by finding a shortest path between $\mathbf{u}^-$ and $\mathbf{v}^-$, and do a *U-jump* to avoid the faulty edge on the path.

If $|F^F| \geq 2$, we assume two of F^F edges to be healthy, and by induction find disjoint paths between $\mathbf{u}^-$ and $\mathbf{v}^-$ in Q_{k-1}. Choose one of the shortest path that has no original faulty edges on as the one we need.

If $|F^F| \leq 1$, w.l.o.g., suppose $|F_{k-1}^{\mathbf{u}^-}| \geq |F_{k-1}^{\mathbf{v}^-}|$. Choose two of $\mathbf{u}^-$'s faulty links to be healthy, and find disjoint paths by induction in Q_{k-1}. Choose one of the paths that has no original faulty edges on as the one we need.

(A1.2-2.1.2) $|F_{k-1}| = 2n-3$.

If $|F_{k-1}^F| \geq 1$, we assume one F_{k-1}^F edge to be healthy, and by induction find disjoint paths between $\mathbf{u}^-$ and $\mathbf{v}^-$ in Q_{k-1}. Choose one of the paths that has no original faulty edges on as the one we need.

If $F_{k-1}^F = \emptyset$, W.l.o.g., suppose $|F_{k-1}^{\mathbf{u}^-}| \geq |F_{k-1}^{\mathbf{v}^-}|$. Choose one of $\mathbf{u}^-$'s faulty links to be healthy, and find disjoint paths by induction in Q_{k-1}. Choose one of the paths that has no original faulty edges on as the one we need.

(A1.2-2.2) $|F_{k-1}| \leq 2n-4$ and $|F_1| \leq 2n-4$.

By induction, find disjoint paths in Q_{k-1} and Q_1, and each choose a shortest path as the one we need.

Case 2 $v_1 \neq 0$.

W.l.o.g., suppose $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}|$, and $v_1 \leq \lfloor \frac{k}{2} \rfloor$.

Case 2.1 $dist(\mathbf{u}, \mathbf{v}^{-v_1}) = 0$.

(A21-1) $\forall e \in N_{\mathbf{u}}$, if $e \notin F$, then $e^{v_1} \notin F$.

The paths can be built as follows: start from $\mathbf{u}$, followed by a healthy link and then straight down to a neighbor of $\mathbf{v}$ then reach $\mathbf{v}$. Find another two paths: $\sigma_1 : [\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \ldots, \mathbf{v}^-, \mathbf{v}]$ and $\sigma_2 : [\mathbf{u}, \mathbf{u}^{k-1}, \mathbf{u}^{k-2} \ldots, \mathbf{v}^+, \mathbf{v}]$.

The faulty edge $f = (\mathbf{r}, \mathbf{r}^+)$ may block one of the disjoint paths. If $v_1 = 1$ and $f = (\mathbf{u}, \mathbf{v})$, this is a trivial case as we can just remove the path $[\mathbf{u}, \mathbf{v}]$. In what follows, we assume $f \neq (\mathbf{u}, \mathbf{v})$, or if f is any of $(\mathbf{u}, \mathbf{u}^+)$ and $(\mathbf{v}, \mathbf{v}^-)$, we have $v_1 \geq 2$.

(A21-1.1) $\mathbf{r} = \mathbf{u}^h, h \in \{1, 2, \ldots, v_1 - 2\} (v_1 - 2 \geq 1)$.

There must exists $\mathbf{t} \in N_{\mathbf{u}}^0, (\mathbf{u}, \mathbf{t}) \notin F$, such that all the edges incident to $\mathbf{t}^i$ are healthy, for $i = 0, 1, \ldots, k-1$. This is true as there are at most $|F_0| + |F_{v_1}| \leq 2n-3$, and there are $2n-2$ neighbors for each of the node in Q_i, for $i = 0, 1, 2, \ldots, k-1$.

We remove the path that passes $\mathbf{t}$, and build the following path: $[\mathbf{u}, \mathbf{t}, \mathbf{s}, \mathbf{s}^+, \mathbf{s}^2, \ldots, \mathbf{s}^{v_1}, \mathbf{t}^{v_1}, \mathbf{v}]$, where $\mathbf{s}$ is $\mathbf{t}$'s neighbor. The length of this path is at most $\lfloor \frac{k}{2} \rfloor + 4$. Another path can be built as: $[\mathbf{u}, \mathbf{u}^+, \ldots, \mathbf{u}^h, \mathbf{t}^h, \mathbf{t}^{h+1}, \mathbf{u}^{h+1}, \mathbf{u}^{h+2}, \ldots, \mathbf{v}^-, \mathbf{v}]$. The length of this path is at most $\lfloor \frac{k}{2} \rfloor + 2$.

(A21-1.2) $\mathbf{r} = \mathbf{u}^h, h \in \{v_1+1, v_1+2, \ldots, k-2\}$.

For this case, we need to modify the path σ_2 to be: $[\mathbf{u}^{v_1}(=\mathbf{v}), \mathbf{u}^{v_1+1}, \ldots, \mathbf{u}^h, \mathbf{t}^h, \mathbf{t}^{h+1}, \mathbf{v}^{h+1}, \mathbf{v}^{h+2}, \ldots, \mathbf{v}^{k-1}, \mathbf{u}]$, where $\mathbf{t}$ is a neighbor of $\mathbf{u}$, such that $(\mathbf{u}^h, \mathbf{t}^h) \notin F$ and $(\mathbf{u}^{h+1}, \mathbf{t}^{h+1}) \notin F$. This is also true as there are at most $2n-3$ faulty edges, and each node has $2n-2$ links in each subgraph $Q_i, i = 0, 1, 2, \ldots, k-1$.

(A21-1.3) $\mathbf{r} = \mathbf{u}$. In this case, we must have $|F^{\mathbf{u}}| > |F^{\mathbf{v}}|$. There is no need to do any change but remove the path $[\mathbf{u}, \mathbf{u}^+, \ldots, \mathbf{v}]$.

(A21-1.4) $\mathbf{r} = \mathbf{u}^-$. In this case, we must have $|F^{\mathbf{u}}| > |F^{\mathbf{v}}|$. There is no need to do any change but remove the path $[\mathbf{u}, \mathbf{u}^+, \ldots, \mathbf{v}]$.

(A21-1.5) $\mathbf{r} = \mathbf{v}^+$. If there exists $(\mathbf{u}, \mathbf{t}) \in F$, such that $(\mathbf{v}, \mathbf{t}^{v_1}) \notin F$, adjust the path σ_2 as follows: $[\mathbf{u}, \mathbf{u}^-, \ldots, \mathbf{v}^+, \rho, \mathbf{t}^{v_1+1}, \mathbf{t}^{b_1}, \mathbf{v}]$, where ρ is a path between $\mathbf{t}^{v_1+1}$ and $\mathbf{v}^+$. This path can be easily build by induction, and the length is at most L_{n-1}. Hence the total length of path σ_2 is $L_{n-1} + k + 1$.

(A21-1.6) $\mathbf{r} = \mathbf{v}^-$. Similar to (A21-1.2) above, suppose $\mathbf{t}$ is such a node that links with $\mathbf{u}$, and all its neighbor links and their parallel edges are healthy. If $v_1 \geq 2$, and there exists $(\mathbf{u}, \mathbf{s}) \in F$, such that $(\mathbf{v}, \mathbf{s}^{v_1}) \notin F$, remove the path that passes $\mathbf{s}$ and the path that passes $\mathbf{t}$. Build the following two paths: $[\mathbf{u}, \mathbf{t}, \mathbf{t}^-, \rho, \mathbf{s}^-, \mathbf{s}, \mathbf{s}^+, \ldots, \mathbf{s}^{v_1}, \mathbf{v}]$, and $[\mathbf{u}, \mathbf{t}^+, \mathbf{r}^+, \mathbf{s}^2, \ldots, \mathbf{r}^{v_1}, \mathbf{v}]$, where $\rho \in Q_{k-1}$ can be build by finding disjoint paths between $\mathbf{u}^-$ and $\mathbf{r}^-$ by induction.

(A21-1.7) $\mathbf{r} = \mathbf{s}^h$, where $\mathbf{s}$ is $\mathbf{u}$'s neighbor for some $h \in \{0, 1, \ldots, v_1 - 1\}$ and f is blocking some path ρ. There exists some node $\mathbf{t}$ that is a neighbor of $\mathbf{s}$ and different from $\mathbf{u}$, such that $(\mathbf{s}^j, \mathbf{t}^j) \notin F$, for all $j \in \{0, 1, \ldots, k - 1\}$. The path ρ can be adjusted as follows: $[\mathbf{u}, \mathbf{s}, \mathbf{s}^1, \ldots, \mathbf{r}, \mathbf{t}^h, \mathbf{t}^{h+1}, \ldots, \mathbf{t}^{v_1}, \mathbf{s}^{v_1}, \mathbf{v}]$.

(A21-2) $\exists e_1 = (\mathbf{u}, \mathbf{t}) \notin F^{\mathbf{u}}$ and $e^{v_1} \in F$, and $\exists e_2 = (\mathbf{u}, \mathbf{s}) \in F^{\mathbf{u}}$ and $e^{v_1} \notin F$. In this case, there are at most $2n - 4$ faulty edges in each subgraph $Q_i, i = 0, 1, \ldots, k - 1$.

Find a node $\mathbf{w}$ in Q_0, such that $\mathbf{r} \neq \mathbf{w}^i$, for every $i = 0, 1, \ldots, k - 1$; $f \notin N^{\mathbf{w}^i}$, for $i = 0, 1, \ldots, k - 1$. Such $\mathbf{w}$ exists. There are k^{n-1} nodes in Q_0, and there are at most $2n - 2$ faulty edges in Q_n^k. Node $\mathbf{u}, \mathbf{r}$ will block $2n - 1$ nodes each. As $\mathbf{u}$ and $\mathbf{v}$ each incidents with at least one faulty link and f has already been considered, we have at most $2n-2-3$ faulty edges left . with each block two nodes. Hence the number of available nodes is at least $k^{n-1} - 2(2n-1) - 2(2n-5) \geq 4$. So, there are at least 4 choices for $\mathbf{w}$.

The steps of building disjoint paths in this case are as follows.

Step 1: Find disjoint paths between $\mathbf{u}$ and $\mathbf{w}$ in Q_0 by induction.

Step 2: Remove $\mathbf{w}$ from each path, and extend the path by straight going down until reach a neighbor of $\mathbf{w}^{v_1}$.

Step 3: Assume the edges are faulty if they are (1) incident with $\mathbf{w}^{v_1}$, (2) not incident with any path built in step 2, and (3) lie inside Q_{v_1}.

If there are at most $2n - 4$ faulty edges in Q_{v_1}, we find disjoint paths between $\mathbf{v}$ and $\mathbf{w}^{v_1}$by induction, and link these paths with the paths built in step 2.

If there are $2n - 3$ faulty edges in Q_{v_1}, there must exist $e \in Q_{v_1}$, such that it is not incident with $\mathbf{v}$ and $\mathbf{w}^{v_1}$. Obviously, there are no faulty edges in Q_i for $i \neq 0, v_1$. Assume e is healthy, and find disjoint paths between $\mathbf{v}$ and $\mathbf{w}^{v_1}$by induction. If e lies on some path, do *D-jump(e)*.

Step 4: Build one or two more paths.

(A21-2.1) $f = (\mathbf{u}, \mathbf{u}^-)$. If $|F^{\mathbf{u}}| > |F^{\mathbf{v}}|$, then we only need to build one more path: (P1): $[\mathbf{u}, \mathbf{u}^+, \ldots, \mathbf{v}]$. If $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$, then we need to build two more paths: one is path (P1) as above. There have an healthy edge $(\mathbf{u}, \mathbf{s})$, that is not on any path. We build the following path: (P2): $[\mathbf{u}, \mathbf{s}, \mathbf{s}^-, \rho_1, \mathbf{u}^-, \mathbf{u}^{-2}, \ldots, \mathbf{u}^{v_1+1}, \mathbf{v}]$, where ρ_1 is a path lies in Q_{k-1} that can be found by induction.

(A21-2.2) $f = (\mathbf{u}, \mathbf{u}^+)$. Path (P2) as built above is kept for this case.

If $v_1 = 1$, or $v_1 > 1$ and $|F^{\mathbf{u}}| > |F^{\mathbf{v}}|$, then no need to find more path.

If $v_1 > 1$ and $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$, then we go back to Step 3. Before finding disjoint paths in Q_{v_1} by induction, we assume one of $\mathbf{v}$'s faulty edge is healthy. After finding disjoint paths, name the path passes this faulty edge as ρ. Suppose $\rho \cap Q_{v_1-1} = \mathbf{s}^{v_1-1}$. In Q_{v_1-1}, find a path to link $\mathbf{v}^-$ and $\mathbf{s}^{v_1-1}$ and disjoint with all other paths. This can be done by carefully setting some of $\mathbf{w}^{v_1-1}$'s links as faulty and using induction to find disjoint paths between $\mathbf{v}^-$ and $\mathbf{w}^{v_1-1}$. Path ρ is now changed to be: $\rho : [\mathbf{u}, \ldots, \mathbf{s}, \mathbf{s}^{v_1-1}, \ldots, \mathbf{v}^-, \mathbf{v}]$.

(A21-2.3) $f = (\mathbf{v}, \mathbf{v}^-)$. If $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$, similar to above (A21-2.2) to build a new path. Otherwise, we don't need any more path.

(A21-2.4) $f = (\mathbf{v}, \mathbf{v}^+)$. If $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$, similar to above (A21-2.1) to build a new path. Otherwise, we don't need any more path.

Case 2.2 $dist(\mathbf{u}, \mathbf{v}^{-v_1}) = 1$ has been discussed in the main body of the paper.

Case 2.3 $dist(\mathbf{u}, \mathbf{v}^{-v_1}) > 1$.

We expand the definition for F^F: if we are considering finding disjoint paths between node $\mathbf{w}$ and node $\mathbf{z}$ in a sub-graph Q_i, F^F is defined as the set of faulty edges that are neither incident with $\mathbf{w}$ nor incident with $\mathbf{z}$. This expansion also applies to appendix C.

General steps in building disjoint paths:

G1. Do $map(\mathbf{v}, 0)$.

G2. Do $DP(\mathbf{u}, \mathbf{v}^-)$.

G3. Extend the paths to $\mathbf{v}$.

G4. Find two more paths: (Ga): [$\mathbf{u}$, $\mathbf{u}^-$, ρ_1, $\mathbf{v}^{-v_1-1}$, $\mathbf{v}^{-v_1}$, ..., $\mathbf{v}$]; (Gb): [$\mathbf{u}$, $\mathbf{u}^+$, ..., $\mathbf{u}^{v_1}$, $\mathbf{u}^{v_1+1}$, ρ_2, $\mathbf{v}^+$, $\mathbf{v}$].

If all of these paths are not blocked, then we have done. However, there will have some case that we need to avoid faults. Below is the detailed discussion.

(A23-1) $v_1 = 1$.

(A23-1.1) $f = (\mathbf{u}, \mathbf{u}^+)$

(A23-1.1.1) $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$

Do $map(\mathbf{v}, 0)$. There exists a faulty edge e, incident with $\mathbf{v}^-$. Assume e is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Suppose $\mathbf{w}$ lies on the path that e lies on. Remove this path and extend all other paths to $\mathbf{v}$. Build the following two more paths: (a): $[\mathbf{u}, \mathbf{w}, \mathbf{w}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$, where ρ_1 can be found by induction in Q_{k-1} on $\mathbf{w}^-$ and $\mathbf{v}^{-2}$ to avoid $\mathbf{u}^-$; (b): $[\mathbf{u}, \mathbf{u}^-, \dots, \mathbf{u}^{v_1+1}, \rho_2, \mathbf{v}^+, \mathbf{v}]$.

(A23-1.1.2) $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 1$

Do $map(\mathbf{v}, 0)$.

If there are at most $2n - 4$ faulty edges in Q_0, do $DP(\mathbf{u}, \mathbf{v}^-)$, and extend the disjoint paths to $\mathbf{v}$. Build one more path: (a): $[\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$, $\rho_1 \in Q_{k-1}$.

If there are $2n - 3$ faulty edges in Q_0, we have two cases to consider: (i) $|F^{\mathbf{u}}| = |F^{\mathbf{v}}| + 1$. This means that $F^F \geq 1$. Let $e \in F^F$, and assume e is healthy, do $DP(\mathbf{u}, \mathbf{v}^-)$. If e lies on some path, do a *U-jump*. Build path (a) as above but need to avoid existing paths (the jumped up edge) and node $\mathbf{u}^-$. (ii) $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 2$. Assume e, an faulty incident with $\mathbf{u}$, is healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Remove this path and extend all of the other paths to $\mathbf{v}$.

(A23-1.2) $f = (\mathbf{u}, \mathbf{u}^-)$

(A23-1.2.1) $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$

Do $map(\mathbf{v}, 0)$. There exists e, an faulty edge incident with $\mathbf{v}^-$. We assume it is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Suppose $\mathbf{w}$ lies on the path that passes e. Remove this path and extend all other paths to $\mathbf{v}$. Build the following two paths: (a): $[\mathbf{u}, \mathbf{w}, \mathbf{w}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$; (b): $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{v}^+, \mathbf{v}]$.

(A23-1.2.2) $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 1$

Do $map(\mathbf{v}, 0)$. If there are at most $2n - 4$ faulty edges in Q_0, do $DP(\mathbf{u}, \mathbf{v}^-)$, and extend all paths to $\mathbf{v}$. Find one more path as follows: (a): $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{v}^+, \mathbf{v}]$.

If there are $2n - 3$ faulty edges in Q_0, and $|F^{\mathbf{u}}| = |F^{\mathbf{v}}| + 1$, there exists $e \in F^F$. Assume e is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. If e lies on some path, do a *U-jump(e)*. One more paths is built as above (a).

If there are $2n - 3$ faulty edges in Q_0, and $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 2$, assume e, a faulty edge incident with $\mathbf{u}$, is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Remove the path that e lies on, and extend all other paths to $\mathbf{v}$. One more paths is built as above (a).

(A23-1.3) $f = (\mathbf{v}, \mathbf{v}^-)$

Do $map(\mathbf{v}, 0)$. Assume $e \in F$, incident with $\mathbf{u}$, is healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Suppose $\mathbf{w}$, a neighbor of $\mathbf{v}^-$ lies on the path that passes edge e. Remove this path, and extend all other paths to $\mathbf{v}$. Build two more paths: (a): $\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{w}^-, \mathbf{w}, \mathbf{w}^+, \mathbf{v}$; (b): $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{v}^+, \mathbf{v}]$.

(A23-1.4) $f = (\mathbf{v}, \mathbf{v}^+)$

Compare to case (A23-1.3), we only need to adjust paths (a) and (b) as follows: (a): $\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^{-2}, \mathbf{v}^{-1}, \mathbf{v}$; (b): $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{w}^2, \mathbf{w}^+, \mathbf{v}]$.

(A23-1.5) $f = (\mathbf{v}^-, \mathbf{v}^{-2})$

After the four general steps G1, G2, G3, and G4, the path (Ga) is blocked by f. To avoid the faulty edge f, we choose one of the disjoint paths, such that it includes the following sub-path: $\mathbf{u}, \mathbf{w}, \rho_1, \mathbf{t}$, where $\mathbf{t}$ is a neighbor of $\mathbf{v}^-$, and ρ_1^+, ρ_1^- are faulty free. Remove this path and build another three paths: (a): $\mathbf{u}, \mathbf{u}^-, \rho_1^-, \mathbf{t}^-, \mathbf{t}, \mathbf{v}^-, \mathbf{v}$; (b): $[\mathbf{u}, \mathbf{w}, \mathbf{w}^+, \rho_1^+, \mathbf{t}^+, \mathbf{v}]$; (c): $[\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \rho_2, \mathbf{v}^+, \mathbf{v}]$.

(A23-1.6) $f = (\mathbf{u}^+, \mathbf{u}^2)$

After the 4 general steps, path (Gb) is blocked by f. There must exist $(\mathbf{u}^+, \mathbf{w}^+) \in Q_1$ is healthy. An alternative path is built as follows: $[\mathbf{u}, \mathbf{u}^+, \mathbf{w}^+, \rho_2, \mathbf{v}^+, \mathbf{v}]$, where $\rho_2 \in Q_1$.

(A23-2) $v_1 \geq 2$.

This case can be analyzed similarly to case (A23-1), and will not discussed here.

APPENDIX C.
THERE IS NO DIMENSION ONE FAULTY EDGE

W.l.o.g., suppose $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}|$

Case 1 $v_1 = 0$.

Case 1.1 $|F_0| = 2n - 2$.

(B11-1) $|F^F| \geq 2$.

Let $e_1, e_2 \in F^F$, and assume them are healthy. Find disjoint paths by induction in Q_0. If e_1 (or e_2) is on some path, do *U-jump(e_1)* (or *U-jump(e_2)*). Build two more paths as follows: (a):

$\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^-, \mathbf{v}$; (b): $\mathbf{u}, \mathbf{u}^+, \rho_2, \mathbf{v}^+, \mathbf{v}$; Both ρ_1 and ρ_2 are disjoint with all existing paths, and they are easy to find.

(B11-2) $|F^F| = 1$.

Assume $e_1 \in F^F$ is healthy.

If $(\mathbf{u}, \mathbf{v}) \in F$, then assume it is healthy, and find disjoint paths by induction in Q_0. Remove the path $[\mathbf{u}, \mathbf{v}]$. If e_1 is on some path, do *U-jump(e_1)*. Find another two more paths: $\sigma_1 : [\mathbf{u}, \mathbf{u}^-, \rho_1, \mathbf{v}^-, \mathbf{v}]$; $\sigma_2 : [\mathbf{u}, \mathbf{u}^+, \rho_2, \mathbf{v}^+, \mathbf{v}]$, where ρ_1, ρ_2 are all disjoint with existing paths.

If $(\mathbf{u}, \mathbf{v}) \notin F$, then $\exists e_2 \in F^{\mathbf{u}}$. Assume e_2 is healthy and find disjoint paths in Q_0 by induction. If e_1 lies on some path, do *U-jump(e_1)*. If e_2 lies on some path, remove this path. Build two more paths: σ_1, and σ_2 as in (B11-2) above.

(B11-3) $|F^F| = 0$.

If $(\mathbf{u}, \mathbf{v}) \in F$, then $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 1$. Assume $e \in F^{\mathbf{u}}$ and $(\mathbf{u}, \mathbf{v})$ are healthy. Find disjoint paths by induction in Q_0, remove the paths that pass the assumed faulty edges. Build two more paths: σ_1, and σ_2 as in (B11-2) above.

If $(\mathbf{u}, \mathbf{v}) \notin F$, then either $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 2$ or $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$. In the case that $|F^{\mathbf{u}}| \geq |F^{\mathbf{v}}| + 2$, we can assume two of $F^{\mathbf{u}}$'s faulty edges are healthy and find disjoint paths by induction in Q_0. Remove the two paths that pass the assumed faulty edges. Build two more paths: σ_1, and σ_2 as in (B11-2) above.

In the case that $|F^{\mathbf{u}}| = |F^{\mathbf{v}}|$, we assume the following faulty edge are healthy: $e_1 \in F^{\mathbf{u}}$ and $e_2 \in F^{\mathbf{v}}$. Find disjoint paths by induction. Build path σ_1, and σ_2 as in (B11-2) above if e_1 and e_2 are not lie on any path or lie on a same path. If e_1 and e_2 lie on different paths, do *D-jump(e_i)* for $i = 1, 2$, and remove the path σ_2.

Case 1.2 $|F_0| = 2n - 3$.

This means there is one faulty edge f lies in $Q_i, i \neq 0$.

(B12-1) $|F^F| \geq 1$.

Assume $e \in F^F$ is healthy. Find disjoint paths by induction in Q_0. If e lies on some path, to avoid it, either do *U-jump(e)* or *D-jump(e)* so as to avoid f as well. Build two more paths σ_1, and σ_2 similar to (B11-2) above; but need to avoid existing paths and faulty edge f.

(B12-2) $|F^F| = 0$.

If $(\mathbf{u}, \mathbf{v}) \in F$, assume it is healthy, and build disjoint paths by induction. Remove the path $[\mathbf{u}, \mathbf{v}]$ and build two more paths σ_1, and σ_2 similar to (B11-2) above; but need to avoid faulty edge f.

If $(\mathbf{u}, \mathbf{v}) \notin F$, then assume $e \in F^{\mathbf{u}}$ is healthy. Find disjoint paths by induction in Q_0. Remove the path if it include e. Find two more paths σ_1, and σ_2 similar to (B11-2) above; but need to avoid faulty edge f.

Case 1.3 $|F_0| \leq 2n - 4$.

Find disjoint paths by induction in Q_0. Find two more paths σ_1 and σ_2 similar to (B11-2) above; but need to avoid faulty edges. σ_1 and σ_2 can be obtained by induction even though there may have more than $2n - 4$ faulty edges lie in Q_{k-1} or Q_1. An exceptional case is, for example, when all of the $2n - 2$ edges in $N^{\mathbf{u}^-}$ are faulty. We can not find σ_1 in Q_{k-1}. This can be solved by doing a *U-jump(e)* on σ_1, where $e \in N^{\mathbf{u}^-}$.

Case 2 $v_1 \neq 0$

Case 2.1 $dist(\mathbf{u}, \mathbf{v}^{-v_1}) = 0$.

We will always include the following two paths in the disjoint paths set unless otherwise stated: $\sigma_1 : [\mathbf{u}, \mathbf{u}^-, \ldots, \mathbf{u}^{v_1+1}, \mathbf{v}]$, and $\sigma_2 : [\mathbf{u}, \mathbf{u}^+, \ldots, \mathbf{u}^{v_1-1}, \mathbf{v}]$.

(B21-1) $\forall e \in N_{\mathbf{u}}$, if $e \notin F$, then $e^{v_1} \notin F$.

To find disjoint paths for this case is straightforward.

(B21-2) $\exists e_1 \in N_{\mathbf{u}}, e_2 \in N_{\mathbf{v}}$, such that $e_1 \in F, e_1^{v_1} \notin F$, and $e_2 \notin F, e_2^{v_1} \in F$.

If $|F^{\mathbf{u}}| = 2n - 3$, we can easily find 3 disjoint paths between $\mathbf{u}$ and $\mathbf{v}$ as follows: $[\mathbf{u}, \mathbf{u}^+, \ldots, \mathbf{v}]$; $[\mathbf{u}, \mathbf{u}^-, \mathbf{r}^-, \mathbf{r}^{-2}, \ldots, \mathbf{r}^{v_1}, \mathbf{v}]$ (suppose $e_1 = (\mathbf{u}, \mathbf{r})$); and $[\mathbf{u}, \mathbf{s}, \mathbf{s}^-, \mathbf{u}^-, \mathbf{r}^{-2}, \ldots, \mathbf{u}^{v_1+1}, \mathbf{v}]$ (suppose $e_2 = (\mathbf{v}, \mathbf{s}^{-v_1})$). Similarly, we can find 3 paths for the case when $|F^{\mathbf{v}}| = 2n - 3$.

Now, we suppose $|F^{\mathbf{u}}| \leq 2n - 4$ and $|F^{\mathbf{v}}| \leq 2n - 4$.

Find a node $\mathbf{w} \in Q_0$, such that $N^{\mathbf{w}} \cap F = \emptyset$. By a simple calculation ($k^{n-1} - [(2n-2) + 1 + 2(2n-2) - 1] \geq 4$), we claim that such $\mathbf{w}$ exists.

Step 1.

If $|F_0| = 2n - 3$ (we have $F^F \neq \emptyset$), assume $e \in F^F$ is healthy; otherwise, no need to make any assumption. Find disjoint paths by induction in Q_0 between $\mathbf{u}$ and $\mathbf{w}$. If e lies on some path, do a *U-jump(e)*.

Step 2. Remove $\mathbf{w}$ and extend these paths each to a neighbor of $\mathbf{w}^{v_1}$

Step 3. Assume those edges in $N^{\mathbf{w}^{v_1}}$ that are not incident with any of the paths built in Step 2 are faulty.

(B21-2.1) If this results in at most $2n-4$ faulty edges in Q_{v_1}, find disjoint paths between $\mathbf{v}$ and $\mathbf{w}^{v_1}$ in Q_{v_1} by induction. Remove the node $\mathbf{w}^{v_1}$ from each of the path and thus links with the path built in Step 2.

(B21-2.2) If this results in $2n-3$ faulty edges in Q_{v_1}, and all faulty edges are incident with one of $\mathbf{v}$ and $\mathbf{w}^{v_1}$, then assume$e_3 \in F^{\mathbf{w}^{v_1}}$ is healthy. Otherwise, there exists $e_4 \in F$ not incident with any of $\mathbf{v}$ and $\mathbf{w}^{v_1}$, and assume e_4 is healthy. Find disjoint paths in Q_{v_1} by induction. Remove the path if it passes edge e_3, or do *D-jump(e_4)* if it passes edge e_4. All remained paths are incident with the paths built in step 2.

(B21-2.3) If this results in $2n-2$ faulty edges in Q_{v_1}, we will have several cases. (1) All faulty edges are incident with either $\mathbf{v}$ or $\mathbf{w}^{v_1}$. If $|F^{\mathbf{v}}| = |F^{\mathbf{w}^{v_1}}|$, for each of $\mathbf{v}$ and $\mathbf{w}^{v_1}$, we assume $e_3 \in F^{\mathbf{v}}$ and $e_4 \in F^{\mathbf{w}^{v_1}}$ are healthy and find disjoint paths by induction. Do *D-jump(e_3)* or *D-jump(e_4)* to avoid the faulty edge e_3 or e_4 if it lies on some path. If they lie on a same path, then remove that path. All remained paths are incident with the paths built in step 2.

(2) One of the faulty edge e_3 is neither incident with $\mathbf{v}$ nor $\mathbf{w}^{v_1}$. We must have $|F^{\mathbf{v}}|+1 \leq |F^{\mathbf{w}^{v_1}}|$. Thus we assume e_3 and $e_4 \in F^{\mathbf{w}^{v_1}}$ are healthy, and find disjoint paths between $\mathbf{w}^{v_1}$ and $\mathbf{v}$ by induction. If e_3 lies on some path, then do *D-jump(e_3)*; if e_4 lies on some path, then remove this path. All remained paths are incident with the paths built in step 2.

(3) At least two faulty edges e_3 and e_4 are neither incident with $\mathbf{v}$ nor $\mathbf{w}^{v_1}$. Assume e_3 and e_4 are healthy, and find disjoint paths between $\mathbf{w}^{v_1}$ and $\mathbf{v}$ by induction. If e_i lies on some path, then do a *D-jump(e_i)*, for $i = 3, 4$.

Case 2.2 $v_1 \neq 0, dist(\mathbf{u}, \mathbf{v}^{-v_1}) = 1$.

Need to consider that if $(\mathbf{u}, \mathbf{v}^-)$ and/or $(\mathbf{u}^+, \mathbf{v})$ are faulty.

If $(\mathbf{u}, \mathbf{v}^-)$ is healthy but $(\mathbf{u}^+, \mathbf{v})$ is faulty, we have the following cases.

If $(\mathbf{u}, \mathbf{v}^-)$ is faulty but $(\mathbf{u}^+, \mathbf{v})$ is healthy, we have the following cases.

If they are both faulty, we will discuss it in the following cases.

(B22-1) $v_1 = 1$.

Define two paths as: $\sigma_1 : [\mathbf{u}, \mathbf{u}^+, \mathbf{v}]$; $\sigma_2 : [\mathbf{u}, \mathbf{v}^-, \mathbf{v}]$. $\sigma_3 : [\mathbf{u}, \mathbf{u}^+, \mathbf{v}^{k-2}, \mathbf{v}^{k-3}, \ldots, \mathbf{v}^+, \mathbf{v}]$; $\sigma_4 : [\mathbf{u}, \mathbf{u}^+, \mathbf{u}^2, \mathbf{v}^+, \mathbf{v}]$; $\sigma_5 : [\mathbf{u}, \mathbf{u}^-, \mathbf{v}^{-2}, \mathbf{v}^-, \mathbf{v}]$.

(B22-1.1) After $map(\mathbf{v}, 0)$, we have $|F_0| = 2n-2$.

We have either at most one of $(\mathbf{u}, \mathbf{v}^-)$ and $(\mathbf{u}^+, \mathbf{v})$ is faulty.

We choose two faulty edges e_1, e_2 by the following rules:

(1) if $|F^F| \geq 2$, let $e_1, e_2 \in F^F$; (2) If $|F^F| = 1$, let $e_1 \in F^F$, and $e_2 \in F^{\mathbf{u}} \setminus F^{\mathbf{v}^-}$; (3) If $|F^F| = 0$ and $|F^{\mathbf{u}}| > |F^{\mathbf{u}}|$, we choose $e_1, e_2 \in F^{\mathbf{u}} \setminus F^{\mathbf{v}^-}$; (4) If $|F^F| = 0$ and $|F^{\mathbf{u}}| = |F^{\mathbf{v}^-}|$, we choose $e_1 \in F^{\mathbf{u}} \setminus F^{\mathbf{v}^-}, e_2 \in F^{\mathbf{v}^-} \setminus F^{\mathbf{u}}$.

Assume e_1, e_2 are healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$, and extend the disjoint paths to $\mathbf{v}$.

If e_1, e_2 are chosen from above case (4), and they both lie on a same path, just remove this path and include σ_1, σ_2, if $\sigma_i, i = 1, 2$ is faulty free, and σ_3 in disjoint paths; if they lie on different paths, do *U-jump(e_i)*, for $i = 1, 2$ and include σ_1 and σ_2, if $\sigma_i, i = 1, 2$ is faulty free, in disjoint paths.

Otherwise, e_1, e_2 are chosen from above case (1–3) if e_i lies on a path and $e_i \in F^F$, do *U-jump(e_i)*, $i \in \{1, 2\}$; if e_i lies on a path and $e_i \in F^{\mathbf{u}}$, $i \in \{1, 2\}$, remove the path. Include σ_1, σ_2 and σ_3, if $\sigma_i, i \in \{1, 2, 3\}$ exists, in disjoint paths.

(B22-1.2) After doing $map(\mathbf{v}, 0)$, $|F_0| = 2n-3$.

If $\exists e \notin F^{\mathbf{u}} \cup F^{\mathbf{v}^-}$, assume e is healthy, and do $DP(\mathbf{u}, \mathbf{v}^-)$. Extend the disjoint paths to $\mathbf{v}$. Do *U-jump(e)* or *D-jump(e)* to avoid faults if necessary. Two more paths can be σ_1 and σ_2. However, if there is one faulty edge e lies on σ_1 or σ_2, just do *U-jump(e)* or *D-jump(e)*.

If $|F^{\mathbf{u}} \cup F^{\mathbf{v}^-}| = 2n-3$, let choose a faulty edge e from $F^{\mathbf{u}}$, and assume it is healthy. Do $DP(\mathbf{u}, \mathbf{v}^-)$. Extend the disjoint paths to $\mathbf{v}$. If e lies on some path, just remove the path. Include

σ_1 and σ_2 as disjoint paths. Similarly, if there is one faulty edge e lies on σ_1 (or σ_2), just do ***U-jump(***e***)*** (or ***D-jump(***e***)***).

(B22-1.3) After doing $map(\mathbf{v}, 0)$, $|F_0| \leq 2n-2$.

Do $DP(\mathbf{u}, \mathbf{v}^-)$. Extend the disjoint paths to $\mathbf{v}$. Include σ_1 and σ_2, if exist, as disjoint paths. However, if σ_1 is blocked from outside Q_0 and Q_1, we can easily find an alternative path that is disjoint from all of existing one by replacing the sub-path in Q_{k-1} with a path that can be found by induction. The case that σ_2 is blocked from outside Q_0 and Q_1 can be dealt with similarly.

(B22-2) $v_1 \geq 2$.

This case can be analyzed similar to case (B22-1), and we will not go into details.

Case 2.3 $v_1 \neq 0, dist(\mathbf{u}, \mathbf{v}^{-v_1}) \geq 2$.

This can be dealt with similar to **Case 2.2**.