Quantified Constraints on Directed Graphs

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Abstract

We study the quantified H-colouring problem for directed graphs. We prove that the quantified H-colouring problem is tractable when H is a directed cycle or a semicomplete digraph with at most one cycle. We give sufficient criteria for the quantified H-colouring problem to be in NP, and thus infer NP-completeness of the quantified H-colouring problem when H is semicomplete with more than one cycle and both a source and a sink.

1 Introduction

A very natural generalisation of digraph colouring problems is defined in terms of graph homomorphism; the problem takes as input a digraph G and accepts it if, and only if, there exists a homomorphism into a fixed digraph H. This problem is known as the *H-colouring problem*[1]. In [10], Hell and Nešetřil proved that the class of H-colouring problems exhibits dichotomy when H is undirected and antireflexive: the problem is tractable if H is bipartite and NP-complete otherwise. In [1], Bang-Jenson, Hell and MacGillavray studied the H-colouring problem for semicomplete H, again deriving a dichotomy: the problem is tractable if H has at most one cycle, and is NP-complete otherwise. *Constraint satisfaction Problems* (CSPs), when parameterised by their constraint language – also known as *non-uniform* CSPs – are closely related to the H-colouring problem. Specifically, every H-colouring problem is an example of a (non-uniform) CSP, and every (non-uniform) CSP is polynomially equivalent to some H-colouring problem [8]. In the case of CSPs with boolean domains, known as *generalised satisfiability*, Schaefer proved a dichotomy by an exhaustive analysis of the types of expressible relations [14]. An algebraic approach has been successful in identifying certain tractable and NP-complete cases (see for example [12, 4]), and has enabled Schaefer's dichotomy to be extended to domains of size three [3]. However, the *dichotomy conjecture* [8], that states that every CSP is either tractable or NP-complete, is still open.

Building on the result from [14], dichotomy results were proved independently in [6] and [7] for *quantified generalised satisfiability* problems without

[1] Traditionally, this is defined only for undirected graphs. However the extension to digraphs is natural and merits no special introduction.

constants (In [14], the result was proved only in the case where the boolean constants were included): they are either tractable or Pspace-complete. The algebraic approach to the CSP has successfully been applied to *quantified constraint satisfaction problems* (QCSPs), to determine sufficient conditions for tractability and Pspace-completeness (see [2, 5]). Partial trichotomy results have been proved for QCSP (respectively, quantified H-colouring) in [2] (respectively, [13]): for two restricted classes the authors prove that every problem is either tractable, NP-complete or Pspace-complete.

In this paper we explore the quantified H-colouring problem for certain digraph templates H. We prove that the the quantified H-colouring problem is tractable when H is a directed cycle or is a semicomplete digraph with at most one cycle. This demonstrates that those semicomplete digraph templates which give rise to tractable CSPs also give rise to tractable QCSPs. Further, we prove that the quantified H-colouring problem is in NP whenever H has both a source and a sink. As a corollary, we derive that the quantified H-colouring problem is NP-complete when H is a semicomplete digraph with more than one cycle and both a source and a sink.

2 Preliminaries

In this paper, we consider only finite antireflexive directed graphs. In a digraph, a vertex x is said to be a source (respectively, sink) if it has in-degree (respectively, out-degree) zero. For $n \geq 2$, we define the directed n-cycle DC_n to be the digraph with vertices $\{0, \ldots, n-1\}$ and edge set $\{(i,j) : j = i+1 \bmod n\}$. An *oriented n-path*[2] is a digraph with vertices $\{0, \ldots, n\}$ and, for $1 \leq i \leq n$, exactly one of the edges $(i, i+1)$ or $(i+1, i)$. The *net length* of this oriented path is the number of instances of edges $(i, i+1)$ (forward-edges) minus the number of instances of edges $(i+1, i)$ (backward-edges). An oriented path in a digraph G is a (not necessarily induced) subgraph of G that is (isomorphic to) some oriented n-path. A semicomplete digraph of size n consists of vertex set $\{0, \ldots, n-1\}$ and, for $1 \leq i, j < n$ and $i \neq j$, at least one of the edges (i,j) or (j,i). A semicomplete digraph such that for no i, j are both (i,j) and (j,i) edges is called a tournament. A semicomplete digraph such that for all $i \neq j$ both (i,j) and (j,i) are edges, is called a clique. The unique clique of size n will be denoted K_n. Note that the directed 2-cycle DC_2 and the 2-clique K_2 coincide. The unique acyclic tournament of size n is known as the transitive n-tournament, and is denoted T^t_n. For more on these definitions see [11].

Given digraphs G and H, a *homomorphism* f from G to H, denoted $G \xrightarrow{f} H$, is a vertex-mapping function $f : V(G) \to V(H)$ such that $(x,y) \in E(G)$ implies $(f(x), f(y)) \in E(H)$. We write $G \longrightarrow H$ if there exists a homomorphism from G to H. The *H-colouring problem* takes as input a digraph G, which is a yes-instance if, and only if, there exists a homomorphism from G to H. For $n \geq 0$, an *n-partitioned digraph* $\mathscr{G}$ consists of a digraph G together with a partition

[2] Note that, while n-cycles have n vertices, n-paths have $n+1$ vertices.

$\{U_1, X_2, U_3, X_4, \ldots, U_{2n+1}, X_{2n+2}\}$ of $V(G)$. In the following, G will always designate the underlying digraph of $\mathscr{G}$. Let $\mathscr{G}$ be an n-partitioned digraph and H be a (non-partitioned) digraph. The $(\mathscr{G}, H)$-*game* is a two-player game, that pitches Adversary (male) against Prover (female). Adversary plays on the universal partitions (the sets U_i) and Prover plays on the existential partitions (the sets X_i). They play alternate partitions, in ascending order, until all the partitions have been played. The game goes as follows. For $0 \leq i \leq n$:

- for every vertex in partition U_{2i+1}, Adversary chooses a vertex in H: i.e. he gives a function $f_{U_{2i+1}} : U_{2i+1} \rightarrow V(H)$; and,
- for every vertex in partition X_{2i+2}, Prover chooses a vertex in H: i.e. she gives a function $f_{X_{2i+2}} : X_{2i+2} \rightarrow V(H)$.

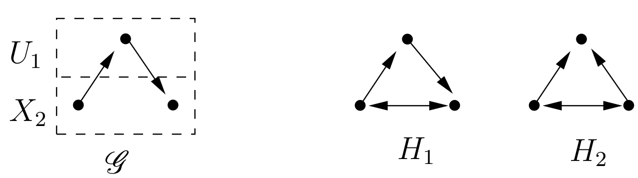
Prover wins if, and only if, the function $f := f_{U_1} \cup f_{X_2} \cup \ldots \cup f_{X_{2n+2}}$ is a homomorphism from G to H. We say that there exists an *alternating-homomorphism* from the n-partitioned digraph $\mathscr{G}$ to the (non-partitioned) digraph H, and we write $\mathscr{G} \xrightarrow{alt} H$ if, and only if, for all functions $f_{U_1} : U_1 \rightarrow V(H)$, there exists a function $f_{X_2} : X_2 \rightarrow V(H)$, such that, ..., for all functions $f_{U_{2n+1}} : U_{2n+1} \rightarrow V(H)$, there exists a function $f_{X_{2n+2}} : X_{2n+2} \rightarrow V(H)$, such that, $f_{U_1} \cup f_{X_2} \cup \ldots f_{U_{2n+1}} \cup f_{X_{2n+2}}$ is a homomorphism from G to H.

If the n-partitioned digraph $\mathscr{G}$ is viewed as a quantified sentence, then our game is exactly a model-checking, or Hintikka, game [9] over the model H. In this guise our game is closely related to that used in the analysis of QCSP by Chen [5]. In any case, the following is a direct consequence of the above definitions.

Proposition 1. *Let $\mathscr{G}$ be an n-partitioned digraph and H be a (non-partitioned) digraph. $\mathscr{G} \xrightarrow{alt} H$ if, and only if, Prover has a winning strategy in the $(\mathscr{G}, H)$-game.*

We define the *quantified H-colouring* problem as the decision problem which takes as input a partitioned digraph $\mathscr{G}$ (n-partitioned, for some n) and whose yes-instances are those $\mathscr{G}$ for which $\mathscr{G} \xrightarrow{alt} H$. We refer to H as the problem's *template*.

Example 1. Consider the following partitioned digraph $\mathscr{G}$ together with the template digraphs H_1 and H_2.



There will be an alternating-homomorphism from $\mathscr{G}$ to a digraph H iff H contains neither a source nor a sink. This is clear since, in the $(\mathscr{G}, H)$-game, Prover must be able to answer any vertex that Adversary chooses in H with both a forward-neighbour and a backward-neighbour. It follows that, while the underlying graph G maps homomorphically into both H_1 and H_2, we have $\mathscr{G} \xrightarrow{alt} H_1$ but $\mathscr{G} \overset{alt}{\not\longrightarrow} H_1$.

2.1 Restricting partitions

We will be particularly interested in templates H whose quantified H-colouring yes-instances are partitioned inputs $\mathscr{G}$ that in some way collapse to be evaluable within NP.

Let $\mathscr{G}$ be a partitioned digraph. We say that $\mathscr{G}$ is in Σ_1*-form* (respectively, Π_2*-form*), if the only non-empty partition is X_2 (respectively, if the only non-empty partitions are among $\{U_1, X_2\}$). If $\mathscr{G}$ is in Π_2-form and there is at most one vertex in U_1, then we say that $\mathscr{G}$ is in Π_2*-fan form*. Finally, we say that $\mathscr{G}$ is in Π_2*-multifan form*, if $\mathscr{G}$ is the disjoint union of digraphs in Π_2-fan form.

Theorem 2. *The restriction of the quantified H-colouring problem to partitioned digraphs in Π_2-multifan form is in NP.*

Proof. Let $\mathscr{G}$ be the disjoint union of $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_m$, all in Π_2-fan form. Note that $\mathscr{G} \xrightarrow{alt} H$ if, and only if, $\mathscr{G}_i \xrightarrow{alt} H$, for every $1 \leq i \leq m$. To test whether $\mathscr{G}_i \xrightarrow{alt} H$ we may consider all possible maps for the vertex in U_1 (if there is one) and then guess the remainder of the homomorphism and verify in polynomial time. □

We describe two partitioned digraphs $\mathscr{G}$ and $\mathscr{G}'$ as *problem-equivalent* if, and only if, for all templates H, $\mathscr{G} \xrightarrow{alt} H$ iff $\mathscr{G}' \xrightarrow{alt} H$. From a partitioned graph $\mathscr{G}$, we derive the *reduced* digraph $\overline{\mathscr{G}}$ by collapsing all universal partitions to U_1 and all existential partitions to X_2. Note that $\mathscr{G}$ and $\overline{\mathscr{G}}$ share the same underlying digraph G.

3 Cases in NP

In this section we will find that, for certain H, every input that is not essentially in Σ_1-form or Π_2-multifan form can be discarded.

3.1 H is a digraph with both a source and a sink

Let H be a digraph with both a source s and a sink t. We begin by proving that certain edges are forbidden in $\mathscr{G}$, due to the presence of both a source and a sink in H. The following may be seen as a generalisation of part of Lemma 1 and Theorem 2 in [13] (note that an isolated vertex is both a source and a sink).

Lemma 3. *Let H be a digraph with both a source and a sink. For any i, j: if there is a forward-edge in $\mathscr{G}$ between any $x \in X_i$ and $y \in U_j$ or between any*

$x \in U_i$ and $y \in U_j$, then $\mathscr{G} \overset{alt}{\not\to} H$; if there is a backward-edge in $\mathscr{G}$ between any $x \in X_i$ and $y \in U_j$ or between any $x \in U_i$ and $y \in U_j$, then $\mathscr{G} \overset{alt}{\not\to} H$.

Proof. For the first part, Adversary plays s for y and wins; for the second part, Adversary plays t for y and wins. ☐

Theorem 4. *If H is an antireflexive digraph with both a source and a sink, then the quantified H-colouring problem is equivalent to the H-colouring problem under logspace reduction.*

Proof. The H-colouring problem reduces trivially to the quantified H-colouring problem.

We define the converse reduction as follows. Let N be a fixed no-instance of the H-colouring problem (say, H augmented with one vertex adjacent to every vertex of H). If $\mathscr{G}$ has an edge as in the previous lemma then we know that it is a no-instance and we reduce $\mathscr{G}$ to N (with all vertices in X_2). If $\mathscr{G}$ has no such edge then every element in a universal partition is isolated, and $\overline{\mathscr{G}}$ is essentially in Σ_1-form. We reduce $\mathscr{G}$ to its underlying digraph G. ☐

Corollary. *For each of the transitive n-tournaments T_n^t, the quantified T_n^t-colouring problem is tractable.*

Proof. Note that T_n^t has both a source and a sink. The result follows from the previous theorem and the fact that the T_n^t-colouring problem is tractable [1]. ☐

Corollary. *If H is a semicomplete digraph with more than one cycle, and both a source and a sink, then the quantified H-colouring problem is NP-complete.*

Proof. It is proved in [1] that the H-colouring problem is NP-complete. The result now follows from the previous theorem. ☐

3.2 Tractable cases

Having established that the quantified T_n^t-colouring problem is tractable, for each transitive n-tournament, we examine other digraph templates that give rise to tractable quantified colouring problems.

3.2.1 Directed cycles

In a directed n-cycle, any oriented path between a vertex and itself must have net length $0 \mod n$. Furthermore, any path between a vertex and its forward-neighbour must have net length 1 *mod* n (and every vertex *has* a forward-neighbour). These facts will allow us to consider only partitioned inputs in Π_2-multifan form since:

Lemma 5. *For $n \geq 2$: if there is a path in $\mathscr{G}$ between any $x \in X_i$ and $y \in U_j$ (for $i < j$) or between any $x \in U_i$ and $y \in U_j$ ($x \neq y$; any i, j), then $\mathscr{G} \overset{alt}{\not\to} DC_n$.*

Proof. We prove the first claim, the proof of the second is similar. If the path has net length 0 *mod* n, then, if Prover plays a for x, Adversary plays the forward-neighbour b of a for y, and wins. If the path has net length other than 0 *mod* n, then, if Prover plays a for x, Adversary also plays a for x, and again wins. □

The following result shows that every partitioned digraph that does not have the paths mentioned in Lemma 5 is essentially in Π_2-multifan form.

Lemma 6. *If there is no path in a partitioned digraph $\mathscr{G}$ between any x in X_i and y in U_j (for $i < j$), or between any x in U_i and y in U_j ($x \neq y$; any i, j), then $\overline{\mathscr{G}}$ is in Π_2-multifan form and is problem-equivalent to $\mathscr{G}$.*

Proof. It suffices to prove, for every connected component $\mathscr{G}'$ of $\mathscr{G}$, that $\overline{\mathscr{G}'}$ is in Π_2-fan form and is problem-equivalent to $\mathscr{G}'$. Let $\mathscr{G}'$ be such a component and let $0 < i \leq n$ be the largest integer such that U_{2i+1} is non-empty. Take x in U_{2i+1}, and let y be any element of $\mathscr{G}'$ connected to x via a path. It follows from the second assumption that y can not be in a universal partition (U_{2i+1} included). Thus, it follows from the first assumption that y belongs to an existential partition of index at least $2i + 2$. It is not hard to see that we can move x to U_1 and all other [existential] vertices of $\mathscr{G}'$ to X_2, preserving problem-equivalence, and generating $\overline{\mathscr{G}'}$ in Π_2-fan form. □

Lemma 7. *Let $\mathscr{G}$ be in Π_2-multifan form. For $n \geq 3$, the following are equivalent:*

- $\mathscr{G} \xrightarrow{alt} DC_n$
- $G \longrightarrow DC_n$

Proof.

(⇒) Trivial.

(⇐) Let $\mathscr{G}_1, \ldots, \mathscr{G}_m$ be the Π_2-fan components of which $\mathscr{G}$ is the disjoint union. If $G \longrightarrow DC_n$ then it follows that $G_i \longrightarrow DC_n$ for each component G_i. We aim to prove that $\mathscr{G}_i \xrightarrow{alt} DC_n$ for each component $\mathscr{G}_i$, which clearly yields $\mathscr{G} \xrightarrow{alt} DC_n$. If $\mathscr{G}_i$ is in Σ_1-form, the result is trivial. Otherwise, let x be the vertex in U_1, and let Adversary play it on some vertex c of DC_n. The result now follows immediately from the symmetry of DC_n: taking the assumed homomorphism $G_i \xrightarrow{h_i} H$, together with the automorphism a_i of DC_n which maps $h_i(x)$ to c, Prover wins by playing the partition X_2 under the strategy $a_i \circ h_i$ (where $\circ$ denotes functional composition). □

Theorem 8. *For $n \geq 2$, the quantified DC_n-colouring problem is tractable.*

Proof. We propose the following algorithm to solve the quantified DC_n-colouring problem. The input $\mathscr{G}$ should be scanned to see if there are any of the forbidden paths of Lemma 6 : if there are any, then the input should be rejected. Otherwise, solve the DC_n-colouring problem, known to be tractable [1], with input G.

This algorithm is clearly polynomial; we prove its correctness. If $\mathscr{G}$ has none of the forbidden paths of Lemma 3, then it is problem-equivalent to the reduced $\overline{\mathscr{G}}$ in Π_2-multifan form, by Lemma 6. The equivalence of $\overline{\mathscr{G}} \xrightarrow{alt} DC_n$ and $G \longrightarrow DC_n$ holds by the previous lemma, and we are done. ☐

3.2.2 The tournaments T^u_{m+3}

We now examine the tournaments T^u_{m+3} which are constructed from the directed 3-cycle by repeatedly adding a source m times. (The superscript u suggests the *unique* cycle.)

Definition. We define T^u_{m+3} inductively:

- Let $T^{(0)} := T^u_3 := DC_3$, the directed 3-cycle.
- From $T^{(r)}$ build $T^{(r+1)}$ by adding a new source, i.e., $V(T^{(r+1)}) := V(T^{(r)}) \uplus \{r+3\}$[3] and $E(T^{(r+1)}) := E(T^{(r)}) \uplus \{\{r+3, i\} : i \in V(T^{(r)})\}$.
- Let $T^u_{m+3} := T^{(m)}$.

Since we have dealt with the case of the directed 3-cycle, we consider $m > 0$, i.e. when T^u_{m+3} has a source.

Lemma 9. *For $m > 0$, if*

- *there is a directed edge in $\mathscr{G}$ between $x \in X_i$ and $y \in U_j$ $(i < j)$, or*
- *there is a directed edge in $\mathscr{G}$ between $x \in U_i$ and $y \in U_j$ (any i, j), or*
- *there is a directed edge in $\mathscr{G}$ from $x \in X_j$ to $y \in U_i$ $(i < j)$,*

then $\mathscr{G}$ is a no-instance of the quantified T^u_{m+3}-colouring problem.

Proof. The first two parts follow from the antireflexivity of T^u_{m+3}: Adversary plays the same vertex for y as Prover plays for x, and wins; or Adversary plays the same vertices for x and y, and wins. For the final part, if Adversary plays y to the source of T^u_{m+3}, Prover can have no reply for x. ☐

Let $\mathscr{G}$ be a partitioned digraph, we define its cousin $\widetilde{\mathscr{G}}$ inductively:

- $\mathscr{G}^{(0)} := \mathscr{G}$.
- From $\mathscr{G}^{(r)}$ we build $\mathscr{G}^{(r+1)}$ by removing *all* sources that are in existential partitions.
- $\widetilde{\mathscr{G}} := \mathscr{G}^{(m)}$.

[3] $T^{(r)}$, being a tournament with $r+3$ vertices, will already have vertex numbers 0 to $r+2$.

Let $Ex(\mathscr{G} - \widetilde{\mathscr{G}})$ be those existential vertices in $\mathscr{G}$ that are not in $\widetilde{\mathscr{G}}$, and let $Ex(\widetilde{\mathscr{G}})$ be those existential vertices in $\mathscr{G}$ that are also in $\widetilde{\mathscr{G}}$ (since $\widetilde{\mathscr{G}} \subseteq \mathscr{G}$ these are the existential vertices of $\widetilde{\mathscr{G}}$). Let $PrEx(\mathscr{G})$ be those vertices of $\mathscr{G}$ in existential partitions *to* which there is a directed path *from* some vertex in a universal partition. Let $Un(\mathscr{G})$ be the set of universal vertices of $\mathscr{G}$. Let $\widetilde{G}$ denote the underlying digraph of $\widetilde{\mathscr{G}}$. We refer to the vertices of T^u_{m+3} that are not in the 3-cycle as the *tail* of T^u_{m+3}.

We will benefit from examining which vertices of the underlying digraph G have been removed in the digraph $\widetilde{G}$. It should be clear that vertices in $Un(\mathscr{G})$ and $PrEx(\mathscr{G})$ can never be removed, and are, therefore, *protected*. Let us consider the sub-partitioned-digraph $\mathscr{G}_1$ of $\mathscr{G}$ induced by the existential vertices that are not protected. $\mathscr{G}_1$ may be put through our given inductive scheme, iteratively removing sources m times, so obtaining $\widetilde{\mathscr{G}_1}$. It should be clear that $\widetilde{\mathscr{G}}$ is that subdigraph of $\mathscr{G}$ induced by the set $Un(\mathscr{G}) \cup PrEx(\mathscr{G}) \cup V(\widetilde{\mathscr{G}_1})$. Apart from the universal vertices and those existential vertices that are protected, our construction is that given for proving the tractability of the T^u_{m+3}-colouring problem in [1]. All of the sets we have defined should now be considered as subsets of $V(\mathscr{G})$ (though some may be subsets of $V(\widetilde{\mathscr{G}})$ too). Before going on we will benefit from the following lemmas.

Lemma 10. *In a winning strategy for Prover in the $(\mathscr{G}, T^u_{m+3})$-game: if Adversary plays all his vertices to the 3-cycle, then Prover must play all of the vertices of $Ex(\widetilde{\mathscr{G}})$ [in $\mathscr{G}$] to the 3-cycle.*

Proof. Again, let $\mathscr{G}_1$ be the sub-partitioned-digraph of $\mathscr{G}$ induced by those existential vertices of $\mathscr{G}$ that are not protected. Recall that $\widetilde{\mathscr{G}}$ is the subdigraph of $\mathscr{G}$ induced by $Un(\mathscr{G}) \cup PrEx(\mathscr{G}) \cup V(\widetilde{\mathscr{G}_1})$. So $Ex(\widetilde{\mathscr{G}})$ is $PrEx(\mathscr{G}) \cup V(\widetilde{\mathscr{G}_1})$. Since universal vertices are played to the 3-cycle, it follows that all vertices of $PrEx(\mathscr{G})$ must be played to the 3-cycle. Furthermore, if any vertex of $V(\widetilde{\mathscr{G}_1})$ [in $\mathscr{G}$] could be played to the tail of T^u_{m+3}, then this could not be extended to a homomorphism from G to T^u_{m+3} – by definition of $\widetilde{G_1}$ – so this could not be a winning strategy for Prover on $(\mathscr{G}, T^u_{m+3})$. The result follows. □

Lemma 11. *Assume that $\mathscr{G}$ has none of the edges of Lemma 9. Then Adversary can win the game on $(\mathscr{G}, T^u_{m+3})$ iff he can win it whilst never playing in the tail of T^u_{m+3}.*

Proof. Since edges of $\mathscr{G}$ from universal partitions only point toward vertices x in higher existential partitions, if Adversary plays in the tail then he allows Prover to answer x with anything on the 3-cycle. However, if he plays on the 3-cycle he limits Prover to a single adjacent vertex of the 3-cycle. It is clear that Adversary gains nothing by playing in the tail. □

Theorem 12. *For $m \geq 0$, the quantified T^u_{m+3}-colouring problem is tractable.*

Proof. The result for $m = 0$ follows from Theorem 8. For $m > 0$ we will solve the quantified T^u_{m+3}-colouring problem by taking any input $\mathscr{G}$ and constructing a given $\mathscr{G}'$. We will prove that

$$\mathscr{G} \xrightarrow{alt} T^u_{m+3} \quad \Leftrightarrow \quad \mathscr{G}' \xrightarrow{alt} T^u_3$$

whereupon we may appeal to the known tractability of the latter problem, and our result will follow.

If $\mathscr{G}$ has any of the edges of Lemma 9 then we set $\mathscr{G}'$ to be any set no-instance of quantified T^u_3-colouring (e.g. the transitive 4-tournament T^t_4 with all vertices in X_2). If $\mathscr{G}$ has none of those edges then we set $\mathscr{G}'$ to be $\widetilde{\mathscr{G}}$. This algorithm is clearly polynomial; it remains for us to prove that it is correct. It is trivially correct if $\mathscr{G}$ has any of the edges of Lemma 9: we assume it does not.

($\Rightarrow$) For a winning strategy σ for Prover in the game on $(\mathscr{G}, T^u_{m+3})$, we claim σ is also a winning strategy for her in the game on $(\widetilde{\mathscr{G}}, T^u_3)$. This follows immediately from Lemma 10.

($\Leftarrow$) From a winning strategy σ for Prover in the game on $(\widetilde{\mathscr{G}}, T^u_3)$, we construct a winning strategy σ' in the game on $(\mathscr{G}, T^u_{m+3})$ where Adversary only plays in the 3-cycle. In that game on $(\mathscr{G}, T^u_{m+3})$, when Adversary plays on the 3-cycle, then Prover answers the vertices in $Ex(\widetilde{\mathscr{G}})$ according to σ, and then maps $Ex(\mathscr{G} - \widetilde{\mathscr{G}})$ to the tail of T^u_{m+3}. The result follows from Lemma 11. □

3.2.3 The tournaments T^u_{3+m}

These tournaments are analagous to the tournaments T^u_{m+3}, but are constructed by the repeated addition of a sink, rather than a source. It follows by a similar argument that, for all m, the quantified T^u_{m+3}-colouring problem is tractable.

3.2.4 The semicomplete digraphs S^u_{m+2}

We now examine the digraphs S^u_{m+2} which are constructed from the directed 2-cycle (the 2-clique) by repeatedly adding a source m times.

Definition. Formally, we define S^u_{m+2} inductively:

- Let $S^{(0)} := S^u_2 := DC_2$, the directed 2-cycle.
- From $S^{(r)}$ build $S^{(r+1)}$ by adding a new source, i.e., $V(S^{(r+1)}) := V(S^{(r)}) \uplus \{r+2\}$ and $E(S^{(r+1)}) := E(S^{(r)}) \uplus \{\{r+2, i\} : i \in V(S^{(r)})\}$.
- Let $S^u_{m+2} := S^{(m)}$.

The reader will not be surprised by the following.

Theorem 13. *For $m \geq 0$, the quantified S^u_{m+2}-colouring problem is tractable.*

Proof. Consider Lemma 9 to Theorem 12 of Section 3.2.2 with all instances of 'T^u_{m+3}' and '3-cycle' substituted by 'S^u_{m+2}' and '2-cycle', respectively. The result follows. □

3.2.5 The semicomplete digraphs S^u_{2+m}

These digraphs are analagous to the digraphs S^u_{m+2}, but are constructed by the repeated addition of a sink, rather than a source. It follows by a similar argument that, for all m, the quantified S^u_{m+2}-colouring problem is tractable.

3.2.6 A tractability result

Theorem 14. *If H is a semicomplete digraph with at most one cycle, then the quantified H-colouring problem is tractable.*

Proof. It follows from standard results about semicomplete digraphs (e.g. see [1]) that H is either a transitive tournament or is either $T^u_3 = DC_3$ or $S^u_2 = DC_2$ with a succession of sources and/or sinks added. If it has both a source and a sink then, by Theorem 4, we can reduce the problem to H-colouring, which is known to be tractable [1]. If it has no sink (resp., no source), then it is one of the digraphs T^u_{m+3} or S^u_{m+2} (resp., T^u_{3+m} or S^u_{2+m}), whose quantified colouring tractability has been detailed in the previous four sections. □

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