EXTENSIONS OF CHURCH’S SYNTHESIS PROBLEM

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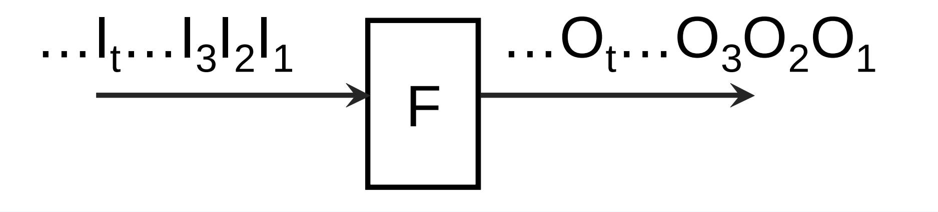
Workshop en l’honneur du de’part a‘ la retraite de
Danie‘le Beauquier et Anatol Slissenko
June, 2009

Church's Problem

Consider a bit by bit transformation of bit streams:



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Given a logical specification of the input-output relation R find a causal mapping (implementation) $F : I \to F(I)$ such that $(I, F(I)) \in R$ for all I.

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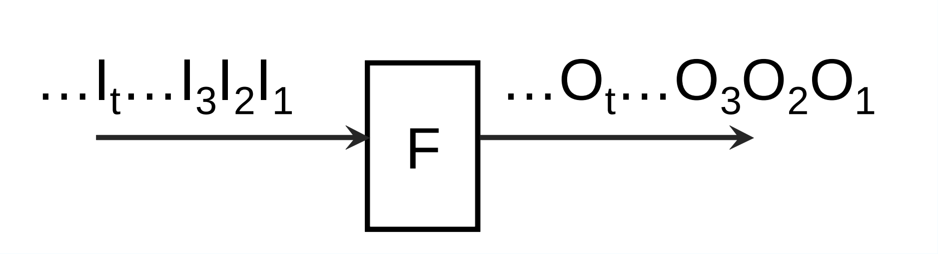

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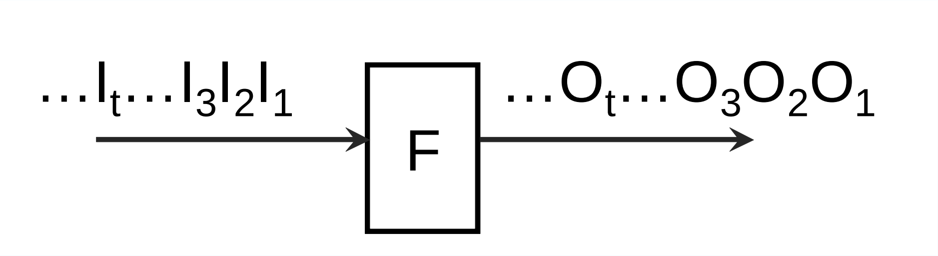
Causal-operator - the output bit O_t at moment t depends only on $I_1 I_2 \ldots I_t$.

Example



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If all $I(t) = 0$ then all $O(t) = 0$; otherwise all $O(t) = 1$.

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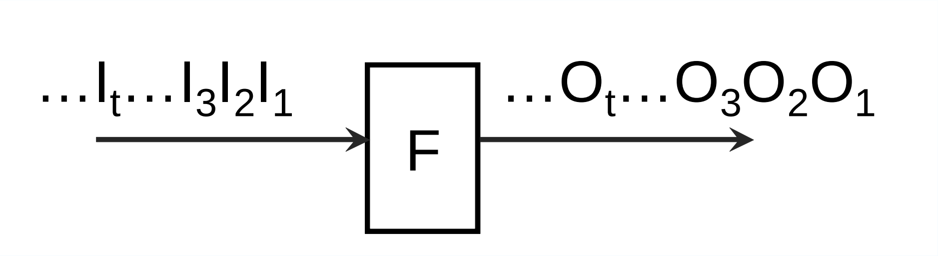

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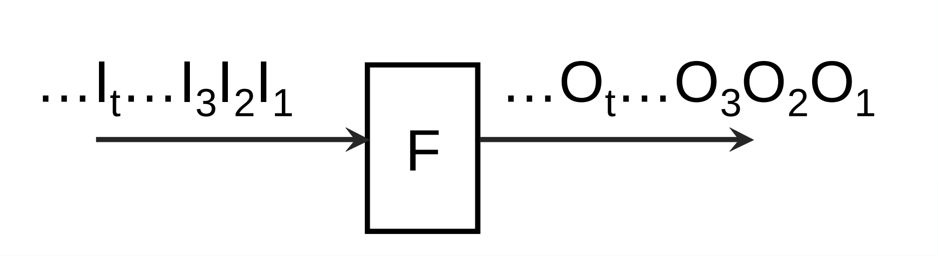

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Consider R defined by the conjunction of three conditions on the input-output stream (I, O):

1. $\forall t(I(t) = 1 \rightarrow O(t) = 1)$
2. never $O(t) = O(t + 1) = 0$
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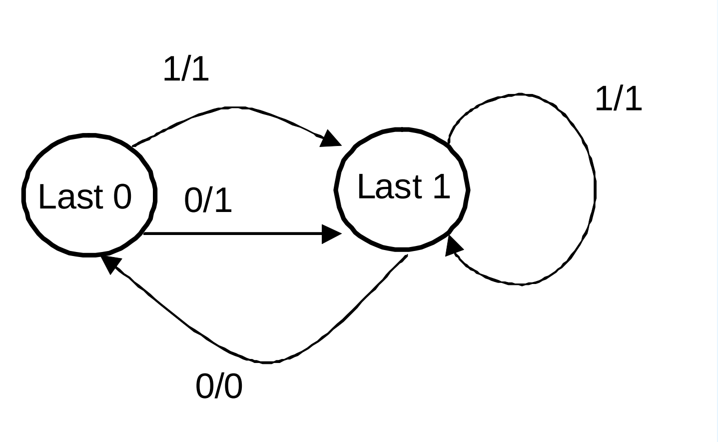

Can be described by a finite state automaton with output.

Büchi-Landweber Theorem

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If such an operator exists then there is a finite state operator which implements ψ.
Moreover, this finite state operator is computable from ψ.

Techniques

Rich interplay of

1. Mathematical logic - Monadic Second-Order Logics
2. Automata theory - automata on infinite objects .
3. Games of infinite length.

Plan

1. McNaughton Games.
2. The Game version of the Büchi-Landweber Theorem.
3. Three orthogonal extensions of the BL theorem:
   - (a) Fragments of MLO
   - (b) The Church Problem with parameters
   - (c) Long Games - the length of games is a countable ordinal
4. Some Open Problems

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At stage $\beta \in \mathbb{N}$: first, Mr. X chooses $\mathbf{X}(\beta) \in \{0, 1\}$; then, Ms. Y chooses $\mathbf{Y}(\beta) \in \{0, 1\}$.

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Winning Conditions At the end of the play two monadic predicates $\mathbf{X}, \mathbf{Y} : \mathbb{N} \to \{0, 1\}$ have been constructed. Then Y wins the play if

$$(\omega, <) \models \varphi(\mathbf{X}, \mathbf{Y});$$

otherwise, X wins the play.

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A strategy for player X: for every finite strings $\langle X(0), Y(0)\rangle$ $\langle X(1), Y(1)\rangle \ldots \langle X(n), Y(n)\rangle$ choose the next bit $X(n+1)$.

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Game Version of Büchi-Landweber Theorem

Theorem (Büchi-Landweber, '69) Let $\varphi(X, Y)$ be an MLO formula. Then:
Determinacy: One of the players has a winning strategy in the game $\mathcal{G}^{\omega}_{\varphi}$.
Decidability: It is decidable which of the players has a winning strategy.
MLO definable strategy: For the player that has a winning strategy, there exists a MLO definable winning strategy.
Synthesis algorithm: There exists an algorithm that given any $\varphi(X, Y)$, constructs a formula ψ that defines a winning strategy for the winning player in $\mathcal{G}^{\omega}_{\varphi}$.

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Remark. A strategy is MLO definable in ω iff it is computable by a finite state automaton with output.

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Remark. The theorem holds when FOMLO is replaced by other natural fragments of MLO. E.g. by WMLO, FOMLO extended by modular quantifiers, and any fragment which includes FOMLO and satisfies the composition theorem.

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Elgot-Rabin (66) The monadic theory of (Nat, <, P) is decidable for the following P:
The set of factorial numbers P=$\{n! \ : \ n \in Nat\}$.
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Many other such unary predicates were found - Semenov, Seifkis, Thomas, Carton, . . .

Games over $(Nat, < P)$

Given a predicate $\mathbf{P} \subseteq \mathbb{N}$ and an MLO formula $\varphi(X, Y, P)$, we define a game $\mathcal{G}^{\omega}_{\varphi}$ as follows.

The game is played by two players Mr. X and Ms. Y.

A play of the game has ω stages.
At stage $\beta \in \mathbb{N}$: first, Mr. X chooses $\mathbf{X}(\beta) \in \{0, 1\}$; then, Ms. Y chooses $\mathbf{Y}(\beta) \in \{0, 1\}$. Both players observe the value of $\mathbf{P}(\beta)$.

Winning Conditions At the end of the play two monadic predicates $\mathbf{X}, \mathbf{Y} : \mathbb{N} \to \{0, 1\}$ have been constructed. Then Y wins the play if

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Let $\mathit{Fact} = \{n! \mid n \in \mathbb{N}\}$ be the set of factorial numbers.
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Y has an MLO definable (in $(\mathbb{N}, <, \mathrm{Fact})$) and recursive winning strategy.

Church's Problem with parameters

Theorem (R. 06) Given a predicate $\mathbf{P} \subseteq \mathbb{N}$ and an MLO formula $\varphi(X, Y, P)$. Then:
Determinacy: One of the players has a winning strategy in the game $\mathcal{G}^{\omega}_{\varphi}$.
MLO definable strategy: For the player that has a winning strategy, there exists a MLO definable (in $(\mathbb{N}, <, \mathbf{P})$) winning strategy.
Synthesis algorithm: There exists an algorithm that given any $\varphi(X, Y, P)$, constructs a formula ψ that defines a winning strategy for the winning player in $\mathcal{G}^{\omega}_{\varphi}$.
Decidability: $\mathrm{MTh}(\mathbb{N}, <, \mathbf{P})$ is decidable if and only if it is decidable which of the players has a winning strategy.
Determinacy by recursive strategies: $\mathrm{MTh}(\mathbb{N}, <, \mathbf{P})$ is decidable if and only if for every $\varphi(X, Y, P)$ one of the players has a recursive winning strategy in the game $\mathcal{G}^{\omega}_{\varphi}$.

Decidability of MLO over ordinals

MLO can be interpreted over an arbitrary linear order. In particular, over ordinals.

Theorem (Büchi) Validity problem is decidable for second-order monadic logic over the class of countable ordinals.

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Each ordinal has a unique representation
$\alpha = \omega^{\omega}\beta + \omega^{n}a_n + \omega^{n-1}a_{n-1} + \cdots + a_0$

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Proof Technique: Reduction to finite-state automata over strings of ordinal length.

McNaughton Games of length α

Given an ordinal α and a formula $\varphi(X, Y)$.

The game $\mathcal{G}_{\varphi}^{\alpha}$ is played by two players.

A play of the game has α stages.
At stage $\beta < \alpha$: first, Mr. X chooses $\mathbf{X}(\beta) \in \{0, 1\}$; then, Ms. Y chooses $\mathbf{Y}(\beta) \in \{0, 1\}$.

Winning Conditions At the end of the play two monadic predicates $\mathbf{X}, \mathbf{Y} \ : \ \alpha \rightarrow \{0, 1\}$ have been constructed. Then Y wins the play if $(\alpha, <) \models \varphi(\mathbf{X}, \mathbf{Y})$; otherwise, X wins the play.

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Büchi-Landweber (69) state that their results on the Church Problem can be extended to the class of countable ordinals.

However, we show that this is not the case.

BL theorem over countable ordinals

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Theorem(A. Shomrat and R. 07) For each $\alpha \geq \omega^{\omega}$, there is an MLO formula φ such that Mr. X has a winning strategy in $\mathcal{G}_{\varphi}^{\alpha}$; however, he has no MLO definable winning strategy.

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This leads to Definable winning strategy problem for $\alpha \geq \omega^\omega$.
Input: An MLO formula $\varphi(X, Y)$.
Question: Does Mr. X have an MLO definable winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$?

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Theorem For each countable ordinal α it is decidable whether the winner of $\mathcal{G}^{\alpha}_{\varphi}$. has an MLO definable winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$.

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For each countable ordinal α, there is a natural notion of a strategy definable by a finite-state automaton on strings of α-length.
If a strategy is finite-state over α then it is MLO definable (in α).
For $\alpha \leq \omega^{\omega}$ the classes of finite-state and MLO definable strategies are the same.
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Theorem(Larson and Shelah 08) There is an MLO formula φ such that each of the following statements is consistent with ZFC:

1. None of the players has a winning strategy in $\mathcal{G}_{\varphi}^{\omega_1}$.
2. Ms. Y has a winning strategy in $\mathcal{G}_{\varphi}^{\omega_1}$.
3. Mr. X has a winning strategy in $\mathcal{G}_{\varphi}^{\omega_1}$.

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We proved it for a large class of predicates which includes:
The set of factorial numbers P=$\{n! \;:\; n \in Nat\}$.
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The set of values of every polynom g - $P_g = \{g(n) \ : \ n \in Nat\}$.

Conjuncture: For each countable ordinal α, and $\varphi \in$FOMLO it is decidable whether the winner of $\mathcal{G}^{\alpha}_{\varphi}$ has an FOMLO definable winning strategy.

Conjuncture:(Determinacy for FOMLO over ω_1) For each FOMLO formula φ the game $\mathcal{G}^{\omega_1}_{\varphi}$ is determinate.

Daniele and Anatol

We wish you the most enjoyable and fulfilling retirement which will just contribute to your continuous scientific endeavor.